DISCRETE MATHEMATICS

LECTURE 7

Plan:

Propositional logic (contd.)

- Sequents
- Proofs of sequents
- Natural deduction rules

SEQUENTS.

Example 1: Four siblings go shopping with their father.

- If Abhay gets shoes, then Asha does not get a necklace.
- If Arun gets a T-shirt, then Aditi gets bangles.
- If Abhay does not get shoes or Aditi gets bangles, the mother will be happy.
- Mother is not happy.

Prove that Asha did not get a necklace and Arun did not get a T-shirt.

Propositions:

p : Abhay gets shoes

q : Asha gets necklace

r : Arun gets T-shirt

s : Aditi gets bangles

m : Mother is happy

$$p \rightarrow \neg q, \quad r \rightarrow s, \quad \neg p \vee s \rightarrow m, \quad \neg m \quad \vdash \quad \neg q \wedge r$$

sequent

derives / entails

Example 2: Five friends A, B, C, D, E have access to a chat room. Is it possible to determine who is chatting if the following information is known?

- Either A or B, or both are chatting.
- Either C or D, but not both, are chatting.
- If E is chatting, then C is also chatting.
- D and A are either both chatting or neither is.
- If B is chatting, then so are A and E.

Explain your reasoning.

A: A is chatting

B:

C:

D:

E:

Prove that A and D are chatting

$$A \vee B, \quad C \vee D, \quad \neg(C \wedge D), \quad E \to C, \quad D \to A, \quad A \to D, \quad B \to A \wedge E \quad \vdash \quad A \wedge D$$

Example 3: Smullyan puzzle.

There is an island that has two kinds of inhabitants

- knights who always tell the truth
- knaves who always lie.

You encounter two people A and B. What are A and B if

A says "B is a knight" and

B says "The two of us are opposite types".

A : A is knight

B: B is knight

Prove that A and B are knaves.

Conclusion

$$\underbrace{A \to B,\ \neg A \to \neg B,\ B \to \neg A,\ \neg B \to \neg A}_{\text{L.H.S} \\ \text{Premises}} \vdash \neg A \wedge \neg B$$

Sequent

Sequent:

$$\underbrace{\phi_1, \phi_2, \ldots \phi_n}_{\text{premises}} \vdash \underbrace{\psi}_{\text{conclusion}}$$

PROOFS OF SEQUENTS

Certain arguments keep recurring:

- If 'p' is true and '$p \to q$' is true, then 'q' is true.
- Assume 'p'. If we arrive at a contradiction, ie., both 'q' and '$\neg q$' are derived to be true, then our assumption is wrong.
  - Hence $\neg p$ holds.
- If 'p' is false, but '$p \vee q$' is true, then 'q' is true.

Natural deduction:

- A set of rules which can be applied to prove sequents.

Instead of writing English text, proofs will be a repeated application of natural deduction rules.

$$\varphi_1 \quad \varphi_2 \quad \dots \quad \vdash$$

$\wedge, \vee, \neg, \bot, \rightarrow$

formulas have operators $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\bot$

Rules for conjunction $\wedge$:

	introduction	elimination
$\wedge$	$\dfrac{\phi, \quad \psi}{\phi \wedge \psi} \wedge_i$	$\dfrac{\phi \wedge \psi}{\phi} \wedge_{e_1}$ $\quad$ $\dfrac{\phi \wedge \psi}{\psi} \wedge_{e_2}$

Example : $p \wedge q, \quad r \vdash q \wedge r$ (sequent)

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$, 1
4.	$q \wedge r$	$\wedge i$, 3, 2

proof of the sequent

Example: $(p \wedge q) \wedge r, \quad s \wedge t \quad \vdash \quad q \wedge s$

1.	$\underline{(p \wedge q)} \wedge r$	premise
2.	$s \wedge t$	premise
3.	$p \wedge q$	$\wedge_{e_1}$, 1
4.	q	$\wedge_{e_2}$, 3
5.	s	$\wedge_{e_1}$, 2
6.	$q \wedge s$	$\wedge_i$, 4, 5

Rules for implication $\to$:

	introduction	elimination
$\to$	$\dfrac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to_i$	$\dfrac{\phi \quad , \quad \phi \to \psi}{\psi} \to_e$ Modus Ponens

Example:

$$p, \quad p \to q, \quad p \to (q \to r) \vdash r$$

	introduction	elimination
$\to$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \to \psi} \to_i$	$\dfrac{\phi, \quad \phi \to \psi}{\psi} \to_e$ Modus Ponens

1.	p	premise
2.	$p \to q$	premise
3.	$p \to (q \to r)$	premise
4.	q	$\to_e$ 1, 2
5.	$q \to r$	$\to_e$, 1, 3
6.	r	$\to_e$, 4, 5

Example: $(p \wedge q) \to r \vdash p \to (q \to r)$

	introduction	elimination
$\to$	$\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \psi\end{matrix}}}{\phi \to \psi} \to_i$	$\dfrac{\phi, \quad \phi \to \psi}{\psi} \to_e$ Modus Ponens

1. $p \wedge q \to r$ — premise
2. p — assumption
3. q — assumption
4. $p \wedge q$ — $\wedge_i$ 2, 3
5. r — $\to_e$, 1, 4
6. $q \to r$ — $\to_i$, 3–5
7. $p \to (q \to r)$ — $\to_i$, 2–6

Example:

$$p \to (q \to r) \vdash p \wedge q \to r$$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$, 2
4.	q	$\wedge e_2$, 2
5.	$q \to r$	$\to e$, 1, 3
6.	r	$\to e$, 5, 4
7.	$p \wedge q \to r$	$\to i$, 2–6

Rules for disjunction $\vee$:

	introduction	elimination
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee_{i_1}$ $\quad$ $\dfrac{\psi}{\phi \vee \psi}\ \vee_{i_2}$	$\dfrac{\phi \vee \psi \qquad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \qquad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\ \vee_e$

Example:

$$p \vee q \vdash q \vee p$$

	introduction	elimination
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\,\vee_{i_1}$ $\quad$ $\dfrac{\psi}{\phi \vee \psi}\,\vee_{i_2}$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\,\vee_e$

1. $p \vee q$ premise
2. p assume
3. $q \vee p$ $\vee_{i_2}$, 2
4. q assume
5. $q \vee p$ $\vee_{i_1}$, 4
6. $q \vee p$ $\vee_e$: 1, 2-3, 4-5

Example: $(p \vee q) \vee r \vdash p \vee (q \vee r)$

	introduction	elimination
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\vee_{i_1}$ $\quad$ $\dfrac{\psi}{\phi \vee \psi}\vee_{i_2}$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \quad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi}\vee_e$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	assume
3.	p	assume
4.	$p \vee (q \vee r)$	$\vee_{i_2}$, 3
5.	q	assume
6.	$q \vee r$	$\vee_{i_1}$, 5
7.	$p \vee (q \vee r)$	$\vee_{i_2}$, 6
8.	$p \vee (q \vee r)$	$\vee_e$, 2, 3–4, 5–7
9.	r	assume
10.	$q \vee r$	$\vee_{i_2}$, 9
11.	$p \vee (q \vee r)$	$\vee_{i_2}$, 10
12.	$p \vee (q \vee r)$	$\vee_e$, 1, 2–8, 9–11

Example: $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

$$\frac{\phi}{\phi \vee \psi}$$

1.	$p \wedge (q \vee r)$	premise
2.	p	$\wedge e_1$, 1
3.	$q \vee r$	$\wedge e_2$, 1
4.	q	assume
5.	$(p \wedge q)$	$\wedge i$, 2, 4
6.	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$, 5
7.	r	assume
8.	$p \wedge r$	$\wedge i$ 2, 7
9.	$(p \wedge q) \vee (p \wedge r)$	$\vee i$, 8
10.	$(p \wedge q) \vee (p \wedge r)$	$\vee e$, 3, 4-6, 7-9

Example: $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$

Exercise

Summary:

- Sequents
- Proofs of sequents
- Natural deduction rules.

So far no rules to deal with negation.

We will see more rules in the next class