DISCRETE MATHEMATICS

LECTURE 5

Plan:

- Introduction to logic
- Rules for natural deduction

Reference:

Sections 1.1 and 1.2.1 of

LOGIC IN COMPUTER SCIENCE

(second edition)

by Michael HUTH & Mark RYAN

Goal: a mechanism to infer "statements" from a given set of "statements".

Example 1: IF the train arrives late AND there are NO taxis at the station,

THEN John is late for his meeting.

John is NOT late for his meeting.

The train did arrive late.

What can you infer?

There are taxis at the station

Example 2: IF it is raining AND

Jane does NOT have her umbrella with her

THEN she will get wet.

Jane is NOT wet.

It is raining.

What can you infer?

Jane has her umbrella with her.

' Both examples have the same " structure".

IF the train arrives late AND
there are no taxis at the station,
THEN John is late for his meeting.
John is NOT late for his meeting.
The train did arrive late.

IF it is raining AND
Jane does not have her umbrella with her
THEN she will get wet.
Jane is NOT wet.
It is raining.

IF p AND NOT q THEN r. NOT r. p.

THEREFORE q.

Both examples have the same "structure".

IF the train arrives late AND there are no taxis at the station, THEN John is late for his meeting.	IF it is raining AND Jane does not have her umbrella with her THEN she will get wet.
John is NOT late for his meeting.	Jane is NOT wet.
The train did arrive late.	It is raining.

IF p AND NOT q THEN r. NOT r. p.

THEREFORE q.

PROPOSITIONS:

the train arrives late, it is raining } p

there are taxis at the station, Jane has her umbrella with her } q

Propositional variables

PROPOSITIONAL LOGIC:

a system of reasoning over propositional statements.

Basic blocks: Propositional variable.

From propositions we can form compound statements using operators.

IF p AND NOT q THEN r

Propositions: $p, q, r, s, t,$ etc

Propositional formulas:

Basic formulas: all propositions

+ 2 special symbols $\top$ (top) and $\bot$ (bottom)

Compound formulas:

$$\underset{\text{AND}}{\phi \wedge \psi} \;\Big|\; \underset{\text{OR}}{\phi \vee \psi} \;\Big|\; \underset{\text{IMPLIES}}{\phi \rightarrow \psi} \;\Big|\; \underset{\text{NOT}}{\neg \phi}$$

where ϕ, ψ are themselves propositional formulas

Example: $p \wedge q$, $\quad p \rightarrow \underbrace{(q \vee r)}_{\psi}$, $\quad (p \wedge q) \rightarrow (\neg r \rightarrow s)$,

$\bot \rightarrow p$

Meaning of propositional formulas:

Propositional variables can take either true or false

- also denoted as 1 or 0

Meaning of propositional formulas is given by a truth table

Example: $p \wedge q$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Truth tables induced by the operators:

AND : $\phi \wedge \psi$

ϕ	ψ	$\phi \wedge \psi$
0	0	0
0	1	0
1	0	0
1	1	1

OR : $\phi \vee \psi$

ϕ	ψ	$\phi \vee \psi$
0	0	0
1	0	1
0	1	1
1	1	1

NOT : $\neg \phi$

ϕ	$\neg\phi$
1	0
0	1

$\top$:

$\top$
1

$\bot$

$\bot$
0

IMPLIES : $\phi \to \psi$

ϕ	ψ	$\phi \to \psi$
0	0	1
0	1	1
1	0	0
1	1	1

IMPLIES : $\phi \to \psi$

ϕ	ψ	$\phi \to \psi$
0	0	1
0	1	1
1	0	0
1	1	1

IF ϕ is true THEN ψ is true

$\phi \to \psi$ is equivalent to $\neg\phi \vee \psi$

Truth tables of more complex formulas:

Construct the truth tables for the following formulas:

1. $p \to \neg q$
2. $\neg p \to q$
3. $(p \vee q) \to r$
4. $\bot \to p$
5. $\top \to p$
6. $\neg p \vee q$

$p \to \neg q$

p	q	$p \to \neg q$
0	0	1
0	1	1
1	0	1
1	1	0

$\neg p \to q$

p	q	$\phi = \neg p$	$\psi = q$	$\phi \to \psi$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	1	1

$\perp \to p$

$\perp$	p	$\underbrace{\perp}_{\phi} \to \underbrace{p}_{\psi}$
0	0	1
0	1	1

$\top \to p$

$\top$	p	$\top \to p$
1	0	0
1	1	1

$(p \vee q) \rightarrow r$

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

$\neg p \vee q$

p	q	$\neg p \vee q$
0	0	1
0	1	1
1	0	0
1	1	1

Binding rules :

$p \wedge q \rightarrow \neg r \vee s$ - How to read it when there are no brackets?

stands for: $(p \wedge q) \rightarrow ((\neg r) \vee s)$

Binding priority:

$$\neg$$
$$\vee \quad \wedge$$
$$\rightarrow$$

- An operand should be associated with the symbol of higher priori
- If operand is surrounded by symbols with equal priority, then associate the operand to the right.

Examples:

$\neg p \vee q$: $(\neg p) \vee q$

$p \vee q \vee r$: $p \vee (q \vee r)$

$p \vee q \wedge r$: $p \vee (q \wedge r)$

$\neg p \rightarrow q$: $(\neg p) \rightarrow q$

$p \rightarrow q \rightarrow r$: $p \rightarrow (q \rightarrow r)$

More examples: Paranthesize the following formulas

$\neg p \vee \neg q \wedge \neg r$ – $(\neg p) \vee ((\neg q) \wedge \neg r)$ – $\neg p \vee (\neg q \wedge \neg r)$

$\neg p \rightarrow \neg q \vee r$ – $(\neg p) \rightarrow ((\neg q) \vee r)$

$p \wedge q \rightarrow \neg r$ – $(p \wedge q) \rightarrow (\neg r)$

$p \wedge q \wedge r \rightarrow s$ – $(p \wedge (q \wedge r)) \rightarrow s$

$\neg p \rightarrow q \rightarrow r \wedge s$ $(\neg p) \rightarrow (q \rightarrow (r \wedge s))$

$\neg p \rightarrow (q \rightarrow (r \wedge s))$

What we have seen so far?

- Propositional logic
- Syntax : constructing formulas from propositions
  binding rules
- Semantics: meaning of propositional formulas
  given by truth tables

Next: a system of inference rules called
Natural Deduction

Natural deduction

$$\underbrace{\phi_1, \phi_2, \ldots, \phi_n}_{\text{premises}} \vdash \underbrace{\psi}_{\text{conclusion}} \quad \text{(sequent)}$$

Natural deduction is a set of rules that can be applied to prove valid sequents.

For valid sequents, the truth table of $\phi_1 \wedge \phi_2 \wedge \ldots \phi_n \rightarrow \psi$ has 1 in all the rows

Examples of valid sequents:

(entails)

$p, \; p \to q \;\vdash\; q$

$p, \; q, \; r \;\vdash\; p \land q \land r$

$p \land q, \; r \;\vdash\; q \land r$

$p \to q \to r, \; p \land q \;\vdash\; r$

$(p \land \neg q) \to r, \; \neg r, \; p \;\vdash\; q$

Invalid sequent: $p \;\vdash\; q$

$(p \land (p \to q)) \to q$

p	q	

$p \to q$

p	q	$p \to q$
0	0	1
0	1	1
1	0	$\boxed{0}$
1	1	1

$p, \quad p \to q \quad \vdash q$

$$(p \land (p \to q)) \to q$$

p	$\overset{\psi}{q}$	$p \to q$	$\overbrace{p \land p \to q}^{\phi}$	$\overbrace{(p \land p \to q) \to q}^{\phi \to \psi}$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

$\phi \to \psi$

$\phi = 0, \quad \psi = 1$

Coming next! Natural deduction rules

Rules for conjunction $\wedge$:

	introduction	elimination
$\wedge$	$\dfrac{\phi, \quad \psi}{\phi \wedge \psi} \wedge_i$	$\dfrac{\phi \wedge \psi}{\phi} \wedge_{e_1}$ $\quad$ $\dfrac{\phi \wedge \psi}{\psi} \wedge_{e_2}$

Example :

$p \land q, \quad r \vdash q \land r$

1.	$p \land q$	premise
2.	r	premise
3.	q	$\land e_2$, 1
4.	$q \land r$	$\land i$, 3, 2

$$\frac{\phi \land \psi}{\psi} \land e_2$$

$$\frac{\phi \quad \psi}{\phi \land \psi} \land i$$

Example: $(p \wedge q) \wedge r,\ s \wedge t \vdash q \wedge s$

1.	$(p \wedge q) \wedge r$	premise
2.	$s \wedge t$	premise
3.	$p \wedge q$	$\wedge e_1$, 1
4.	q	$\wedge e_2$, 3
5.	s	$\wedge e_1$, 2
6.	$q \wedge s$	$\wedge i$, 4, 5

Example:

$$p,\ q \wedge r,\ s \wedge t \ \vdash\ \boxed{p \wedge q \wedge s}$$

$p \wedge \boxed{(q \wedge s)}$

$q \wedge (p \wedge s)$

1.	p	premise
2.	$\boxed{q \wedge r}$	premise
3.	$s \wedge t$	premise
4.	q	$\wedge e_1$, 2
5.	s	$\wedge e_1$, 3
6.	$(q \wedge s)$	$\wedge i$, 4, 5
7.	$p \wedge (q \wedge s)$	$\wedge i$, 1, 6

Summary:

→ Propositional logic (how to write formulas)

→ Truth tables for formulas.

→ Natural deduction.

$$\phi_1, \phi_2, \ldots \phi_n \vdash \psi$$

→ write a proof using rules that shows this sequent is valid.