1. Show that if a system $Ax \leq b$ has an extreme point, then the columns of A are linearly independent. **(3 marks)**

2. Prove or disprove the following statement: if the primal has a unique optimum, then the dual has a unique optimum (an LP is said to have a unique optimum if there is exactly one feasible solution giving the optimum value). **(5 marks)**

3. Suppose x^* is the unique optimum of an LP. Show that the second best extreme point must be adjacent to x^*. **(5 marks)**

4. Consider the problem: minimize $c^T x$ subject to $Ax = b$ and $x \geq 0$, where A is an $n \times n$ square matrix with $A = A^T$, and $c = b$.

   Show that if there exists an x_0 such that $Ax_0 = b$, $x_0 \geq 0$, then x_0 is an optimal point. **(5 marks)**

5. Consider the following problem. **(7 marks)**

$$
\begin{array}{lccccccccccl}
\text{Maximize} & 10x_1 & + & 24x_2 & + & 20x_3 & + & 20x_4 & + & 25x_5 & & \\
\\
\text{Subject to} & x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & + & 5x_5 & \leq 19 & (C1) \\
 & 2x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & \leq 57 & (C2) \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq 0 &
\end{array}
$$

   (a) Write its dual with two variables w_1, w_2 (corresponding to the constraints (C1) and (C2)) and verify that $(w_1, w_2) = (4, 5)$ is a feasible solution.

   (b) Use complementary slackness to show that $(w_1, w_2) = (4, 5)$ gives the optimal solution to the dual.