Solving parity games using progress measures

Lecture notes for the course "Games on Graphs"

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These notes are based on the paper [Jur00] and Section 3.3.3 of the book [AG11]. The paper [Jur00] introduced the small progress measures algorithm for solving parity games. In these notes, we give a more detailed desription of the algorithm along with illustrative examples.

1 Overview

We consider finite graphs $G = (V, E)$, decorated with a priority function $p : V \mapsto \{1, 2, \ldots, d\}$ which labels every vertex with a number between 1 and d. Without loss of generality, we will assume that the number d is even. A cycle in a graph is said to be *even* if the maximum priority occuring in it is even. It is said to be *odd* if the maximum priority is odd.

Goal: Given a graph, we want to attach a "quantity" to every vertex so that if the quantities satisfy some *local property* at every vertex, the graph satisfies some *global property*. For us, the global property is that all cycles in the graph are even. By local property, we mean that the quantity at each vertex satisfies a condition with respect to the quantities of its neighbours. The goal is to come up with an appropriate definition of the quantity and the local property. This can then be extended to characterize 0-dominions in a parity game graph.

Let us start with a simple cycle G_1: $v_0 \to v_1 \to \cdots \to v_n \to v_0$. Let us further assume that $d = 2$, that is, the only priorities are 1 and 2. We now want a witness to the fact that G_1 contains a vertex of priority 2 - in other words, it is not the case that all vertices have priority 1. Attach a natural number $\xi_1(v_i)$ to each vertex constrained with the following local property for every vertex v_i (assume that $v_{n+1} = v_0$):

$$\begin{aligned} &\text{if } p(v_i) = 1, \text{ then} \quad \xi_1(v_i) > \xi_1(v_{i+1}) \\ &\text{if } p(v_i) = 2, \text{ then} \quad \text{no constraints} \end{aligned} \tag{1.1}$$

Note that if we are able to attach a ξ_1 satisfying (1.1) to G_1, then the cycle G_1 should necessarily have a 2. If all vertices had priority 1, then (1.1) requires that $\xi_1(v_0) > \xi_1(v_1) > \cdots > \xi_1(v_n) > \xi_1(v_0)$, which is not possible. The cycle G_1 needs to have a 2 to stop this strictly decreasing sequence. Conversely, if G_1 is a cycle containing a 2, we would be able to attach a ξ_1 satisfying the local constraints.

Let us now assume that $d = 4$, that is the possible priorities are $1, 2, 3$ and 4. The condition (1.1) says that if 1 occurs in G_1, then 2 should necessarily occur. However, it does not say anything about priority 3 and allows cycles containing only priorities 2 and 3, for instance. We need the additional constraint that if 3 is present, then 4 should necessarily be present in the cycle. To achieve this, an additional number ξ_3 is attached to every vertex. This number decreases whenever there is a priority smaller than 3. To reset this decreasing sequence, a 4 needs to be present. This gives the following local property for each vertex v_i:

$$\begin{array}{lll} \text{if } p(v_i) = 1, \text{ then} & \xi_1(v_i) > \xi_1(v_{i+1}) \text{ and} & \xi_3(v_i) \geq \xi_3(v_{i+1}) \\ \text{if } p(v_i) = 2, \text{ then} & \text{no constraint on } \xi_1 \text{ and} & \xi_3(v_i) \geq \xi_3(v_{i+1}) \\ \text{if } p(v_i) = 3, \text{ then} & \text{no constraint on } \xi_1 \text{ and} & \xi_3(v_i) > \xi_3(v_{i+1}) \\ \text{if } p(v_i) = 4, \text{ then} & \text{no constraint on } \xi_1 \text{ and } \xi_3 & \end{array} \tag{1.2}$$

The above condition (1.2) can be read as: if 1 is present in the cycle, then either $2, 3$ or 4 should necessarily be present and if 3 is present in the cycle, then 4 should necessarily be present. This entails that we can attach the pair $\langle \xi_3, \xi_1 \rangle$ to G_1 iff the maximum priority occuring in G_1 is even.

This observation can be extended to the case when the priorities are $1, 2, \ldots, d$ (recall that d is assumed to even). A tuple $\langle \xi_{d-1}, \xi_{d-3}, \ldots, \xi_3, \xi_1 \rangle$ needs to be attached to every vertex. This contains a representative for every possible odd priority. The quantity ξ_k should decrease in all vertices with priority smaller than k and at k it should strictly decrease. To put it differently, at a vertex v with priority j:

$$\begin{array}{ll} \text{if } j \text{ is odd, then} & \xi_j(v_i) > \xi_j(v_{i+1}), \\ & \xi_{j+2}(v_i) \geq \xi_{j+2}(v_{i+1}), \\ & \quad \cdots \\ & \xi_{d-1}(v_i) \geq \xi_{d-1}(v_{i+1}) \\ & \\ \text{if } j \text{ is even, then} & \xi_{j+1}(v_i) \geq \xi_{j+1}(v_{i+1}), \\ & \xi_{j+3}(v_i) \geq \xi_{j+3}(v_{i+1}), \\ & \quad \cdots \\ & \xi_{d-1}(v_i) \geq \xi_{d-1}(v_{i+1}) \end{array}$$

The above condition says that if an odd priority occurs in the cycle, then a higher priority should necessarily occur. This ensures that the highest priority occuring in the cycle is even. Conversely, for any cycle with maximum priority even, we will be able to attach quantities satisfying the above criterion. These quantities (or *measures*) give a characterization for even cycles.

We wish to have a similar characterization for graphs in which every cycle is even. To do this, we will use the core idea: at a vertex of odd priority k the measure $\langle \xi_{d-1}, \xi_{d-3}, \ldots, \xi_k \rangle$ should strictly decrease along its edges (for a suitably defined ordering). Therefore, the only way to get back to an odd priority vertex is by visiting a vertex of higher priority, where this measure can be increased. This entails that there can be no cycles with maximum priority odd.

2 Progress measures

Let d be an even number. Consider tuples of the form $\langle a_{d-1}, a_{d-3}, \ldots, a_3, a_1 \rangle$ where each $a_i \in \mathbb{N}$. The set of such tuples is denoted as $\mathbb{N}^{d/2}$. We will use $\{\leq, <, =, >, \geq\}$ to denote *lexicographic ordering* on tuples. For instance, $\langle 4, 3, 2, 6 \rangle > \langle 4, 2, 5, 9 \rangle$. Let $G = (V, E)$ be a graph and let $p : V \mapsto \{1, \ldots, d\}$ be a priority function. We define a function $\xi : V \mapsto \mathbb{N}^{d/2}$. Following the notation of [AG11], we will define the function ξ using a sequence of functions $\xi_{d-1}, \xi_{d-3}, \ldots, \xi_3, \xi_1 : V \mapsto \mathbb{N}$. Each $\xi(v)$ would hence be a tuple $\langle \xi_{d-1}(v), \xi_{d-3}(v), \ldots, \xi_3(v), \xi_1(v) \rangle$. We denote by $\xi^{[k]}(v)$ the tuple $\langle \xi_{d-1}(v), \xi_{d-3}(v), \ldots, \xi_k(v) \rangle$ if k is odd and the tuple $\langle \xi_{d-1}(v), \xi_{d-3}(v), \ldots, \xi_{k+1}(v) \rangle$ if k is even.

Definition 1 (Parity progress measure) Given a graph $G = (V, E)$, a function $\xi : V \mapsto \mathbb{N}^{d/2}$ is said to be a parity progress measure for G if for all vertices $v \in V$

$$\xi^{[p(v)]}(v) \geq \max_{(v,w) \in E} \xi^{[p(v)]}(w)$$

The inequality is strict if $p(v)$ is odd.

A graph G is said to *admit* a progress measure if such a function ξ exists for G.

Lemma 2 If a graph G admits a progress measure, all cycles in G are even.

Proof
Suppose there is an odd cycle in G: $v_0 \to v_1 \to \cdots \to v_n \to v_0$. Assume that v_0 is the vertex with the maximum priority, say k. By our assumption, k is odd. Let ξ be the progress measure admitted by G. We have:

$$\xi^{[p(v_i)]}(v_i) \geq \xi^{[p(v_i)]}(v_{i+1})$$

Since $p(v_i) \leq k$ for all i, we have that $\xi^{[k]}(v_i) \geq \xi^{[k]}(v_{i+1})$ for all i (note that this is the lexicographic ordering over tuples of the form $(a_{d-1}, a_{d-3}, \ldots, a_k)$). This gives:

$$\xi^{[k]}(v_0) \geq \xi^{[k]}(v_1) \geq \cdots \geq \xi^{[k]}(v_n) \geq \xi^{[k]}(v_0)$$

Additionally the inequality $\xi^{[k]}(v_0) \geq \xi^{[k]}(v_1)$ is strict. This gives a contradiction $\xi^{[k]}(v_0) > \xi^{[k]}(v_0)$. Hence there can be no odd cycles in G. □

The converse to the statement of the above lemma is also true - if all cycles in a graph are even, it admits a progress measure. We can in fact prove some boundedness criterion on the progress measure which it admits. For a graph G, let n_q be the number of vertices of priority q. We denote by M_G the set of tuples $\langle a_{d-1}, a_{d-3}, \ldots, a_3, a_1 \rangle \in \mathbb{N}^{d/2}$ such that each $a_q \leq n_q$.

Definition 3 (Small parity progress measure) A progress measure $\xi : V \mapsto \mathbb{N}^{d/2}$ on a graph $G = (V, E)$ is said to be *small* if $\xi(v) \in M_G$ for each $v \in V$.

The following lemma and its proof appear in [Jur00] with a different notation.

Lemma 4 If all cycles in a graph G are even, G admits a small progress measure.

Proof
We will prove a stronger property about the small progress measure: we can construct a progress measure ξ for G in which for all vertices v with $p(v)$ odd, $\xi^{[p(v)]}(v) > \langle 0, 0, \dots, 0\rangle$. Proof proceeds by induction. For the base case, assume that graph G has a single vertex v. If $p(v)$ is even, assign $\xi(v)$ to $\langle 0, 0, \dots, 0\rangle$. If $p(v)$ is odd, then set $\xi_{p(v)}(v) = 1$ and set the rest to 0.

Now, consider a graph G with n vertices. Assume that the lemma holds for all graphs with at most $n-1$ vertices. Without loss of generality, we can also assume that the set of vertices with priority d or $d-1$ is non-empty.

Suppose G has a vertex u with $p(u) = d$. Consider the subgraphs induced by $G \setminus \{u\}$ and $\{u\}$. By induction hypothesis, we can associate a small progress measure ξ' to $G \setminus u$. Set $\xi(u) := \langle 0, 0, \dots, 0\rangle$, and $\xi(v) := \xi'(v)$ to all $v \in G \setminus \{u\}$. Since $p(v)$ is even and is also the highest priority, the function ξ satisfies the conditions to be a progress measure. Clearly, it is also small.

Suppose G has no vertices with priority d. Pick a vertex u with priority $p(u) = d - 1$. Such a vertex should definitely exist because of our assumption. Let $W_1 := \{v \in V \mid \text{there is a non-trivial path from } u \text{ to } v \text{ in } G\}$. The vertex $u \notin W_1$ since that gives a cycle containing u, and as $p(u) = d - 1$ and there are no vertices of priority d, this cycle would be odd. Therefore the set $W_2 := V \setminus W_1$ is a non-empty set (containing at least u). This gives a partition (W_1, W_2) of G such that there are no edges from W_1 to W_2. Let n'_q, n''_q, n_q be the number of vertices of priority q in W_1, W_2 and G respectively. Clearly, $n_q = n'_q + n''_q$. By induction hypothesis, we can associate small progress measures $\rho_1 : W_1 \mapsto M_{W_1}$ and $\rho_2 : W_2 \mapsto M_{W_2}$. Define the function ξ for G as follows:

$$\xi(v) := \begin{cases} \rho_1(v) & \text{if } v \in W_1 \\ \rho_2(v) + \langle n'_{d-1}, n'_{d-3}, \dots, n'_3, n'_1\rangle & \text{if } v \in W_2 \end{cases}$$

From our additional assumption that $\rho_2(v)^{[p(v)]} > \langle 0, 0, \dots, 0\rangle$, it can be inferred that $\xi(v)$ is a progress measure. The fact that $n_q = n'_q + n''_q$ for every q gives that ξ is a small progress measure. □

The above lemmas give us the following theorem.

Theorem 5 *A graph G has all cycles even iff it admits a small progress measure.*

Small progress measures are therefore a witness for graphs containing only even cycles. Not all graphs have all cycles even. For graphs with odd cycles, we can characterize subgraphs which contain only even cycles by extending the progress measures with a fresh element $\top$. The element $\top$ is defined to be bigger than every tuple. A progress measure is now a function $\xi : V \mapsto \mathbb{N}^{d/2} \cup \{\top\}$ such that for all vertices v with $\xi(v) \neq \top$, the condition given by 1 holds. Similarly a small progress measure is such a function $\xi : V \mapsto M_V \cup \{\top\}$. We define $\text{dom}(\xi) := \{v \in V \mid \xi(v) \neq \top\}$.

Lemma 6 For a progress measure ξ, the graph induced by $\text{dom}(\xi)$ has all cycles even.

Proof
Follows from Lemma 2. □

A converse statement to the above holds too. Let us call a set $G' \subseteq G$ a *trap* if G' is an induced subgraph and there is no edge from a vertex in G' to a vertex in G.

Lemma 7 For every trap $G' = (V', E')$ of G in which all cycles are even, there exists a progress measure ξ such that $\text{dom}(\xi) = V'$.

Proof
The trap G' partitions the graph into $V', V \setminus V'$ such that there are no edges from V' to $V \setminus V'$. Since in V' all cycles are even, we can give a progress measure (without $\top$) thanks to Lemma 4. For vertices in $V \setminus V'$, we can assign $\top$. This is a progress measure since there are no edges from V' to V. Moreover its domain is exactly V'. □

Lemmas 2 and 4 characterize traps in a graph in which all cycles are even. We will now use this idea to characterize dominions of Player 0 in a parity game. Recall that Player 0 wins a play if the *maximum* priority occurring infinitely often is *even.* A set $D \subseteq V$ is said to be a 0-dominion if 0 can win from every vertex in D and moreover 0 can force Player 1 to stay inside D. Due to positional determinacy of parity games, 0 has a positional winning strategy in its dominion. If we remove the edges from 0 vertices in D that are not given by the strategy, the resultant graph would be a trap in which all cycles are even. This now leads us to the definition of a progress measure for a parity game.

Definition 8 (Game parity progress measure) Let $G = (V, E)$ be a parity game graph. A function $\xi : V \mapsto M_V \cup \{\top\}$ is a (small) game parity progress measure if for all vertices $v \in V$:

$$\text{if } v \in V_0 \text{ and } \xi(v) \neq \top \text{ , then } \xi^{[p(v)]} \geq \min_{(v,w)\in E} \xi^{[p(v)]}(w)$$
$$\text{and if } p(v) \text{ is odd the inequality is strict}$$

and

$$\text{if } v \in V_1 \text{ and } \xi(v) \neq \top \text{ , then } \xi^{[p(v)]} \geq \max_{(v,w)\in E} \xi^{[p(v)]}(w)$$
$$\text{and if } p(v) \text{ is odd the inequality is strict}$$

The discussion above leads us to the following lemmas. Fix a game graph $G = (V, E)$.

Lemma 9 The set $\text{dom}(\xi)$ of a game parity progress measure is a 0-dominion.

Proof
We will define a strategy σ for Player 0. For every $v \in V_0 \cap \mathrm{dom}(\xi)$, let $\sigma(v)$ be the w with minimum progress measure. Restrict the graph to vertices in $\mathrm{dom}(\xi)$ and remove all edges from 0 vertices that are not given by σ. Call this graph G'. In this graph all Player 0 vertices have a single outgoing edge. Therefore the function ξ satisfies the condition 1 given for a parity progress measure. From Lemma 2, all cycles in G' are even. Since Player 1 vertices in G' belong to $\mathrm{dom}(\xi)$, all edges out of these vertices lead to vertices in $\mathrm{dom}(\xi)$ and hence in G'. These two observations show that G' is a 0 dominion for which σ is a winning strategy. □

Lemma 10 For every 0-dominion $D \in V$, there exists a game parity progress measure ξ such that $\mathrm{dom}(\xi) = D$.

Proof
Similar to proof of Lemma 7. □

The above two lemmas give a function from the set of progress measures to the set of 0-dominions. We want the progress measure corresponding to the biggest 0-dominion. In the subsequent sections, we will address the following two questions:

1. Which progress measure corresponds to the biggest 0-dominion?

2. How do we compute this progress measure?

2.1 Least progress measure

Fix a graph $G = (V, E)$ and a priority function $p : V \mapsto \{1, 2, \ldots, d\}$ where d is even. For each $i \leq d$, let n_i be the number of vertices in V having priority i. Recall that we denote by M_V the set of tuples $\langle a_{d-1}, a_{d-3}, \ldots, a_3, a_1\rangle$ such that $0 \leq a_i \leq n_i$ for each a_i. Clearly, M_V is a finite set. We consider the lexicographic ordering on the set M_V, denoted by $\leq_{\text{lex}}$. Additionally, we will define the element $\top$ to be strictly bigger than every element of M_V, that is, $s <_{\text{lex}} \top$ for every element $s \in M_V$.

Let $\mathcal{F}$ be the set of functions of the form $V \mapsto M_V \cup \{\top\}$. This set is finite. The lexicographic ordering $\leq_{\text{lex}}$ can now be extended to functions. For two functions $f_1, f_2 \in \mathcal{F}$, we define $f_1 \sqsubseteq f_2$ if for all $v \in V$, we have $f_1(v) \leq_{\text{lex}} f_2(v)$. The relation $\sqsubseteq$ is a partial order (not all pairs of functions are comparable).

Game progress measures are elements of $\mathcal{F}$ which additionally satisfy the local conditions given by Definition 8. As mentioned before, each progress measure can be mapped to a 0-dominion. The following lemma establishes that as the progress measures decrease (in the order $\sqsubseteq$), the 0-dominions become bigger (in the usual subset order).

Lemma 11 Let ρ_1 and ρ_2 be progress measures. Then, $\rho_1 \sqsubseteq \rho_2 \Rightarrow \mathrm{dom}(\rho_1) \supseteq \mathrm{dom}(\rho_2)$.

Proof
Pick $v \in \mathrm{dom}(\rho_2)$. By definition, $\rho_2(v) \neq \top$. As $\rho_2(v) \geq_{\text{lex}} \rho_1(v)$ by assumption, the value $\rho_1(v)$ cannot be $\top$ either. Hence $v \in \mathrm{dom}(\rho_1)$, showing that $\mathrm{dom}(\rho_2) \subseteq \mathrm{dom}(\rho_1)$. □

Definition 12 (Least progress measure) Let $\rho_1, \rho_2, \ldots, \rho_k$ be the set of progress measures for G. Define $\rho^* : V \mapsto M_V \cup \{\top\}$ as:

$$\rho^*(v) := \min_{0 \leq i \leq k} \rho_i(v) \quad \text{for each } v \in V$$

It is not immediate that the function ρ^* defined above satisfies the local conditions needed for it to be a progress measure. The following lemma shows that the local conditions are indeed satisfied.

Lemma 13 The function ρ^* is a progress measure.

Proof
Pick a $v \in V_0$. The value $\rho^*(v)$ equals $\rho_m(v)$ for some m. Since ρ_m is a progress measure, we know that $\rho_m(v) \geq_{\text{lex}} \rho_m(w)$ for some neighbour w of v, and the inequality is strict if $p(v)$ is odd. By definition, $\rho^*(w) \leq \rho_m(w)$. This gives the following sequence of inequalities:

$$\rho^*(v) \ =_{\text{lex}} \ \rho_m(v) \ \geq_{\text{lex}} \ \rho_m(w) \ \geq_{\text{lex}} \ \rho^*(w)$$

The inequality in the middle is strict if $p(v)$ is odd. This shows that v satisfies the local condition if $v \in V_0$. A similar reasoning can be done for the case $v \in V_1$. ☐

From Lemma 11, the smaller the progress measure, the bigger is the dominion. Since ρ^* is the smallest progress measure, it will correspond to the winning region of 0 which is the biggest 0-dominion, as every 0-domininion has an associated progress measure due to Lemma 10.

3 Computing the least progress measure

Algorithm 1.1 gives a method to compute the least progress measure. Consider the procedure *progress-measure-lifting(G)*. It starts by assigning the tuple $(0, \ldots, 0) \in \mathbb{N}^{d/2}$ to every vertex. Call this initial function ρ_0. After each iteration of the while loop, another function is computed. Let ρ_i denote the function resulting after the i^{th} iteration. In each iteration, a vertex which does not satisfy the local progress condition is picked and its value is *lifted* to the least value in $M_V \cup \{\top\}$ so that the local condition is met. The *lift* procedure in Algorithm 1.1 shows the implementation. The algorithm stops if the progress condition is satisfied at every vertex. Therefore, if the algorithm stops, it is clear that the function in the end is a progress measure. We will now show that the algorithm does terminate, and moreover the progress measure obtained in the end is the least progress measure ρ^*.

Lemma 14 For each i, we have $\rho_i \sqsubseteq \rho_{i+1}$.

Proof
If ρ_{i+1} is the same as ρ_i, we are done. Otherwise ρ_{i+1} differs from ρ_i only at a single vertex v. At this vertex, the value $\rho_{i+1}(v)$ is bigger than $\rho_i(v)$ due to the while loop condition in Line 3. This proves the lemma. ☐

The above lemma shows that the algorithm does terminate. The following lemma shows that the computed progress measure is the least.

Algorithm 1.1: Progress measure lifting algorithm

```
algorithm progress-measure-lifting(G)
   for all v ∈ V do ξ(v) ← (0, 0, ..., 0)
   while ξ(v) <_lex lift(ξ, v)
       do ξ(v) ← lift(ξ, v)
   endwhile
   return (dom(ξ), V \ dom(ξ))

algorithm lift(ξ, v)
   if v ∈ V_0 then t ← min_{v→w} ξ^{[p(v)]}(w)
   else t ← max_{v→w} ξ^{[p(v)]}(w)
   endif
   if (t = ⊤) return ⊤
   if p(v) is even return ⟨t_{d−1}, t_{d−3}, ..., t_{p(v)+1}, 0, 0, ..., 0⟩
   else
       if (t_i = n_i for all i ∈ {p(v), ..., d − 3, d − 1})
           return ⊤
       else
           find smallest j ∈ {p(v), ..., d − 3, d − 1} s.t. t_j < n_j
           return ⟨t_{d−1}, t_{d−3}, ..., t_j + 1, 0, ..., 0⟩
       endif
   endif
```

Lemma 15 For each i, we have $\rho_i \sqsubseteq \rho^*$.

Proof
Clearly, $\rho_0 \sqsubseteq \rho^*$. We will now show that if $\rho_i \sqsubseteq \rho^*$, then $\rho_{i+1} \sqsubseteq \rho^*$, which proves the lemma.

Either $\rho_{i+1} = \rho_i$, in which case we are done. Otherwise, ρ_{i+1} differs from ρ_i at a single vertex v. It is therefore enough to show that $\rho_{i+1}(v) \leq_{\text{lex}} \rho^*(v)$. Let us assume that $v \in V_0$. The case when $v \in V_1$ can be shown using the same arguments as below, replacing min with max.

Firstly, note that as $\rho_i \sqsubseteq \rho^*$, we have:

$$\min_{v \to w} \rho_i^{[p(v)]}(w) \;\; \leq_{\text{lex}} \;\; \min_{v \to w} \rho^{*[p(v)]}(w) \tag{1.3}$$

Since ρ^* is a progress measure, we have:

$$\rho^{*[p(v)]}(v) \geq_{\text{lex}} \min_{v \to w} \rho^{*[p(v)]}(w) \quad \text{where } \geq_{\text{lex}} \text{ is strict if } p(v) \text{ is odd} \tag{1.4}$$

From (1.3) and (1.4), we get:

$$\min_{v \to w} \rho_i^{[p(v)]}(w) \;\; \leq_{\text{lex}} \;\; \rho^{*[p(v)]}(v) \quad \text{where } \leq_{\text{lex}} \text{ is strict if } p(v) \text{ is odd} \tag{1.5}$$

Let $\vec{a}$ be the smallest tuple which is $\geq_{\text{lex}}$ than $\min_{v \to w} \rho_i^{[p(v)]}(w)$ with $\geq_{\text{lex}}$ being strict if $p(v)$ is odd. If $\vec{a} = \top$, then from (1.5), we have $\rho^*(v) = \top$. The algorithm sets $\rho_{i+1}(v) := \top$ in this case and hence we get $\rho_{i+1}(v) \sqsubseteq \rho^*(v)$. If $\vec{a} \neq \top$, again from (1.5), we get that $\vec{a} \leq_{\text{lex}} \rho^{*[p(v)]}(v)$. The algorithm sets $\rho_{i+1}^{[p(v)]}(v) := \vec{a}$ and $\rho_{i+1}^k(v) = 0$ for all $k < p(v)$. This shows that $\rho_{i+1}(v) \sqsubseteq \rho^*(v)$.

$\square$

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