Games on Graphs: Tutorial 1

1. Read Section 3.1 of text *"Lectures in Game theory for Computer Scientists"* by Apt and Grädel.

2. Show that for any function $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$:

$$\sup_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} f(x, y) \leq \inf_{y \in \mathbb{R}} \sup_{x \in \mathbb{R}} f(x, y)$$

3. In this question, we use notations as in Section 3.1 of text.

   A game is said to be determined if:

$$\sup_{\chi} \inf_{\mu} \pi(\text{Outcome}(v, \mu, \chi)) = \inf_{\mu} \sup_{\chi} \pi(\text{Outcome}(v, \mu, \chi)) \qquad \text{for all vertices } v \tag{1}$$

   where μ, χ range over strategies of Minimizer and Maximizer respectively, $\text{Outcome}(v, \mu, \chi)$ is the play starting at v induced by μ and χ, and π is a payoff function. A game is said to be *positionally determined* if the sup on the left hand side and the inf on the right hand side range over positional strategies of Maximizer and Minimizer respectively. Show that if a game is positionally determined, it is determined.

4. Exercises 3.1 to 3.3 from the text *"Lectures in Game theory for Computer Scientists"* by Apt and Grädel.

5. Show that (1) holds for reachability games.

6. Consider the algorithm Büchi-win in Figure 3.1 (Page 79) of the text. Design a game G for which Büchi-win makes at least 2 recursive calls.

7. Exercises 3.4 to 3.7 from the text *"Lectures in Game theory for Computer Scientists"* by Apt and Grädel.

8. Design a game G with $O(n)$ vertices for which Büchi-win makes at least n recursive calls.

9. Consider the following algorithm for a mixed Büchi and co-Büchi objective: two disjoint sets T and S are specified; Player 1 wins plays that visit T infinitely often and S only finitely often. Design an algorithm to compute winning regions of each player in this mixed objective game.

10. Consider the mixed Büchi and co-Büchi objective mentioned in the previous question.Give an example of a game graph on which the algorithm given below does not terminate.

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algorithm mixed-win(G)
  if reach_1(T) = V and S = ∅
  then (W_0, W_1) = (∅, V)
  else
      W'_0 = V \ reach_1(T)
      G' = V \ reach_0(W'_0)
      (W''_0, W''_1) = mixed-win(G'')
      (W_0, W_1) = (V \ W''_1, W''_1)
  endif
  return (W_0, W_1)
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11. Consider a parity game in which every vertex is controlled by Player 1. Does he win from every vertex?

12. Exercises 3.7 to 3.11 from the text *"Lectures in Game theory for Computer Scientists"* by Apt and Grädel.

13. Consider a parity game G whose arena is finite. Pick a vertex v_0 in G (need not necessarily be Player 1 vertex). We will now define a new game G' starting from v_0 where all plays are finite. The game stops as soon as a vertex is visited twice. A play is thus a finite path $v_0, \ldots, v_n$ such that $v_0, \ldots, v_{n-1}$ are pairwise distinct and $v_n = v_j$ for some $j < n$. Player P_1 wins if the maximum colour in the loop: $\max\{\chi(v_j), \chi(v_{j+1}), \ldots, \chi(v_n)\}$ is odd.

    Show that the parity game G starting in v_0 and the game G' are equivalent (that is, P_1 wins in G from v_0 iff she wins G'):

    i) Show that if P_1 has a winning strategy in G starting from v_0, then she has a winning strategy in G'.

    ii) Show that if P_1 has a winning strategy in G', then she has a winning strategy in G starting from v_0.