DISCRETE MATHEMATICS

LECTURE 8

Plan:

Propositional logic

- Natural deduction rules (contd.)

Rules for negation $\neg$

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \bot\end{matrix}}}{\neg\phi}\ \neg_i$	$\dfrac{\phi \qquad \neg\phi}{\bot}\ \neg_e$

also called

$\bot$ - introduction

Example:

$$p \to q,\ p \to \neg q \vdash \neg p$$

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \bot\end{matrix}}}{\neg\phi}\ \neg i$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg e$ also called $\bot$ - introduction

1.	$p \to q$	premise
2.	$p \to \neg q$	premise
3.	p	assume
4.	q	$\to e$, 1, 3
5.	$\neg q$	$\to e$, 2, 3
6.	$\bot$	$\bot i$, 4, 5
7	$\neg p$	$\neg i$, 3-6.

Remark: Formulas of the form $\phi \wedge \neg\phi$ are called Contradictions

$$p \wedge \neg p \qquad (p \to q \;\wedge\; \neg(p \to q)) \qquad \text{etc.}$$

The rule

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}$$

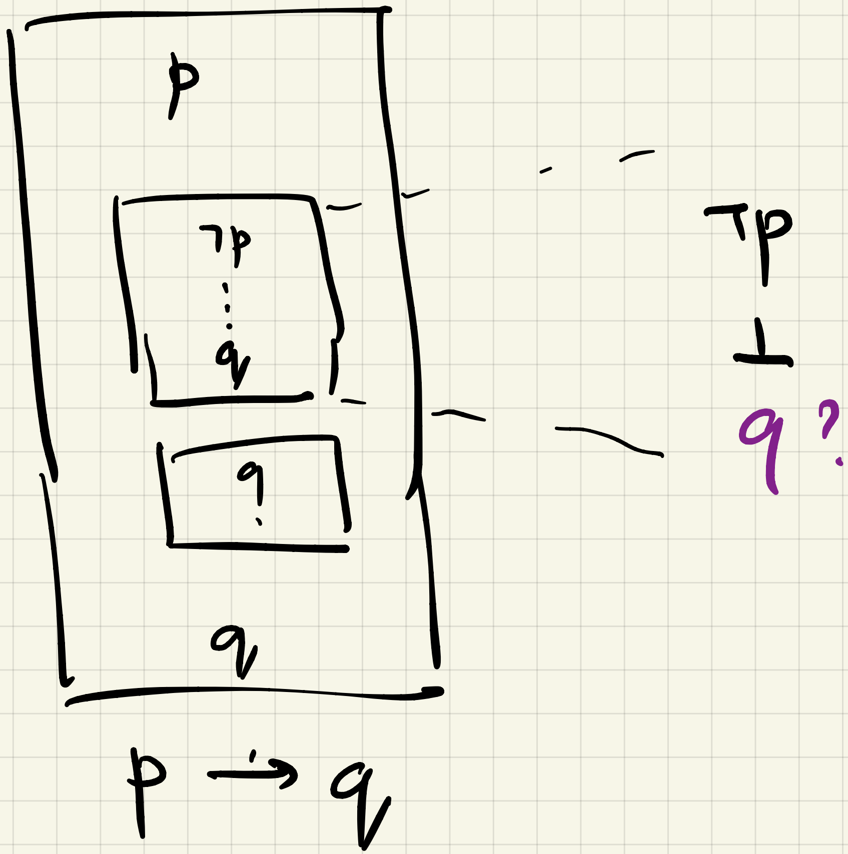
is called Proof by contradiction

Example: $\phi \to \psi, \ \neg\psi \ \vdash \ \neg\phi$ (Modus tollens)

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg_i$	$\dfrac{\phi \qquad \neg\phi}{\bot}\ \neg_e$ also called $\bot$ - introduction

1.	$\phi \to \psi$	premise
2.	$\neg\psi$	premise
3.	ϕ	assume
4.	ψ	$\to_e$, 1, 3
5.	$\bot$	$\bot_i$, 2, 4
6.	$\neg\phi$	$\neg_i$, 3-5

Example: $\neg p \vee q \vdash p \to q$



Rule for $\perp$:

	introduction	elimination
$\perp$	$\dfrac{\phi \quad \neg\phi}{\perp}$ (already seen as $\neg e$)	$\dfrac{\perp}{\phi}$

- $\perp$ can derive any formula:
- This is because $\perp \to \phi$ is always true

Example: $\neg p \lor q \vdash p \to q$

	introduction	elimination
$\perp$	$\dfrac{\phi \quad \neg\phi}{\perp}$ (already seen as $\neg e$)	$\dfrac{\perp}{\phi}\ \perp e$

1.	$\neg p \lor q$	premise
2.	p	assume
3.	$\neg p$	assume
4.	$\perp$	$\perp i$, 2, 3
5.	q	$\perp e$, 4
6.	q	assume
7.	q	copy 6,
8.	q	$\lor e$, 1, 3-5, 6-7
9.	$p \to q$	$\to i$, 2-8

$\neg p \vee q \vdash p \to q$

1.	$\neg p \vee q$	premise
2.	$\neg p$	assume
3.	p	assume
4.	$\perp$	$\perp i$, 2, 3.
5.	q	$\perp e$, 4
6.	$p \to q$	$\to i$, 3-5
7.	q	assume
8.	p	assume
9.	q	copy
10.	$p \to q$	

$p \to q$

Rule for double negation $\neg\neg$

		elimination
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg e$

Example: $\phi \vdash \neg\neg\phi$

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg_i$	$\dfrac{\phi \qquad \neg\phi}{\bot}\ \neg_e$ also called $\bot$ - introduction

$$
\begin{array}{lcl}
1. & \phi & \text{premise} \\
2. & \boxed{\begin{array}{c}\neg\phi \\ \\ \bot\end{array}} & \text{assume} \\
3. & & \bot_i,\ 1, 2. \\
4. & \neg\neg\phi & \neg_i,\ 2\text{–}3
\end{array}
$$

Example:

$$p, \quad \neg\neg(q \wedge r) \quad \vdash \quad \neg\neg p \wedge r$$

1.	p	premise
2.	$\neg\neg(q \wedge r)$	premise
3.	$\boxed{\neg p}$	assume
4.	$\boxed{\bot}$	$\bot i$, 1, 3
5.	$\neg\neg p$	$\neg i$ 3-4
6.	$(q \wedge r)$	$\neg\neg e$, 2
7.	r	$\wedge e_2$, 6
8.	$\neg\neg p \wedge r$	$\wedge i$, 5, 8

Example: $\neg\phi \to \bot \vdash \phi$

1.	$\neg\phi \to \bot$	premise
2.	$\neg\phi$	assume
3.	$\bot$	$\to$e, 1,2
4.	$\neg\neg\phi$	$\neg$i, 2-3
5.	ϕ	$\neg\neg$e, 4

Example: $\neg(p \vee q) \vdash \neg p \wedge \neg q$

1.	$\neg(p \vee q)$	premise
2.	p	assume
3.	$p \vee q$	$\vee_i$, 2
4.	$\perp$	$\perp_i$ 1, 3
5.	$\neg p$	$\neg_i$, 2–4
6.	q	assume
7.	$p \vee q$	$\vee_i$, 6
8.	$\perp$	$\perp_i$, 1, 7
9.	$\neg q$	$\neg_i$, 6–8
10.	$\neg p \wedge \neg q$	$\wedge_i$, 5, 9.

Example: $\vdash \phi \vee \neg\phi$

LEM: Law of Excluded Middle

$\neg(p \vee q) \vdash \neg p \wedge \neg q$

1.	$\neg(p \vee q)$	premise
2.	p	assume
3.	$p \vee q$	$\vee_i$, 2
4.	$\bot$	$\bot_i$ 1, 3
5.	$\neg p$	$\neg_i$, 2–4
6.	q	assume
7.	$p \vee q$	$\vee_i$, 6
8.	$\bot$	$\bot_i$, 1, 7
9.	$\neg q$	$\neg_i$, 6–8
10.	$\neg p \wedge \neg q$	$\wedge_i$, 5, 9

1.	$\neg(\phi \vee \neg\phi)$	assume
2.	$\neg\phi \wedge \neg\neg\phi$	using ideas similar to previous example
3.	$\neg\phi$	$\wedge e$, 2
4.	$\neg\neg\phi$	$\wedge e$, 2
5.	$\bot$	$\bot i$, 3, 4
6.	$\neg\neg(\phi \vee \neg\phi)$	$\neg_i$ 1-5
7.	$\phi \vee \neg\phi$	$\neg\neg e$, 6.

Example:

$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

1.	$\neg(p \wedge q)$	premise
2.	$p \vee \neg p$	LEM
3.	p	assume
4.	q	assume
5.	$p \wedge q$	$\wedge i$, 3, 4
6.	$\bot$	$\bot i$, 1, 5
7.	$\neg q$	$\neg i$, 4–6
8.	$\neg p \vee \neg q$	$\vee i$, 7
9.	$\neg p$	assume
10.	$\neg p \vee \neg q$	$\vee i$, 9
11.	$\neg p \vee \neg q$	$\vee e$, 2, 3–8, 9–10

Summary:

Propositional logic

- a "calculus" for reasoning about statements
- a mechanism to solve logic puzzles
- some logic puzzles can be seen as sequents
- Proofs of sequents using natural deduction rules