Games

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Outline

Finite Duration Games
- Win-Lose Games
- Payoff Games

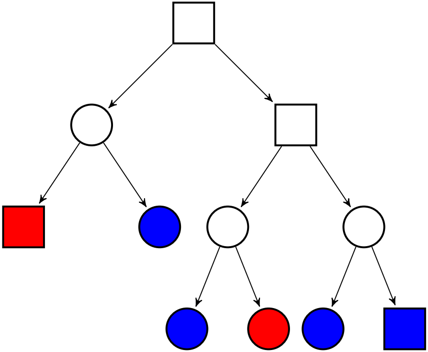
Infinite Duration Games
- Parity Games
- Mean Payoff Games

Simple Stochastic Games

Outline

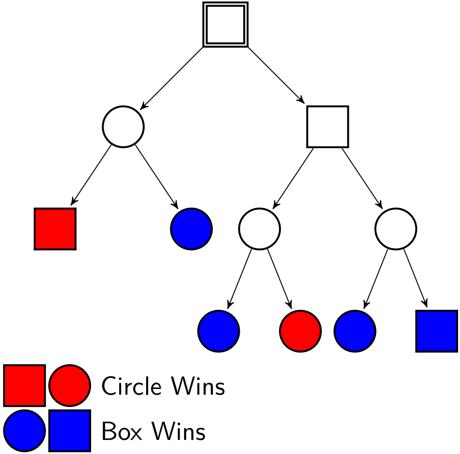
Finite games

Win-Lose game



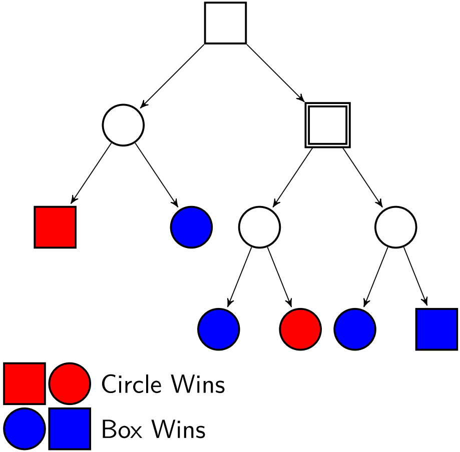
Finite games

Win-Lose game



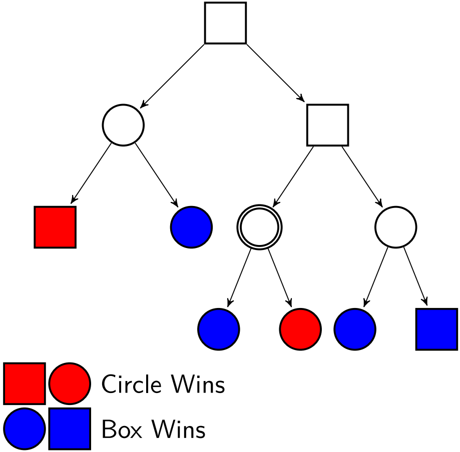
Finite games

Win-Lose game



Finite games

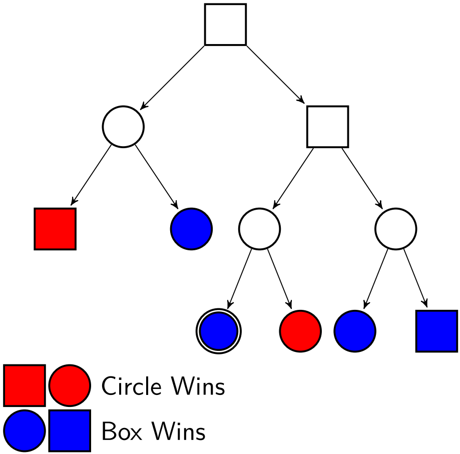
Win-Lose game



Finite games

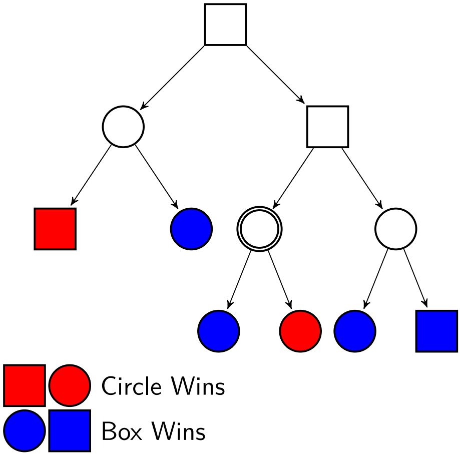
Win-Lose game

Box wins



Finite games

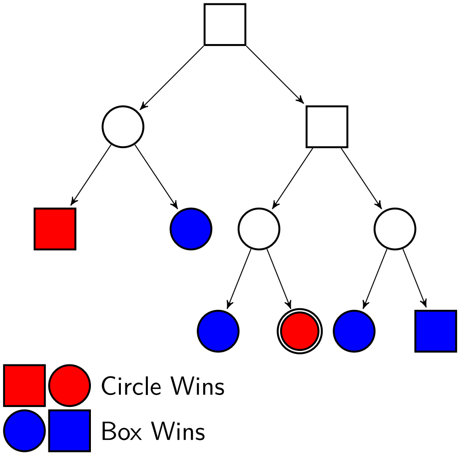
Win-Lose game



Finite games

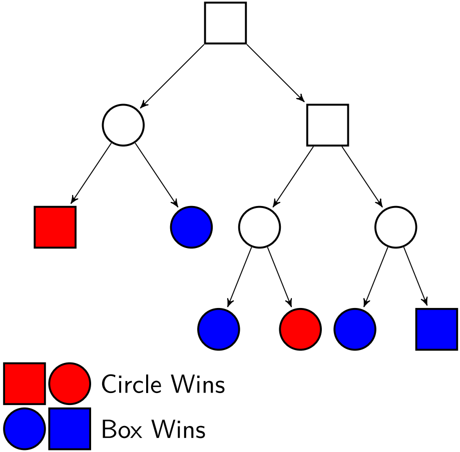
Win-Lose game

Circle wins



Finite games

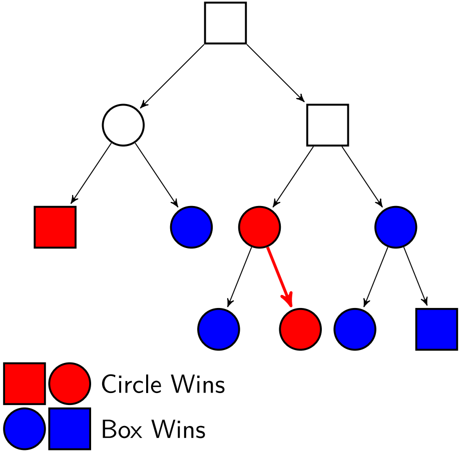
Win-Lose game



Finite games

Win-Lose game

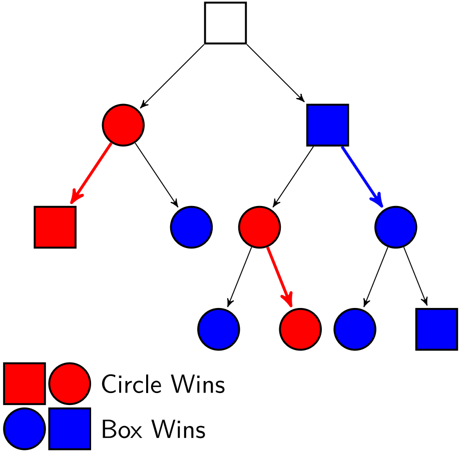
Algorithm for optimal play



Finite games

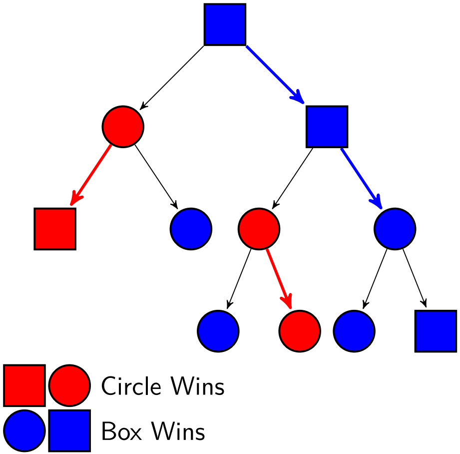
Win-Lose game

Algorithm for optimal play



Finite games

Win-Lose game

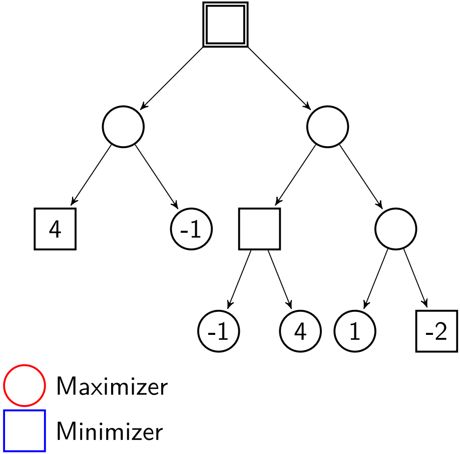


Algorithm for optimal play

Box can always win

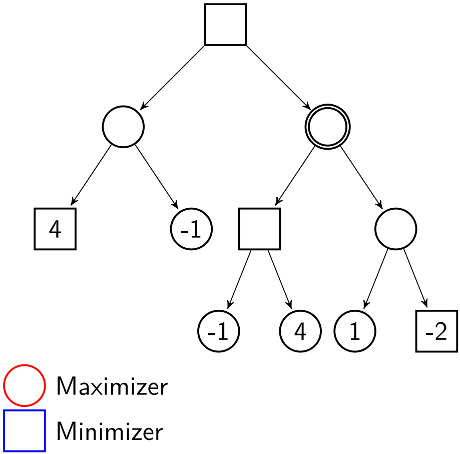
Finite games

Payoff game



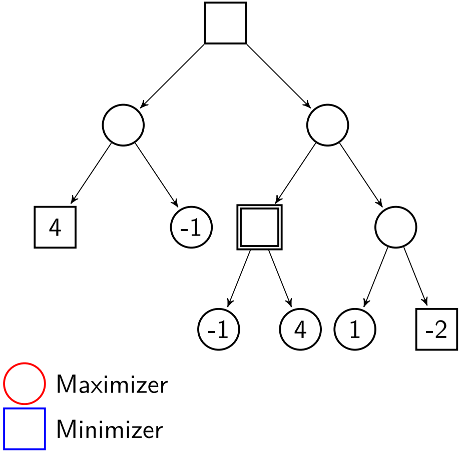
Finite games

Payoff game



Finite games

Payoff game

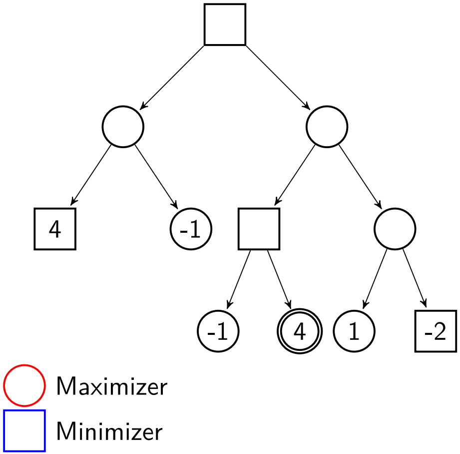


Finite games

Payoff game

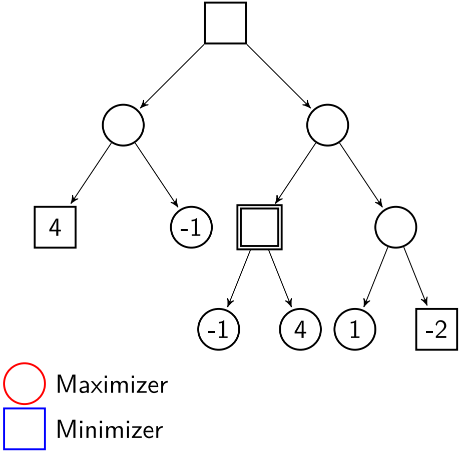
Payoff

Min pays 4 units to Max



Finite games

Payoff game

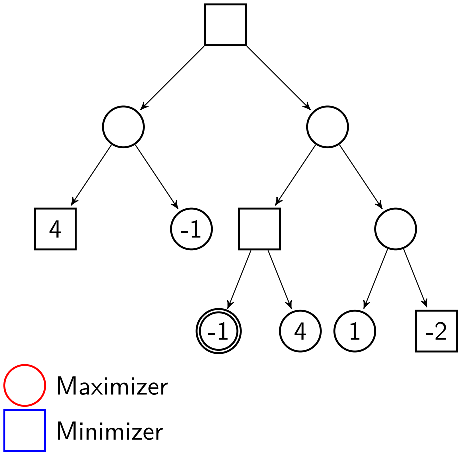


Finite games

Payoff game

Payoff

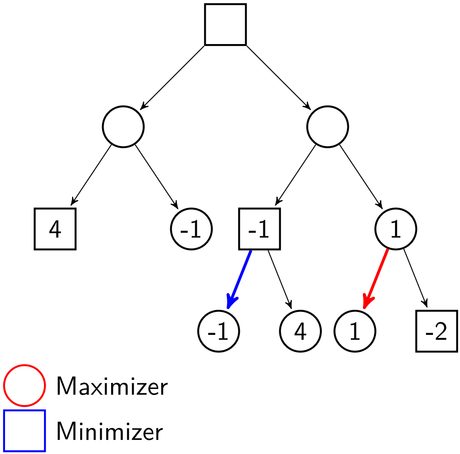
Min pays -1 units to Max



Finite games

Payoff game

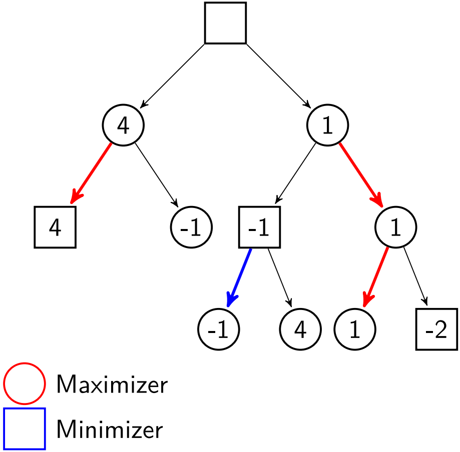
MinMax algorithm



Finite games

Payoff game

MinMax algorithm



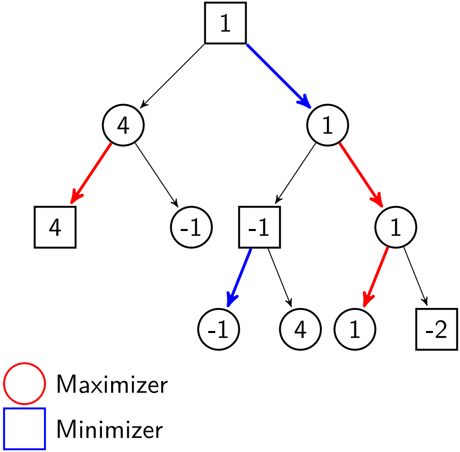
Finite games

Payoff game

MinMax algorithm

Value $= 1$

- Min can ensure a payoff ≤ 1
- Max can ensure a payoff ≥ 1



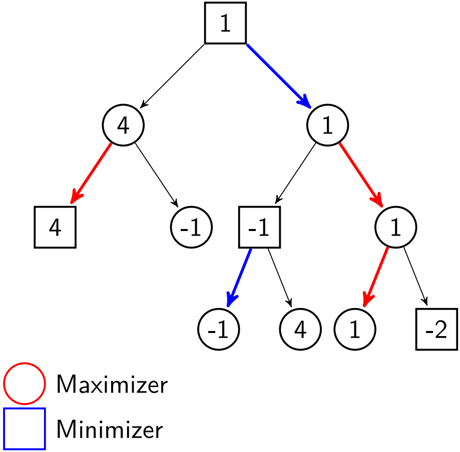
Finite games

Payoff game

MinMax algorithm

Value $= 1$

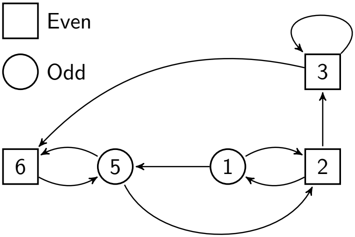
- Min can ensure a payoff ≤ 1
- Max can ensure a payoff ≥ 1
- When both play optimally the payoff is exactly 1.



Outline

Parity Games

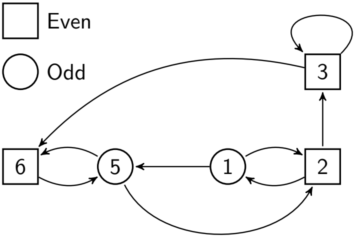
Winning conditions



Parity Games

Winning conditions

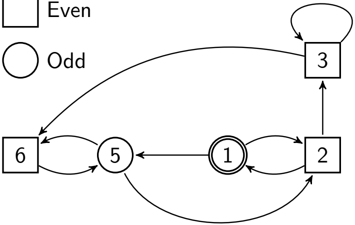
$\pi_1 =$



Parity Games

Winning conditions

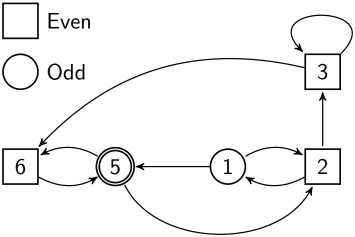
$\pi_1 = \quad 1$



Parity Games

Winning conditions

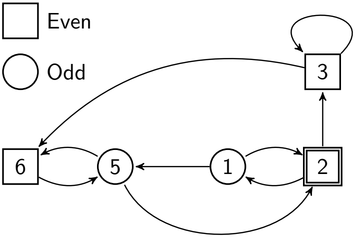
$\pi_1 = \ 1 \ \ 5$



Parity Games

Winning conditions

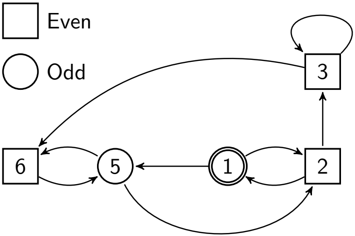
$\pi_1 = \quad 1 \;\; 5 \;\; 2$



Parity Games

Winning conditions

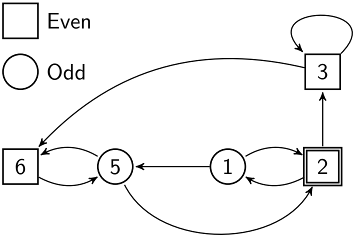
$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1$



Parity Games

Winning conditions

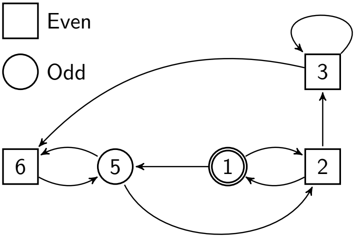
$\pi_1 = \; 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2$



Parity Games

Winning conditions

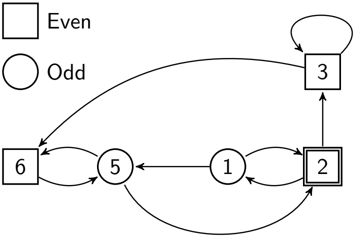
$\pi_1 = \;\; 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$
$\inf(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$
Even wins



Parity Games

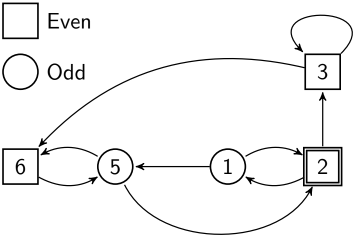
Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

Even wins

$\pi_2 =$



Parity Games

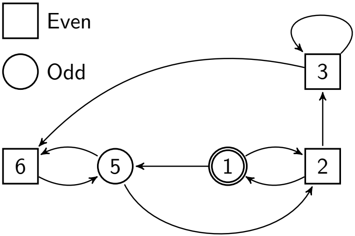
Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\inf(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$

Even wins

$\pi_2 = \quad 1$



Parity Games

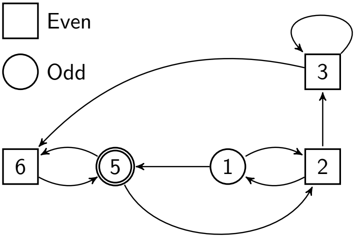
Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathrm{inf}(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$

Even wins

$\pi_2 = \quad 1 \;\; 5$



Parity Games

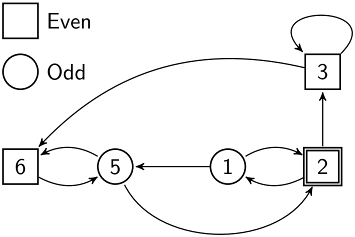
Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2$



Parity Games

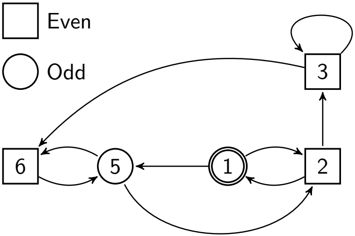
Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1$



Parity Games

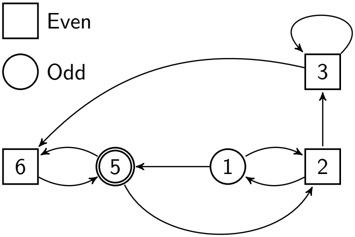
Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\text{inf}(\pi_1) = \{1, 2\} \qquad \max \text{Inf}(\pi_1) = 2$

Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

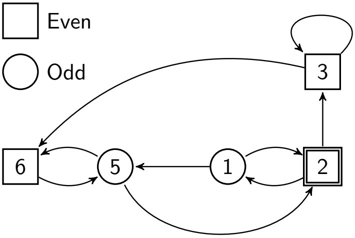
$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_2) = \{1, 2, 5\} \qquad \max \mathsf{Inf}(\pi_2) = 5$

Odd wins



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1,2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

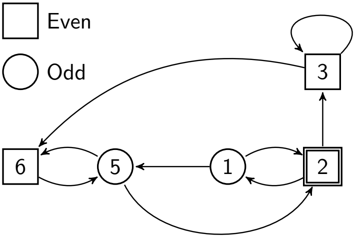
Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_2) = \{1,2,5\} \qquad \max \mathsf{Inf}(\pi_2) = 5$

Odd wins

$\pi =$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

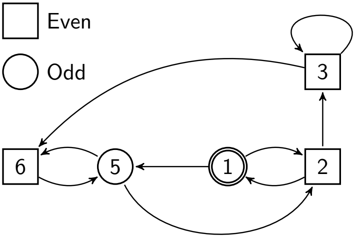
Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_2) = \{1, 2, 5\} \qquad \max \mathsf{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1$



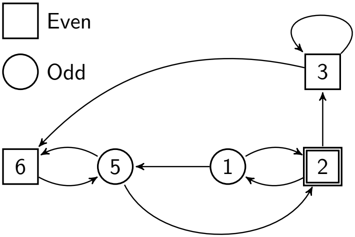
Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$
$\mathrm{inf}(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$
Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$
$\mathrm{inf}(\pi_2) = \{1, 2, 5\} \qquad \max \mathrm{Inf}(\pi_2) = 5$
Odd wins

$\pi = \quad 1 \;\; 2$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\inf(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$

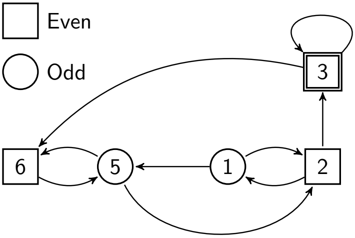
Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\inf(\pi_2) = \{1, 2, 5\} \qquad \max \mathrm{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1 \;\; 2 \;\; 3$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\inf(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$

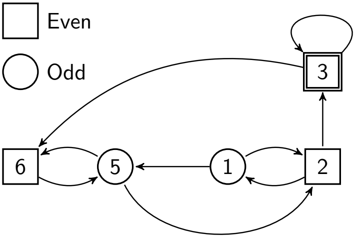
Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\inf(\pi_2) = \{1, 2, 5\} \qquad \max \mathrm{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1 \;\; 2 \;\; 3 \;\; 3$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \ \ 5 \ \ 2 \ \ 1 \ \ 2 \ \ 1 \ \ 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

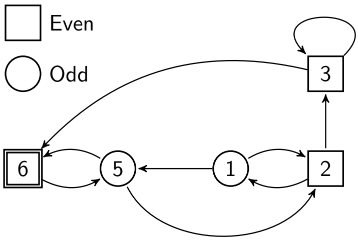
Even wins

$\pi_2 = \quad 1 \ \ 5 \ \ 2 \ \ 1 \ \ 5 \ \ 2 \quad \ldots$

$\mathsf{inf}(\pi_2) = \{1, 2, 5\} \qquad \max \mathsf{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1 \ \ 2 \ \ 3 \ \ 3 \ \ 6$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \quad 5 \quad 2 \quad 1 \quad 2 \quad 1 \quad 2 \quad \ldots$

$\inf(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$

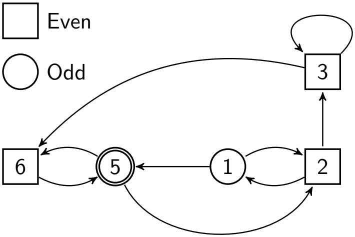
Even wins

$\pi_2 = \quad 1 \quad 5 \quad 2 \quad 1 \quad 5 \quad 2 \quad \ldots$

$\inf(\pi_2) = \{1, 2, 5\} \qquad \max \mathrm{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1 \quad 2 \quad 3 \quad 3 \quad 6 \quad 5$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\inf(\pi_1) = \{1, 2\} \qquad \max \mathrm{Inf}(\pi_1) = 2$

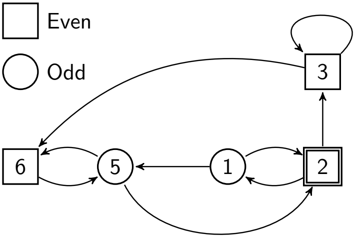
Even wins

$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\inf(\pi_2) = \{1, 2, 5\} \qquad \max \mathrm{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1 \;\; 2 \;\; 3 \;\; 3 \;\; 6 \;\; 5 \;\; 2$



Parity Games

Winning conditions

$\pi_1 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_1) = \{1, 2\} \qquad \max \mathsf{Inf}(\pi_1) = 2$

Even wins

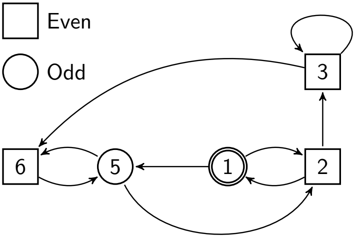
$\pi_2 = \quad 1 \;\; 5 \;\; 2 \;\; 1 \;\; 5 \;\; 2 \quad \ldots$

$\mathsf{inf}(\pi_2) = \{1, 2, 5\} \qquad \max \mathsf{Inf}(\pi_2) = 5$

Odd wins

$\pi = \quad 1 \;\; 2 \;\; 3 \;\; 3 \;\; 6 \;\; 5 \;\; 2 \;\; 1 \quad \ldots$

$\mathsf{Parity}(\max \mathsf{Inf}(\pi))$ wins



Parity Games

Questions

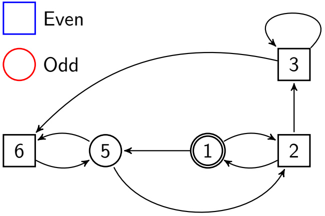
- Does either Even or Odd have a strategy to always win?
- If so, then how to compute the winning strategy?

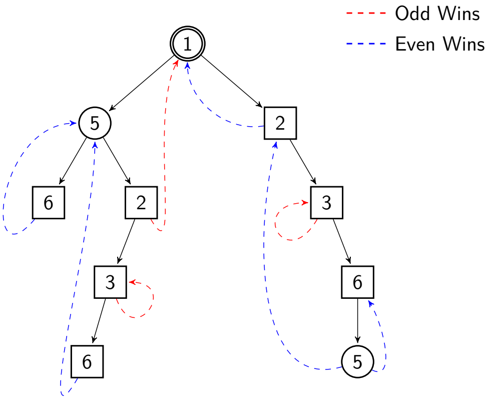
Parity Games

Questions

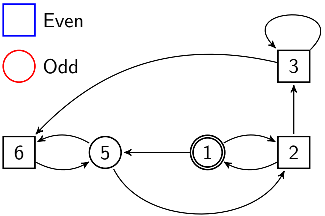
- Does either Even or Odd have a strategy to always win?
  Yes
- If so, then how to compute the winning strategy?
  By reduction to finite duration games

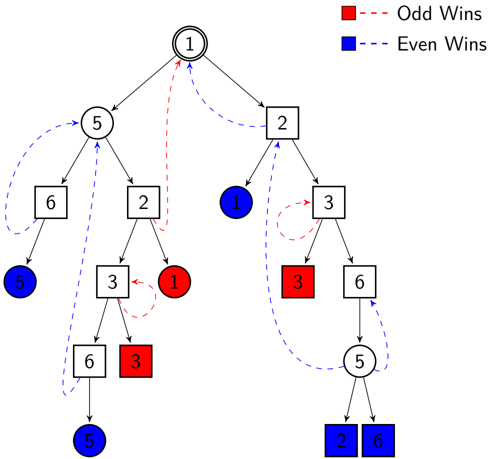
Parity Games



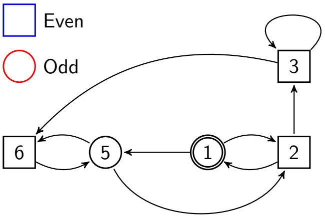


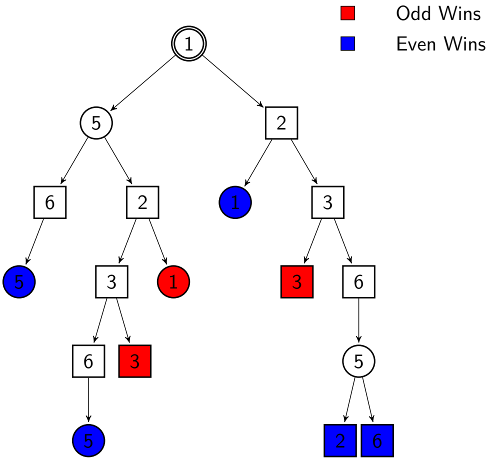
Parity Games



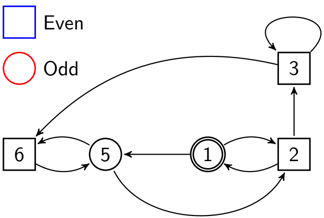


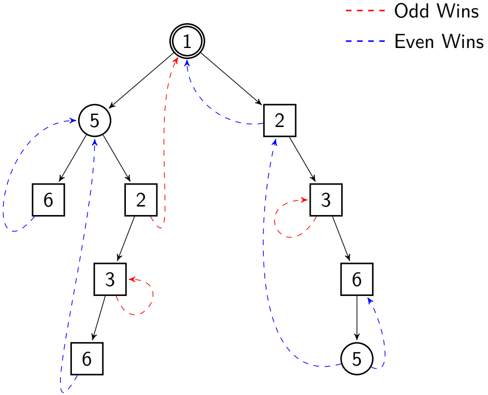
Parity Games



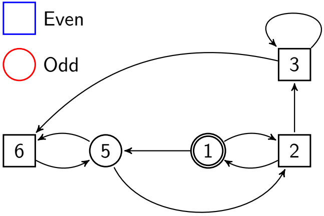


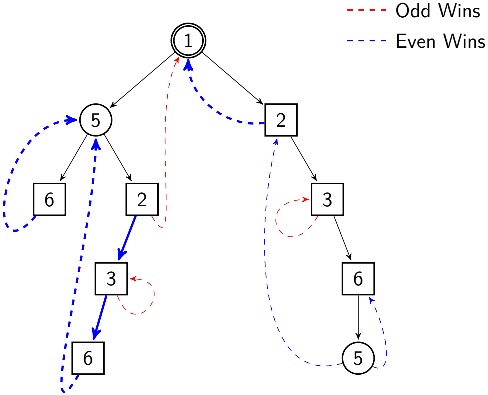
Parity Games





Parity Games

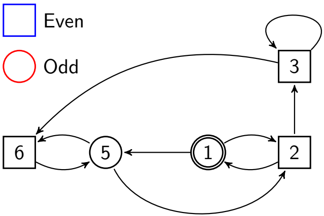


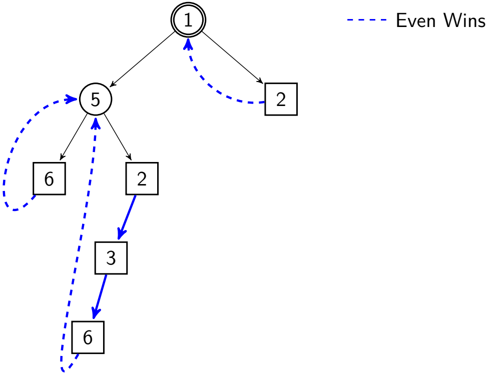


Finite game

Even has a winning strategy

Parity Games

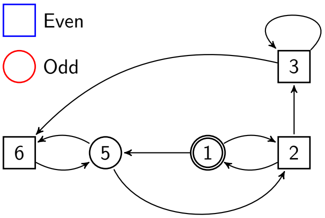


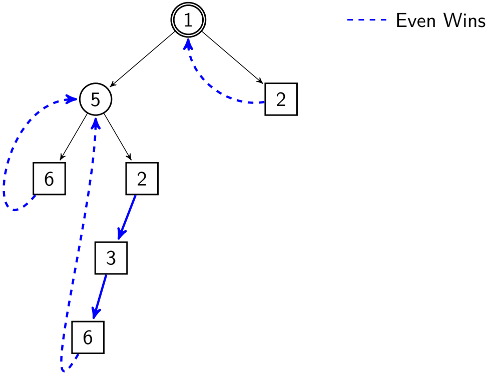


Finite game

Every loop has max priority even

Parity Games



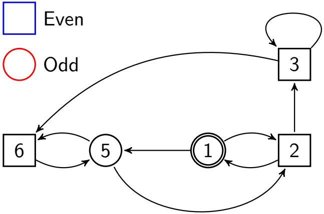


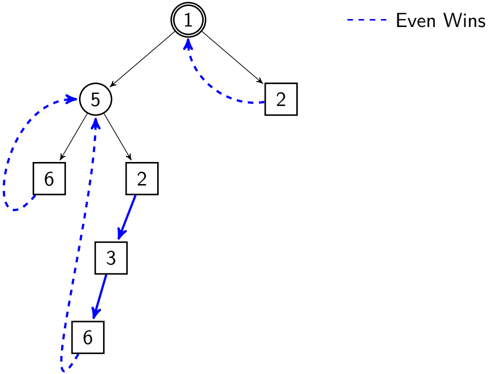
Finite game

Every loop has max priority even

Extension to infinite plays

Parity Games





Finite game

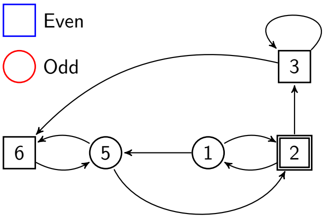
Every loop has max priority even

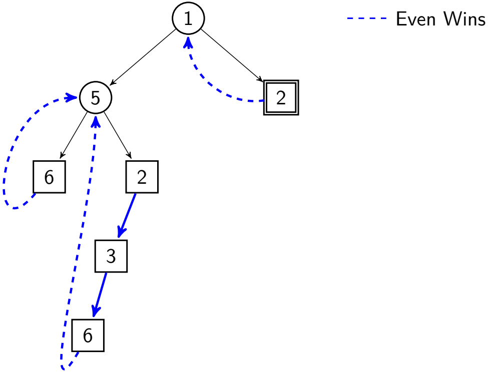
Extension to infinite plays

$\pi = \ 1$

Stack $= 1$

Parity Games





Finite game

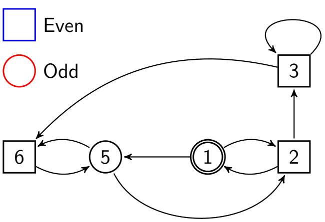
Every loop has max priority even

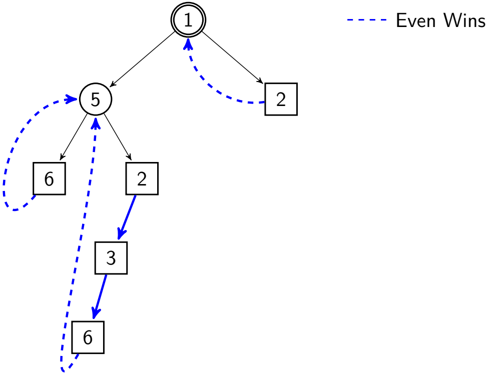
Extension to infinite plays

$\pi = \ 1 \ \ 2$

Stack $= 1 \ \ 2$

Parity Games





Finite game

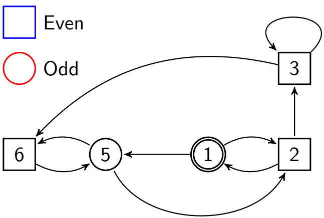
Every loop has max priority even

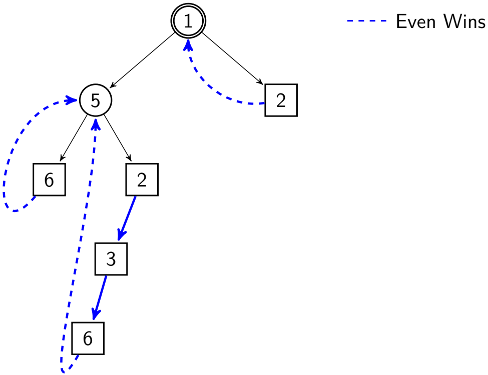
Extension to infinite plays

$\pi =$ 1 2 1

Stack = 1 2 1

Parity Games





Finite game

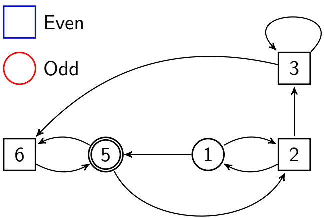
Every loop has max priority even

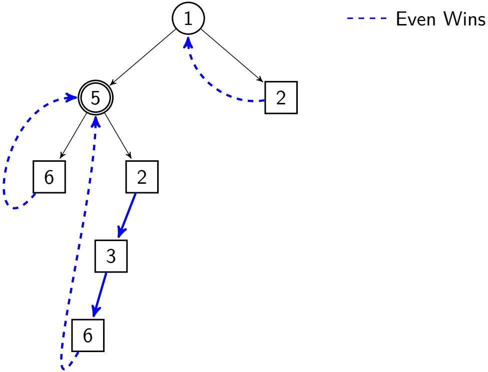
Extension to infinite plays

$\pi = \;\; 1 \;\; 2 \;\; 1$

Stack $= 1$

Parity Games





Finite game

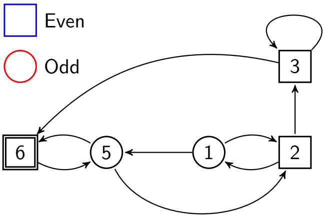
Every loop has max priority even

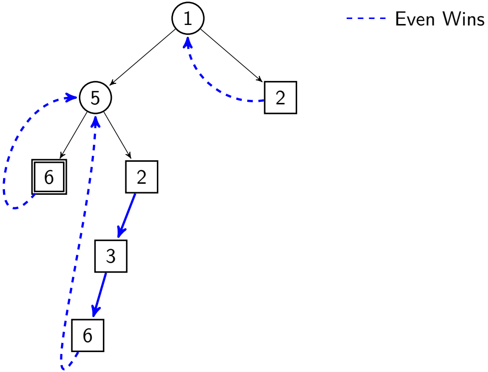
Extension to infinite plays

$\pi =$ 1 2 1 5

Stack = 1 5

Parity Games





Finite game

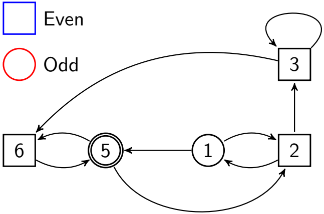
Every loop has max priority even

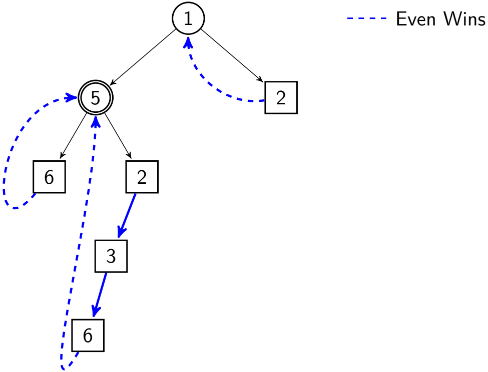
Extension to infinite plays

$\pi = \;\; 1 \;\; 2 \;\; 1 \;\; 5 \;\; 6$

Stack $= 1 \;\; 5 \;\; 6$

Parity Games





Finite game

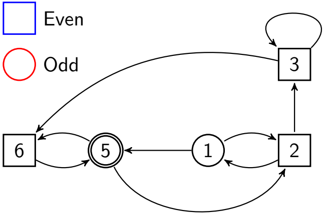
Every loop has max priority even

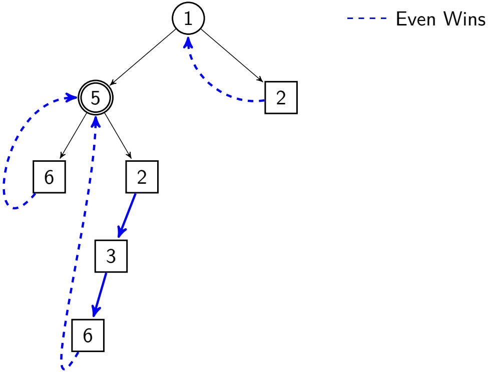
Extension to infinite plays

$\pi =$ 1 2 1 5 6 5

Stack = 1 5 6 5

Parity Games





Finite game

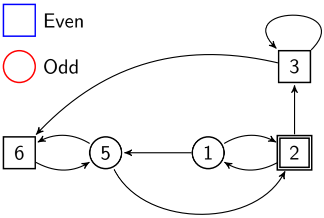
Every loop has max priority even

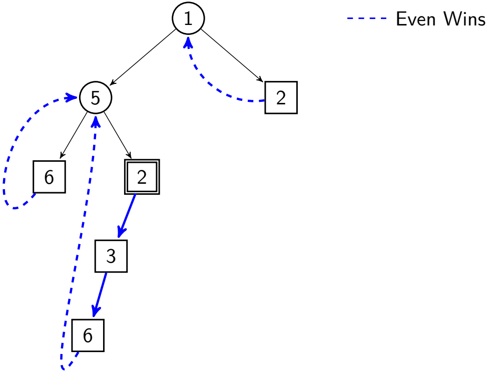
Extension to infinite plays

$\pi =$ 1 2 1 5 6 5

Stack = 1 5

Parity Games





Finite game

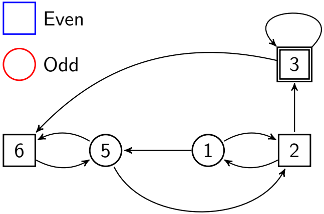
Every loop has max priority even

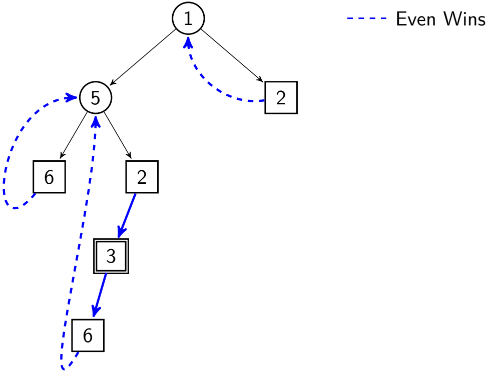
Extension to infinite plays

$\pi = \;\; 1 \;\; 2 \;\; 1 \;\; 5 \;\; 6 \;\; 5 \;\; 2$

Stack = 1 5 2

Parity Games





Finite game

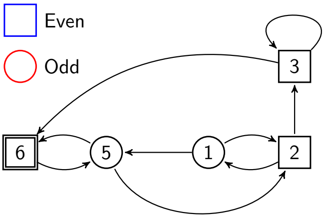
Every loop has max priority even

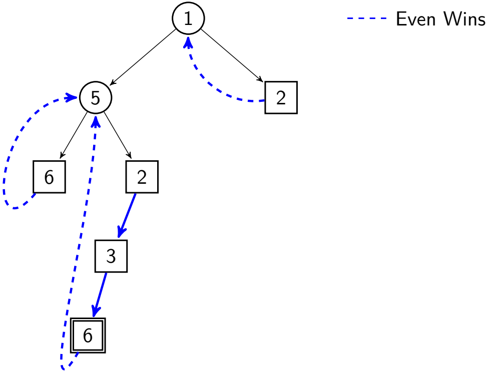
Extension to infinite plays

$\pi = \; 1 \;\; 2 \;\; 1 \;\; 5 \;\; 6 \;\; 5 \;\; 2 \;\; 3$

Stack $= 1 \;\; 5 \;\; 2 \;\; 3$

Parity Games





Finite game

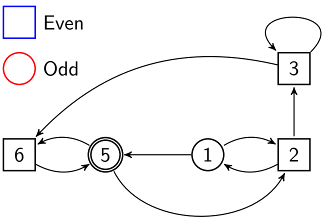
Every loop has max priority even

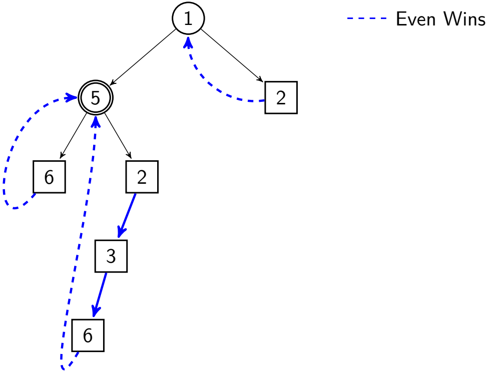
Extension to infinite plays

$\pi =$ 1 2 1 5 6 5 2 3 6

Stack = 1 5 2 3 6

Parity Games





Finite game

Every loop has max priority even

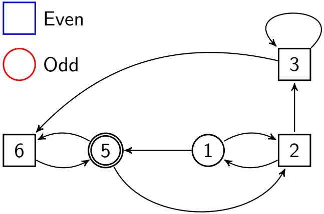
Extension to infinite plays

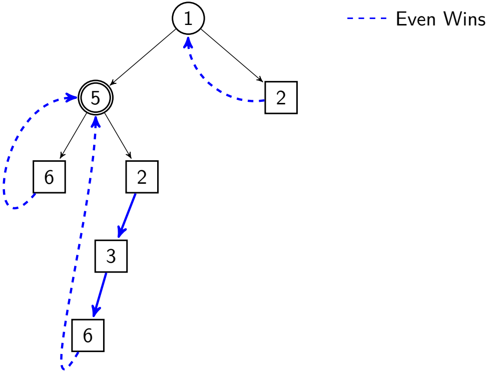
$\pi =$ 1 2 1 5 6 5 2 3 6 5

Stack = 1 5 2 3 6 5

- Every eliminated cycle has max priority even

Parity Games





Finite game

Every loop has max priority even

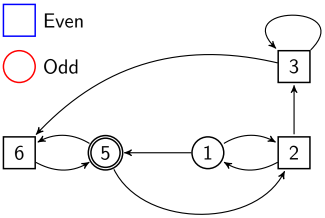
Extension to infinite plays

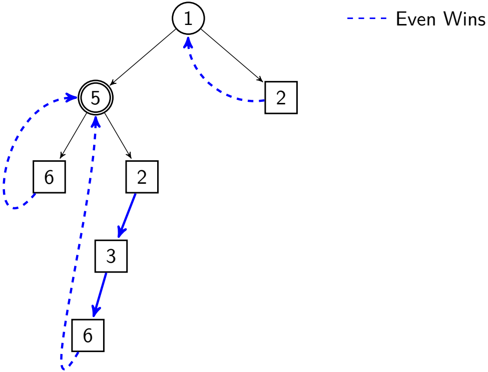
$\pi =$ 1 2 1 5 6 5 2 3 6 5

Stack = 1 5 ...

- Every eliminated cycle has max priority even

Parity Games





Finite game

Every loop has max priority even

Extension to infinite plays

$\pi = \ 1 \ \ 2 \ \ 1 \ \ 5 \ \ 6 \ \ 5 \ \ 2 \ \ 3 \ \ 6 \ \ 5$

Stack $= 1 \ \ 5 \ \ \ldots$

- Every eliminated cycle has max priority even
- Hence max Inf priority in π is Even

Parity Games

Better Algorithms

- Marcin Jurdzinski and Jens Vöge. "A discrete strategy improvement algorithm for solving parity games". In: *Computer Aided Verification*. Springer, 2000, pp. 202–215

$$\text{Upper bound}^{1} : O\left((n/d)^d\right)$$

[1] see also Friedmann, "Exponential Lower Bounds for Solving Infinitary Payoff Games and Linear Programs".

Parity Games

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- Marcin Jurdzinski, Mike Paterson, and Uri Zwick. “A Deterministic Subexponential Algorithm for Solving Parity Games”. In: *SIAM Journal on Computing* 38.4 (Jan. 2008), pp. 1519–1532

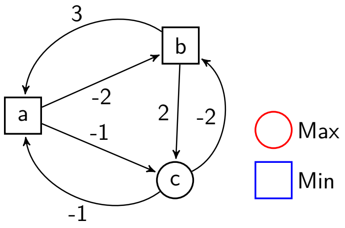
$$n^{O(\sqrt{n})}$$

[1] see also Friedmann, “Exponential Lower Bounds for Solving Infinitary Payoff Games and Linear Programs”.

Outline

Mean Payoff Games

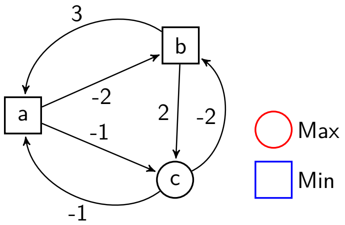
Payoffs



Mean Payoff Games

Payoffs

$(ab)^\omega$



Mean Payoff Games

Payoffs

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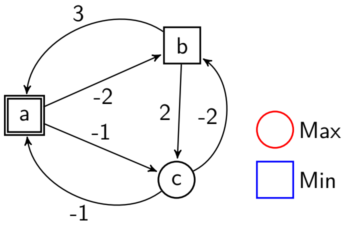

Mean Payoff Games

Payoffs

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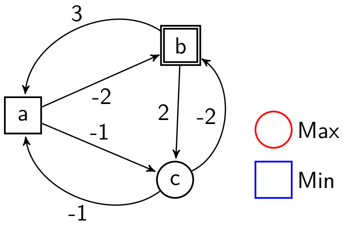
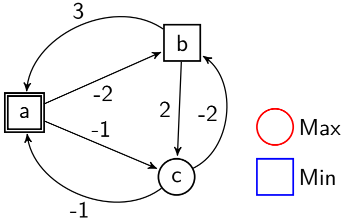


Mean Payoff Games

Payoffs

$(ab)^\omega$



Mean Payoff Games

Payoffs

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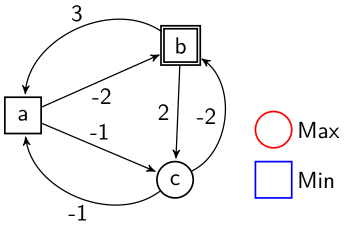
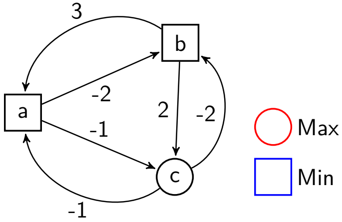


Mean Payoff Games

Payoffs

$(ab)^{\omega}$

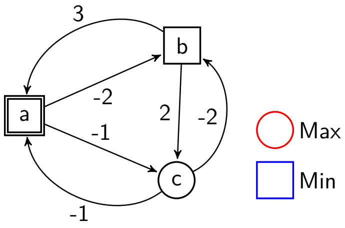


Mean Payoff Games

Payoffs

$(ab)^\omega$

a



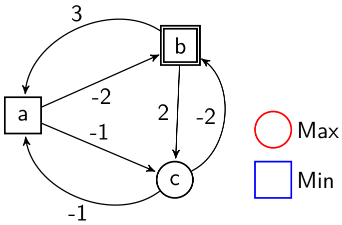
Mean Payoff Games

Payoffs

$(ab)^{\omega}$

$a \xrightarrow{-2} b$

$$\frac{-2}{1}$$



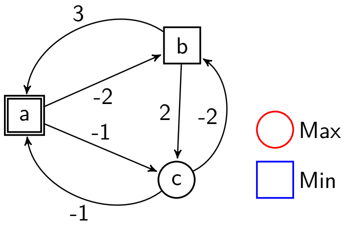
Mean Payoff Games

Payoffs

$(ab)^\omega$

$a \xrightarrow{-2} b \xrightarrow{+3} a$

$$\frac{-2+3}{2}$$



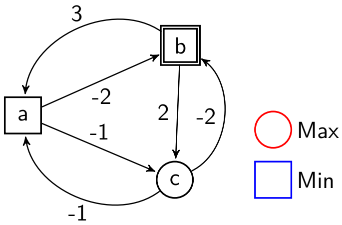
Mean Payoff Games

Payoffs

$(ab)^\omega$

$$\text{a}\xrightarrow{-2}\text{b}\xrightarrow{+3}\text{a}\xrightarrow{-2}\text{b}$$

$$\frac{-2+3-2}{3}$$



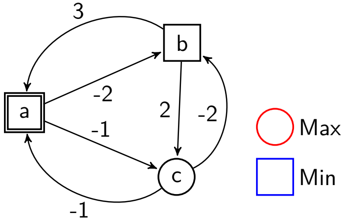
Mean Payoff Games

Payoffs

$(ab)^\omega$

$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a$$

$$\frac{-2+3-2+3}{4}$$



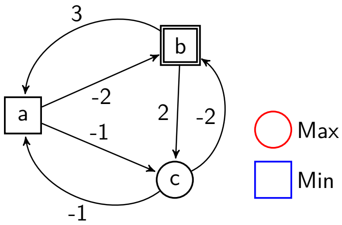
Mean Payoff Games

Payoffs

$(ab)^\omega$

$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b$$

$$\frac{-2+3-2+3-2}{5}$$



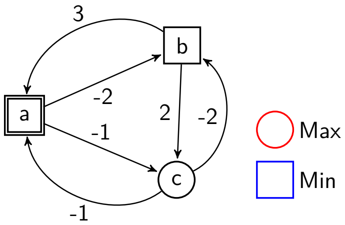
Mean Payoff Games

Payoffs

$(ab)^\omega$

$$\mathsf{a} \xrightarrow{-2} \mathsf{b} \xrightarrow{+3} \mathsf{a} \xrightarrow{-2} \mathsf{b} \xrightarrow{+3} \mathsf{a} \xrightarrow{-2} \mathsf{b} \xrightarrow{+3} \mathsf{a}$$

$$\frac{-2+3-2+3-2+3}{6}$$



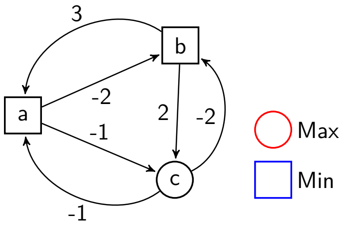
Mean Payoff Games

Payoffs

$(ab)^\omega$ Min pays $\frac{1}{2}$ units to Max

$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \ \ldots$$

$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \rightarrow \frac{1}{2}$$



Mean Payoff Games

Payoffs

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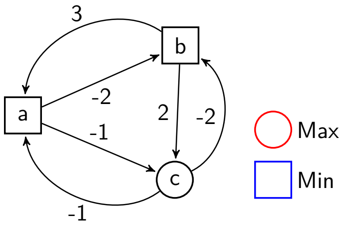
$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \to \frac{1}{2}$$

$(acb)^\omega$



Mean Payoff Games

Payoffs

$(ab)^\omega$ Min pays $\frac{1}{2}$ units to Max

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$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \rightarrow \frac{1}{2}$$

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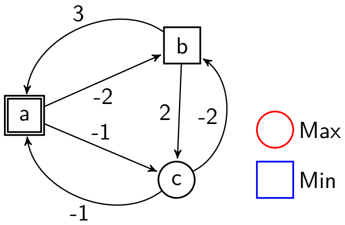

Mean Payoff Games

Payoffs

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$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \ \ldots$$

$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \rightarrow \frac{1}{2}$$

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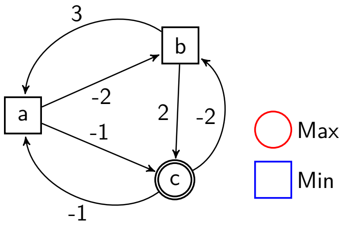


Mean Payoff Games

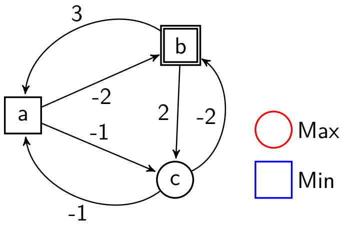
Payoffs

$(ab)^\omega$ Min pays $\frac{1}{2}$ units to Max

$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \ \ldots$$

$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \rightarrow \frac{1}{2}$$

$(acb)^\omega$



Mean Payoff Games

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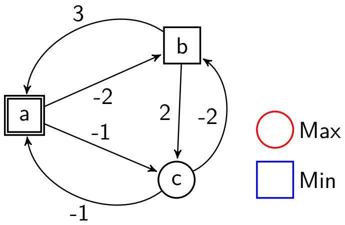

Mean Payoff Games

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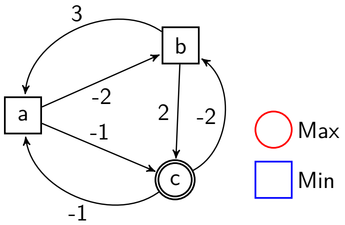

Mean Payoff Games

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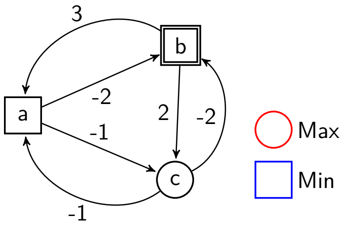

Mean Payoff Games

Payoffs

$(ab)^\omega$ Min pays $\frac{1}{2}$ units to Max

$$\mathsf{a}\xrightarrow{-2}\mathsf{b}\xrightarrow{+3}\mathsf{a}\xrightarrow{-2}\mathsf{b}\xrightarrow{+3}\mathsf{a}\xrightarrow{-2}\mathsf{b}\xrightarrow{+3}\mathsf{a}\ \dots$$

$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \to \frac{1}{2}$$

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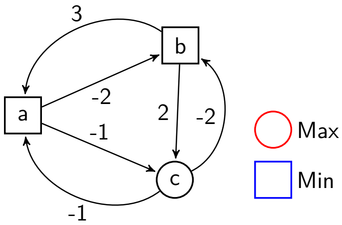

Mean Payoff Games

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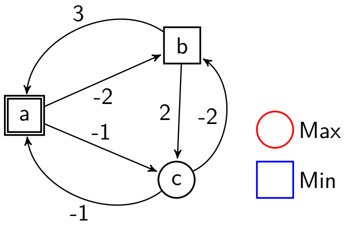
$(acb)^\omega$

a



Mean Payoff Games

Payoffs

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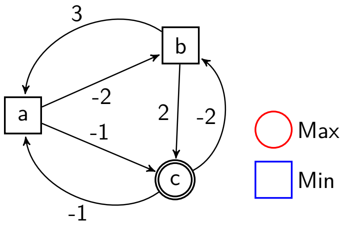
$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \to \frac{1}{2}$$

$(acb)^\omega$

$$a \xrightarrow{-1} c$$

$$\frac{-1}{1}$$



Mean Payoff Games

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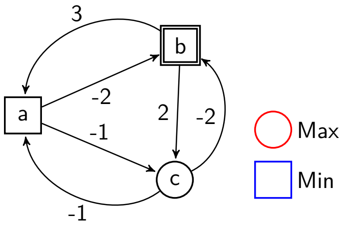
$(acb)^\omega$

$$a \xrightarrow{-1} c \xrightarrow{-2} b$$

$$\frac{-1-2}{2}$$



Mean Payoff Games

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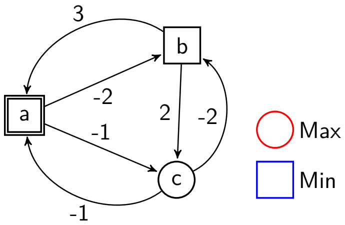
$$\frac{-1-2+3}{3}$$



Mean Payoff Games

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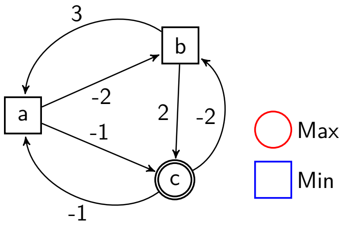
$(acb)^\omega$

$$a \xrightarrow{-1} c \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-1} c$$

$$\frac{-1-2+3-1}{4}$$



Mean Payoff Games

Payoffs

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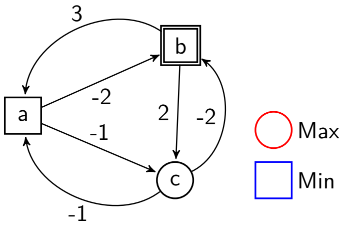
$(acb)^\omega$

$$a \xrightarrow{-1} c \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-1} c \xrightarrow{-2} b$$

$$\frac{-1-2+3-1-2}{5}$$



Mean Payoff Games

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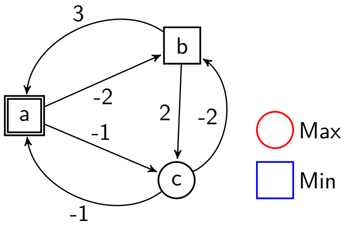


$(acb)^\omega$

$$a \xrightarrow{-1} c \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-1} c \xrightarrow{-2} b \xrightarrow{+3} a$$

$$\frac{-1-2+3-1-2+3}{6}$$

Mean Payoff Games

Payoffs

$(ab)^\omega$ Min pays $\frac{1}{2}$ units to Max

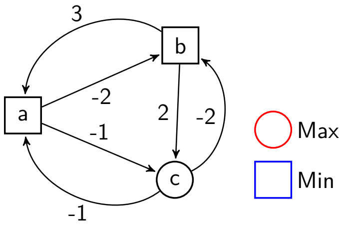
$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \ \ldots$$

$$\frac{-2+3-2+3-2+3}{6} \sim \frac{n(-2+3)}{2n} \to \frac{1}{2}$$

$(acb)^\omega$ Min pays $-\frac{1}{3}$ units to Max

$$a \xrightarrow{-1} c \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-1} c \xrightarrow{-2} b \xrightarrow{+3} a \ \ldots$$

$$\frac{-1-2+3-1-2+3}{6} \sim \frac{n(-1-2+3)}{3n} \to 0$$



Mean Payoff Games

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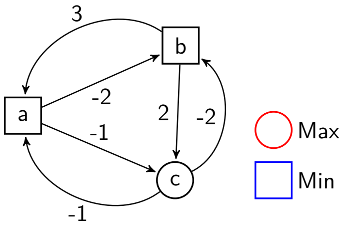

- Min tries to minimize lim
- Max tries to maximize lim

Mean Payoff Games

Payoffs

$(ab)^\omega$ Min pays $\frac{1}{2}$ units to Max

$$a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \xrightarrow{-2} b \xrightarrow{+3} a \ \ldots$$

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$(acb)^\omega$ Min pays $-\frac{1}{3}$ units to Max

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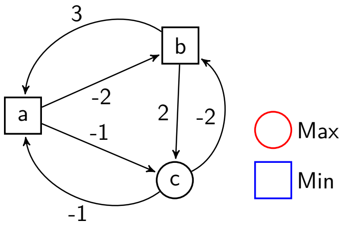

Generally

- Min tries to minimize lim sup
- Max tries to maximize lim inf

Mean Payoff Games

Questions

- Does the game have a value? i.e. is there a v so that
  - Max can ensure $\liminf \geq v$
  - Min can ensure $\limsup \leq v$

Mean Payoff Games

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  Yes
- How to compute the optimal strategies?

Mean Payoff Games

Questions

- Does the game have a value? i.e. is there a v so that
  - Max can ensure $\liminf \geq v$
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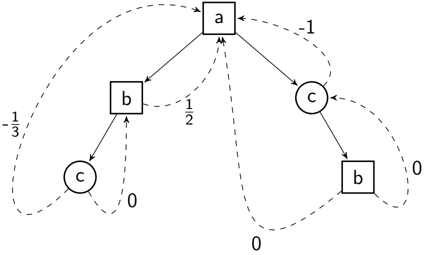
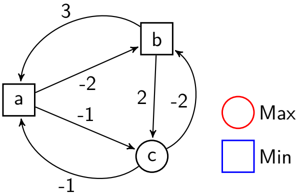
  Yes
- How to compute the optimal strategies?

Solution using the finite game

Mean Payoff

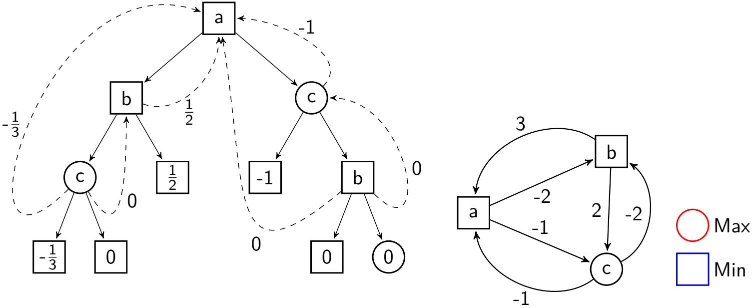
Finite Game





Mean Payoff

Finite Game



Mean Payoff

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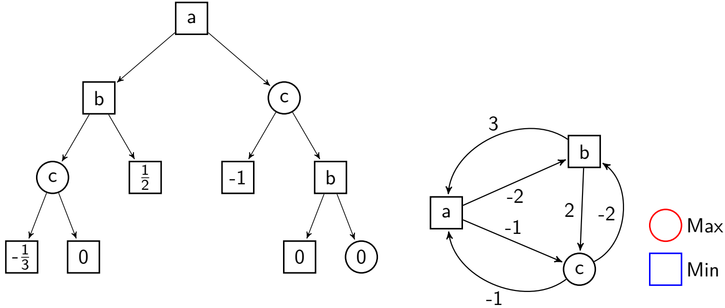

Mean Payoff

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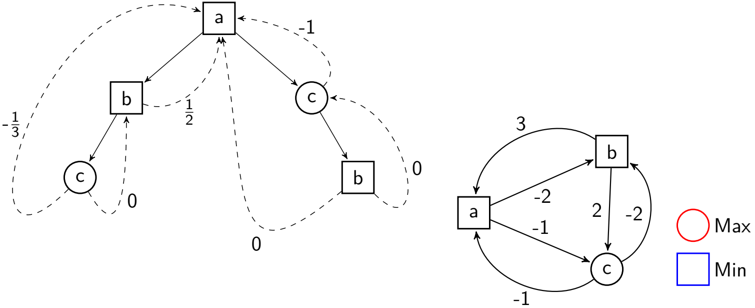

Mean Payoff

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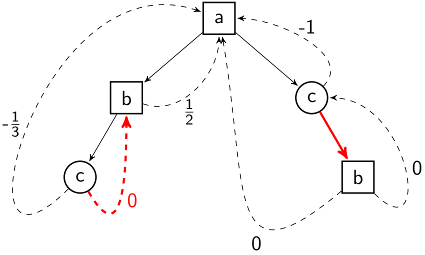
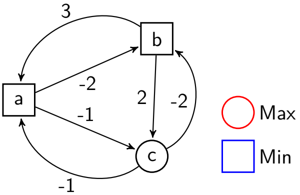
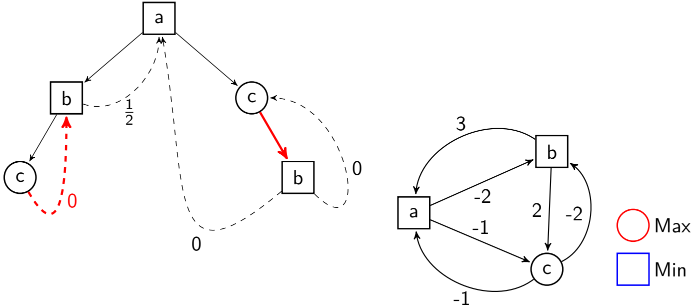




Mean Payoff

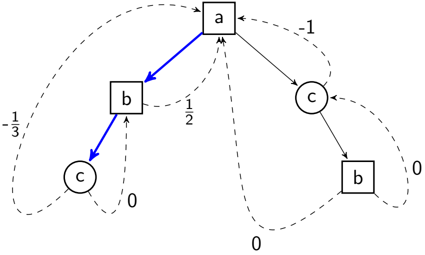
Finite Game

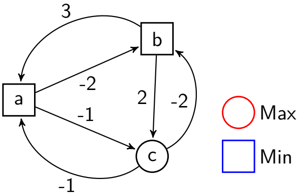


Max can ensure ≥ 0 in the finite game

Mean Payoff

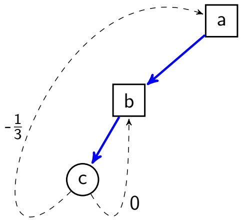
Finite Game

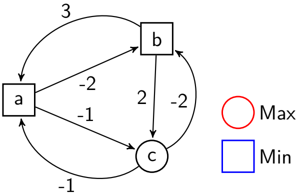




Mean Payoff

Finite Game

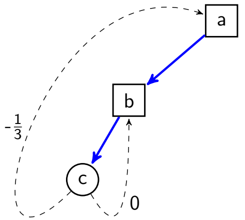


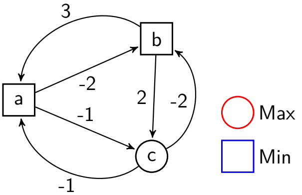


Min can ensure ≤ 0

Mean Payoff

Finite Game

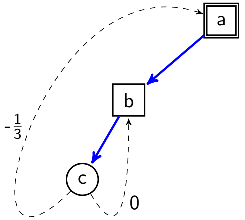


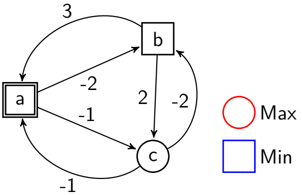


Min can ensure ≤ 0 in the mean payoff game too

Mean Payoff

Finite Game





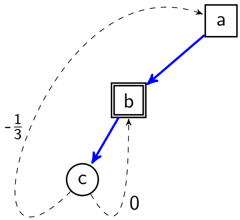
Min can ensure ≤ 0 in the mean payoff game too

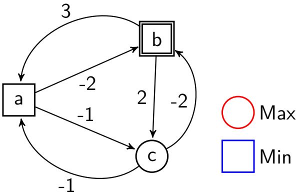
$\pi =$ a

Stack = a

Mean Payoff

Finite Game





Min can ensure ≤ 0 in the mean payoff game too

$\pi =$ a b

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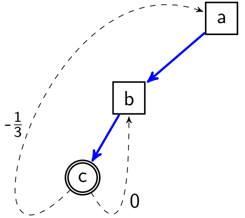
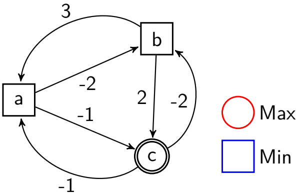
Mean Payoff

Finite Game





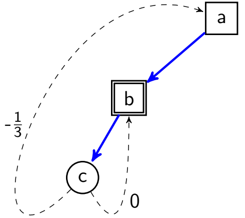
Min can ensure ≤ 0 in the mean payoff game too

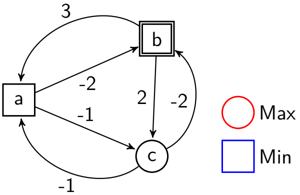
$\pi =$ a b c

Stack = a b c

Mean Payoff

Finite Game





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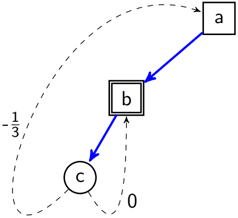
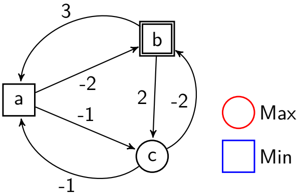
Mean Payoff

Finite Game





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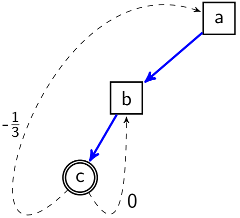
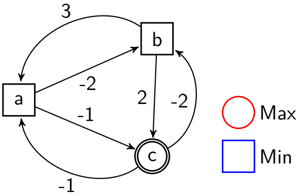
Mean Payoff
Finite Game





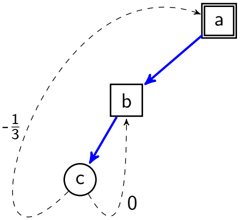
Min can ensure ≤ 0 in the mean payoff game too

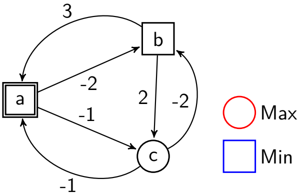
$\pi =$ a b c b c

Stack $=$ a b c

Mean Payoff

Finite Game





Min can ensure ≤ 0 in the mean payoff game too

$\pi =$ a b c b c a

Stack = a b c a

Mean Payoff

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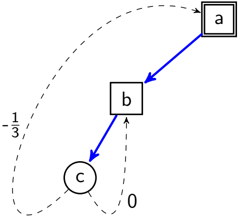

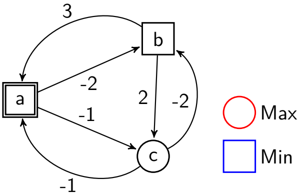

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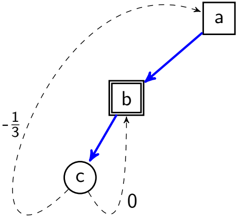

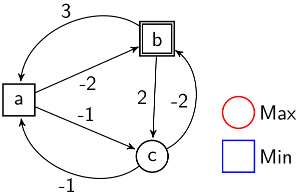

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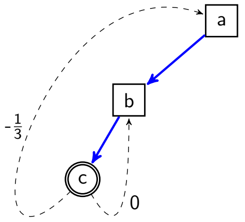
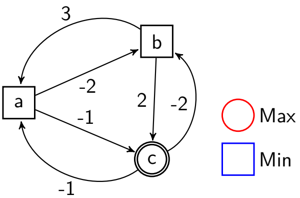
Mean Payoff

Finite Game





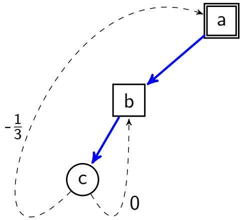
Min can ensure ≤ 0 in the mean payoff game too

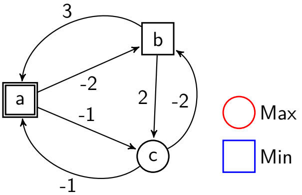
$\pi =$ a b c b c a b c

Stack = a b c

Mean Payoff

Finite Game





Min can ensure ≤ 0 in the mean payoff game too

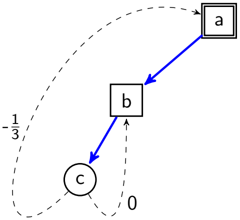
$\pi =$ a b c b c a b c a

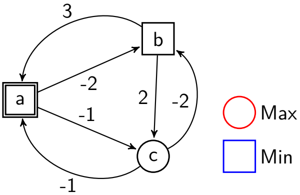
Stack $=$ a b c a

Every time a cycle with average value ≤ 0 is eliminated

Mean Payoff

Finite Game





Min can ensure ≤ 0 in the mean payoff game too

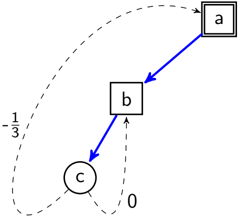
$\pi =$ a b c b c a b c a

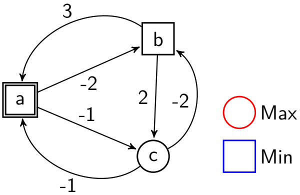
Stack = a

Every time a cycle with average value ≤ 0 is eliminated

Mean Payoff

Finite Game





Min can ensure ≤ 0 in the mean payoff game too

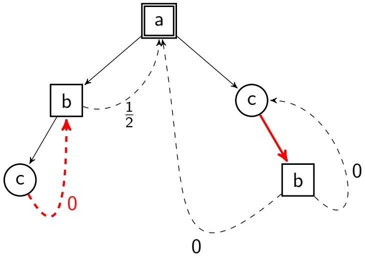
$\pi =$ a b c b c a b c a

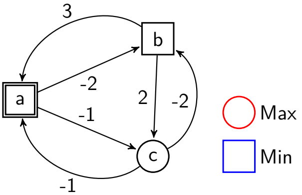
Hence limsup of averages of π is ≤ 0

Stack = a

Mean Payoff

Finite Game





Max can ensure ≥ 0

- Similarly Max can ensure liminf of the average is ≥ 0
- Hence the value of Mean payoff game is 0

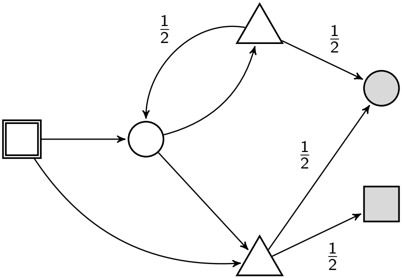
Outline

Finite Duration Games

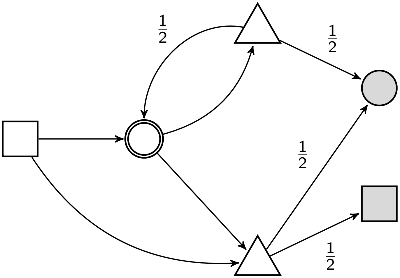
Infinite Duration Games

Simple Stochastic Games

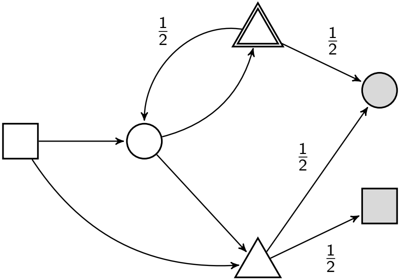
Simple Stochastic Game



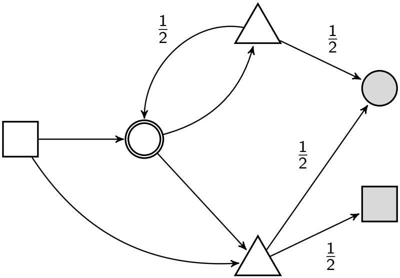
Simple Stochastic Game



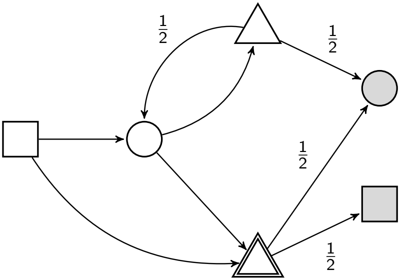
Simple Stochastic Game



Simple Stochastic Game

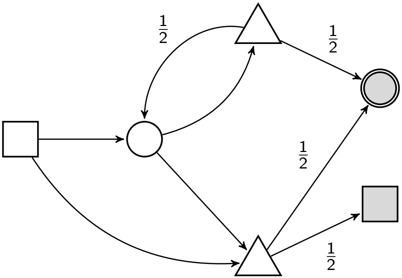


Simple Stochastic Game

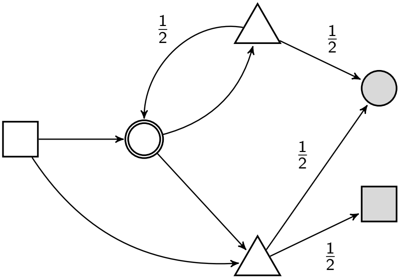


Simple Stochastic Game

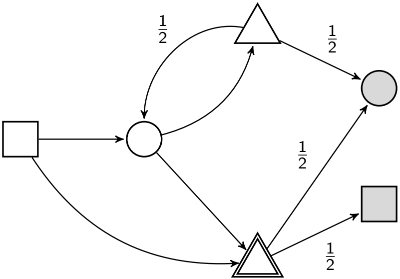
Circle Wins



Simple Stochastic Game

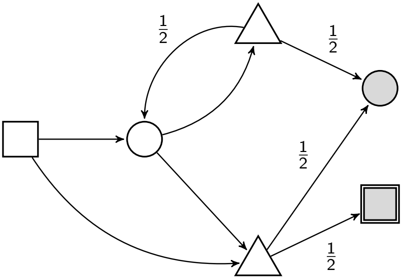


Simple Stochastic Game



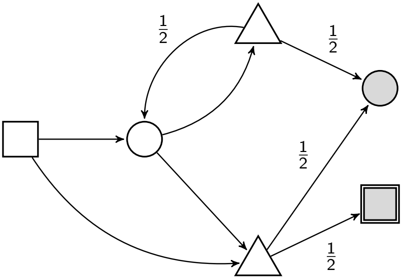
Simple Stochastic Game

Box Wins

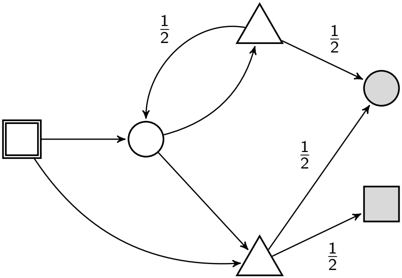


Simple Stochastic Game

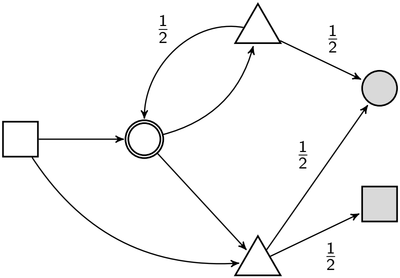
Or



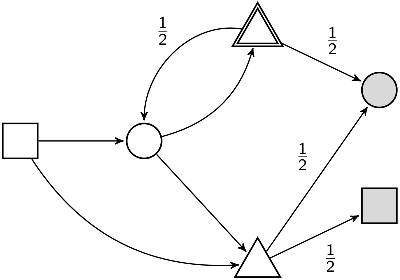
Simple Stochastic Game



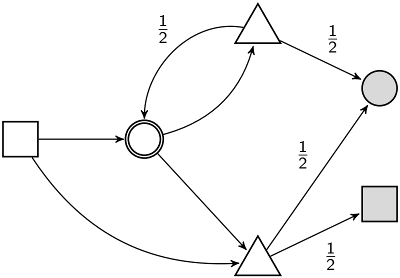
Simple Stochastic Game



Simple Stochastic Game



Simple Stochastic Game



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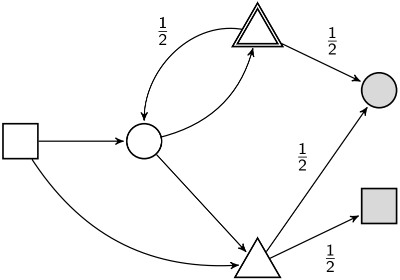

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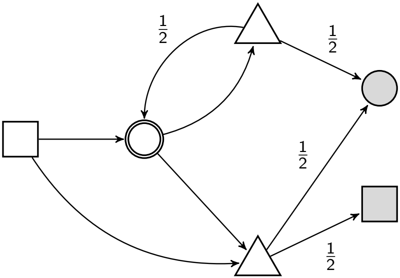

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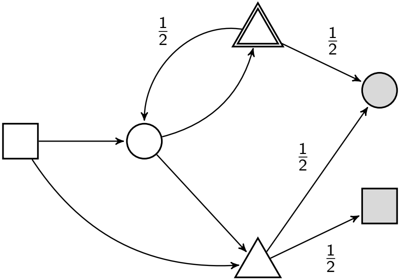
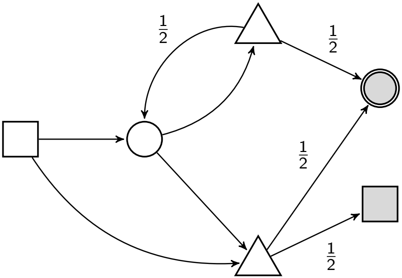
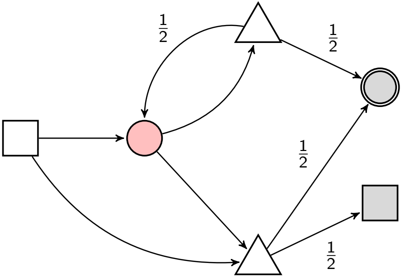


Simple Stochastic Game



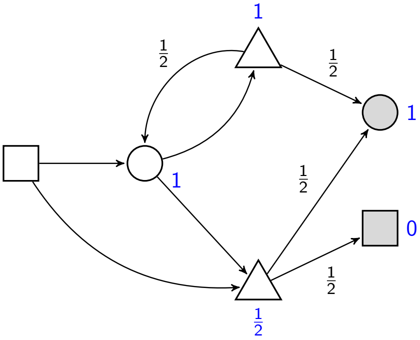
Simple Stochastic Game

Circle can win from ● with probability 1



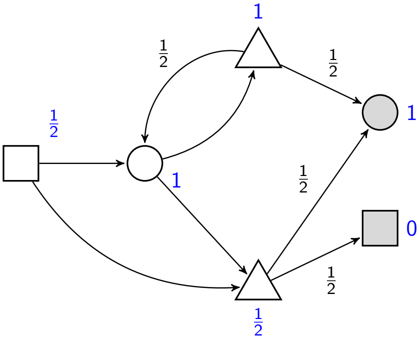
Simple Stochastic Game

Values



Simple Stochastic Game

Values



Simple Stochastic Game

Values

$$v({\color{gray}\bullet}) = 1$$

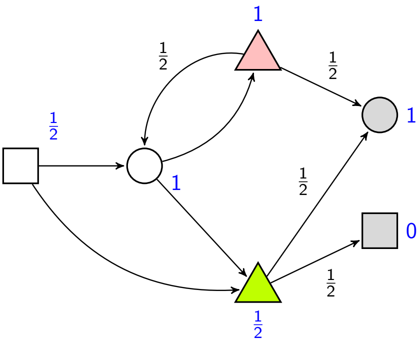
$$v({\color{gray}\blacksquare}) = 0$$

$$v({\color{pink}\blacktriangle}) = \frac{1}{2}(v({\color{gray}\bullet}) + v(\bigcirc))$$

$$v({\color{green}\blacktriangle}) = \frac{1}{2}(v({\color{gray}\bullet}) + v({\color{gray}\blacksquare}))$$

$$v(\bigcirc) = \max\{v({\color{pink}\blacktriangle}), v({\color{green}\blacktriangle})\}$$

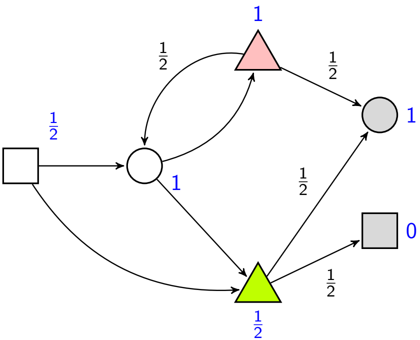
$$v(\square) = \min\{v(\bigcirc), v({\color{green}\blacktriangle})\}$$



Simple Stochastic Game

Values

$$v({\color{gray}\bullet}) = 1$$
$$v({\color{gray}\blacksquare}) = 0$$
$$v({\color{pink}\blacktriangle}) = \frac{1}{2}(v({\color{gray}\bullet}) + v(\circ))$$
$$v({\color{lime}\blacktriangle}) = \frac{1}{2}(v({\color{gray}\bullet}) + v({\color{gray}\blacksquare}))$$
$$v(\circ) = \max\{v({\color{pink}\blacktriangle}), v({\color{lime}\blacktriangle})\}$$
$$v(\square) = \min\{v(\circ), v({\color{lime}\blacktriangle})\}$$

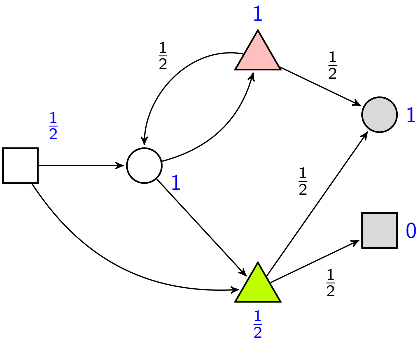


These equations have a unique solution.

Simple Stochastic Game

Values

$$v(\circledcirc) = 1$$
$$v(\boxdot) = 0$$
$$v(\triangle_{\text{pink}}) = \frac{1}{2}(v(\circledcirc) + v(\bigcirc))$$
$$v(\triangle_{\text{green}}) = \frac{1}{2}(v(\circledcirc) + v(\boxdot))$$
$$v(\bigcirc) = \max\{v(\triangle_{\text{pink}}), v(\triangle_{\text{green}})\}$$
$$v(\square) = \min\{v(\bigcirc), v(\triangle_{\text{green}})\}$$



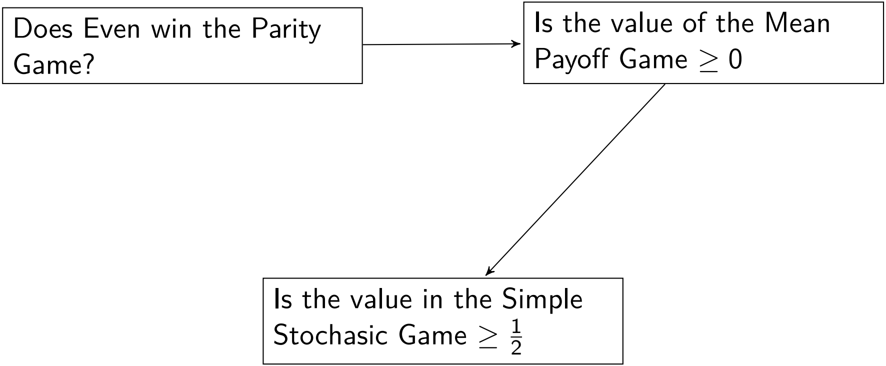
These equations have a unique solution.

From state s -

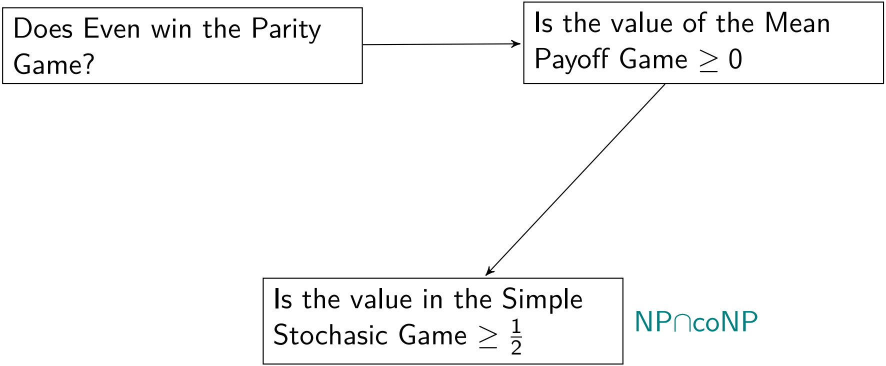
$\bigcirc$ has a strategy to reach $\circledcirc$ with probability $\geq v(s)$

$\square$ has a strategy to reach $\boxdot$ with probability $\geq 1 - v(s)$

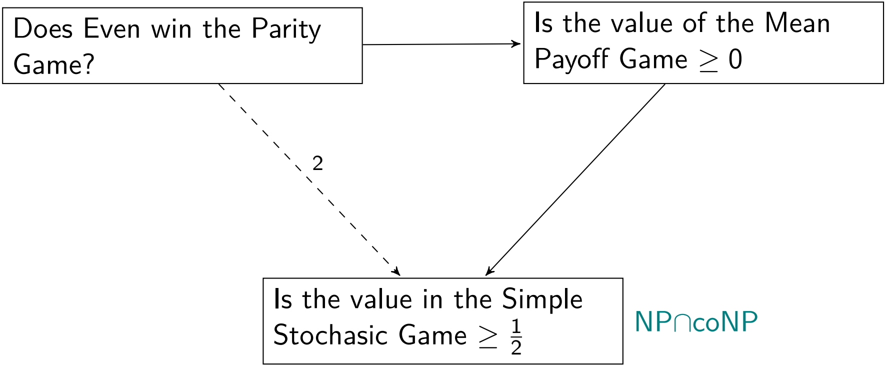
Complexity of solving games



Complexity of solving games

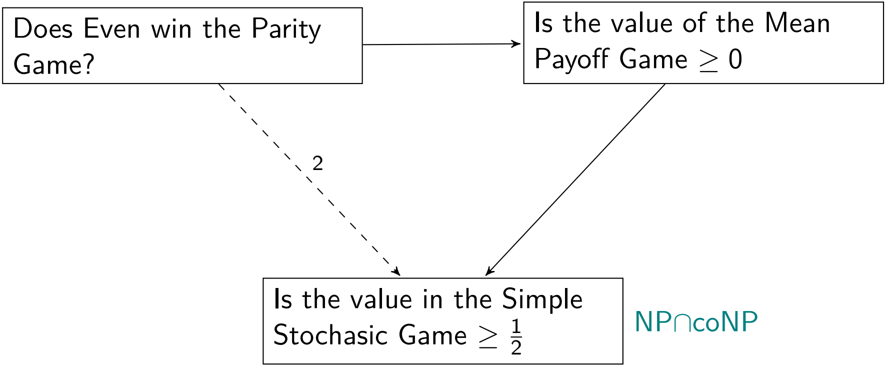


Complexity of solving games



[2] Chatterjee and Fijalkow, "A reduction from parity games to simple stochastic games".

Complexity of solving games



Open Problem

Is there a polynomial time algorithm for any of them?

[2] Chatterjee and Fijalkow, "A reduction from parity games to simple stochastic games".

Timeline

- Lloyd S. Shapley. "Stochastic games". In: *Proceedings of the National Academy of Sciences* 39.10 (1953), pp. 1095–1100
- E.A. Emerson and C.S. Jutla. "Tree automata, mu-calculus and determinacy". In: IEEE Comput. Soc. Press, 1991, pp. 368–377
- Anne Condon. "The complexity of stochastic games". In: *Information and Computation* 96.2 (Feb. 1992), pp. 203–224
- Uri Zwick and Mike Paterson. "The complexity of mean payoff games on graphs". In: *Theoretical Computer Science* 158.1 (May 1996), pp. 343–359
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Thank you