1. Show that in any open interval of the real line having length $1/[n(n-1)]$ there is atmost one rational of the form p/q where $1 \leq q \leq n$.

2. Consider a mean-payoff game with n vertices and integer weights coming from the set $\{-W, \ldots, W\}$. Prove the following statements about the value $v(a)$ of a vertex a:

   i) $-W \leq v(a) \leq W$

   ii) $v(a)$ is a rational number of the form p/q where $1 \leq q \leq n$

3. Consider Figure 1 in page 348 of [ZP96]. Show that the value iteration method arising out of the equations in Theorem 2.1 needs at least $n^3 W$ iterations to come within $1/[2n(n-1)]$ distance of the actual value.

4. Consider a 2-player game with players Max and Min. Assume that the game starts from some initial vertex a. A pair of strategies (σ, τ) (not necessarily positional) of Max and Min respectively determines a play $\pi_{\sigma,\tau}$. Let Payoff$(\pi_{\sigma,\tau})$ denote the payoff of the play. The *value of the game* starting at a, denoted by $v(a)$ is defined as follows:

   Definition. Let $v_1(a) := \sup_\sigma \inf_\tau \text{Payoff}(\pi_{\sigma,\tau})$ and $v_2(a) := \inf_\tau \sup_\sigma \text{Payoff}(\pi_{\sigma,\tau})$. If $v_1(a)$ equals $v_2(a)$, then the value $v(a)$ is defined to be this number $v_1(a)$.

   Suppose there is a number $v'(a)$ satisfying the following two conditions for the game starting at a:

   - there exists a strategy σ_1 for Max s.t. for every strategy τ of Min, we have Payoff$(\pi_{\sigma_1,\tau}) \geq v'(a)$
   - there exists a strategy τ_1 for Min s.t. for every strategy σ of Max, we have Payoff$(\pi_{\sigma,\tau_1}) \leq v'(a)$

   Show that if such a number $v'(a)$ exists, then it equals the value of the game, that is, $v(a) = v'(a)$.

5. Consider the set of equations given for a discounted payoff game in Theorem 5.1 of [ZP96]. Let x_a be the variable corresponding to vertex a in the set of equations. Theorem 5.1 proves that there is a unique solution to the set of equations. Let this unique solution associate the number η to x_a. Show that the value $v(a)$ of the discounted payoff game starting at a equals η.

6. From a parity game $G = (V_0, V_1, E)$ with n vertices, define a mean-payoff game as follows. The graph for the mean-payoff game remains the same. Player 0 and 1 become respectively the Minimizer and Maximizer in the mean-payoff game. Let $p(u)$ be the priority of a vertex u. Add the weight $-(-n)^{p(u)}$ to all outgoing edges of u in the mean-payoff game.

   Player 1 wins the parity game from a vertex a if she can force a play where the maximum priority occuring infinitely often is odd. Show that Player 1 wins the parity game at a vertex a iff the value $v(a) > 0$ in the mean-payoff game.

References

[ZP96] Uri Zwick and Mike Paterson. The complexity of mean payoff games on graphs. *Theoretical Computer Science*, 158(1):343 – 359, 1996.