Problem Set 1
Discrete Mathematics 2020-21

CHENNAI MATHEMATICAL INSTITUTE

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> *If we can't even properly count what we possess, how will we be able to use it?*
>
> Dr Prem Jagyasi, *Author and speaker*

Problem 1. How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

Problem 2. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

Problem 3. Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

Problem 4. On a certain planet in the solar system Tau Cetus, more that half the surface of the planet is dry land. Show that the Tau Cetans can dig a tunnel straight through the centre of the planet, beginning and ending on dry land. (Assume that their technology is sufficiently developed.)

Problem 5. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

> *Stop counting pages… you will never finish the book.*
>
> Deyth Banger, *Author*

Problem 6. A **circular r-permutation of n people** is a seating of r of these n people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table. Find a formula for the number of circular r−permutaions of n people.

Problem 7. What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?

Problem 8. Show that if n and k are integers with $1 \le k \le n$, then $\binom{n}{k} \le n^k/2^{k-1}$.

Problem 9. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r and k are nonnegative integers with $r \le n$ and $k \le r$, using a combinatorial argument.

Problem 10. Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.

Problem 11. Show that the number of paths in the xy plane between the origin $(0,0)$ and point (m,n), where m and n are nonnegative integers, such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward, is $\binom{m+n}{n}$. No moves to the left or downward are allowed.

[Hint: First, show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.]

Problem 12. How many ways are there for a horse race with four horses to finish if ties are possible? [Note: Any number of horses may tie.]

Problem 13. Show that in any group of five people, there are two who have an identical number of friends within the group.

Problem 14. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

Problem 15. Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n.

> *Many of the things you can count, don't count. Many of the things you can't count, really count.*
>
> Albert Einstein, *Theoretical physicist*

Problem 16. How many bit strings of length 8 contain either three consecutive 0s or four consecutive 1s?

***Problem 17.** Let $\{(x_i, y_i) : i = 1, 2, 3, 4, 5\}$ be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

***Problem 18.** Let x be an irrational number. Show that for some positive integer j not exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.

***Problem 19.** How many ways are there to place two bishops on a chessboard so that they do not attack each other?

***Problem 20.** Prove the **hockeystick identity**

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

a) using a combinatorial argument.

b) using Pascal's identity.

Bonus Problem 21. Solve the question 12 for k horses. [Note: The horses are identical.]

Bonus Problem 22. Is it possible to construct a sequence in question 14, of length 17 instead of 16? If yes, give the sequence, else prove why not.