DISCRETE MATHEMATICS

LECTURE 4

Plan:

- Principle of Inclusion-Exclusion
- Applications of this principle
  - No. of onto functions
  - Derangements

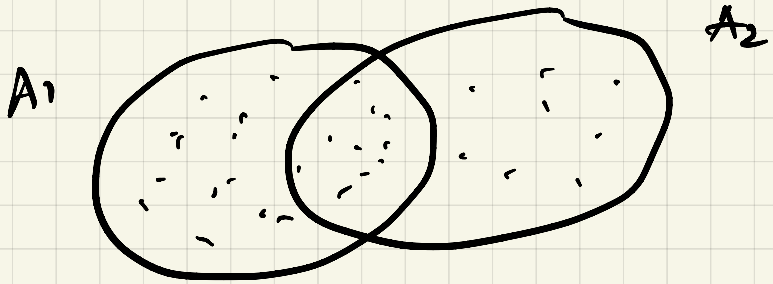
Reference:

Sections 8.5 and 8.6 of book:

DISCRETE MATHEMATICS AND ITS APPLICATIONS (7^{th} edition)

by Kenneth Rosen

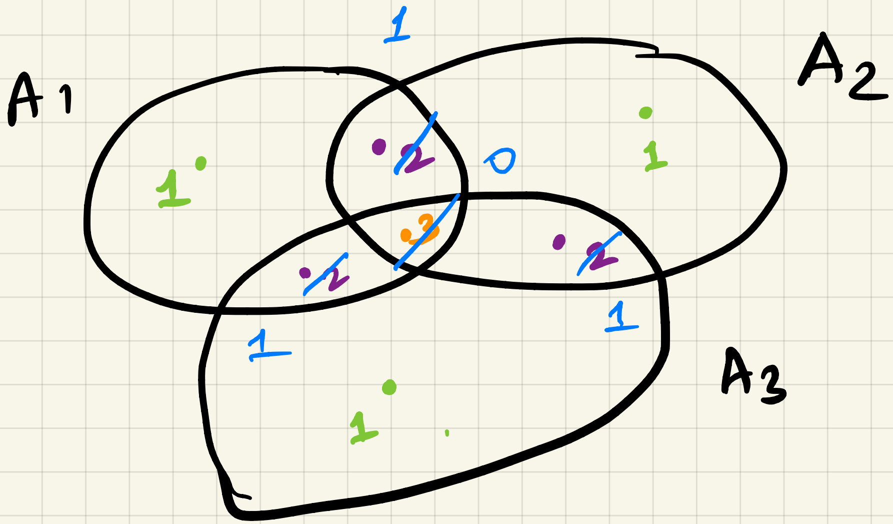
$|A_1 \cup A_2|$:



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

↳ seen earlier as subtraction rule

$|A_1 \cup A_2 \cup A_3|$:



$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1|$$

$$+ |A_1 \cap A_2 \cap A_3|$$

$|A_1 \cup A_2 \cup A_3 \cup A_4|$:

$a \in A_1 \cap A_2 \cap A_3 \cap A_4$

$$\overset{1}{} \quad \overset{1}{} \quad \overset{1}{} \quad \overset{1}{} \quad = 4$$

$$= |A_1| + |A_2| + |A_3| + |A_4|$$

$-6 \qquad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| \cdots - |A_3 \cap A_4|$

$4 \qquad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_2 \cap A_4|$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$|A_1 \cup A_2 \cup \dots \cup A_n|$$

$$= \sum_{i=1}^{n} |A_i|$$

$$- \sum_{1 \le i, j \le n} |A_i \cap A_j| \quad i \ne j$$

$1 \le i < j \le n$

$$+ \sum_{i,j,k} |A_i \cap A_j \cap A_k| \quad i \ne j \ne k$$

$1 \le i < j < k \le n$

$\vdots$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

Why is this correct?

- Every element $a \in A_1 \cup A_2 \cup \dots \cup A_n$ is counted exactly once

$a \in A_1 \cap A_2 \cap \dots A_j$

$a \notin$ other A_k's.

- $|A_i|$ — $+j$ times

$|A_i \cap A_j|$ $-\binom{j}{2}$

$|A_i \cap A_j \cap A_k|$ $+\binom{j}{3}$

$\vdots$

$(-1)^{j+1}$

Contd. from previous slide:

In all intersections of more than $j+1$ sets, 'a' is not counted.

- Expression giving the count of 'a' is:

$$\binom{j}{1} - \binom{j}{2} + \binom{j}{3} \cdots\cdots + (-1)^{j+1}\binom{j}{j}$$

- What is the value of this expression?

- 1 (why?)

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

$x=1$, $y=-1$

$$0 = \sum_{i=0}^{n} \binom{n}{i} 1 \cdot (-1)^i$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots (-1)^n \binom{n}{n}$$

$$\Rightarrow \binom{n}{1} - \binom{n}{2} + \binom{n}{3} \cdots (-1)^{n+1} \binom{n}{n} = 1$$

$|A_1 \cup A_2 \cup \dots \cup A_n|$:

Principle of Inclusion - Exclusion:

Let $A_1, A_2, \dots, A_n$ be finite sets

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|$$

$$- \sum_{i,j} |A_i \cap A_j|$$

$$+ \sum_{i,j,k} |A_i \cap A_j \cap A_k|$$

$$- \dots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Example 1: How many bit strings of length 8 contain 6 consecutive zeroes?

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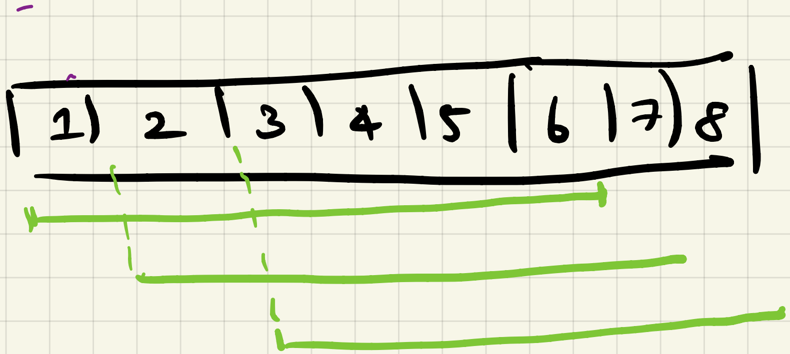
Bit strings of length 8
with 6 consecutive zeroes.



A_1 = { strings of length 8 that start with 6 consecutive zeroes }

A_2 = { strings ______ that have 6 consecutive zeroes starting from position 2 }

A_3 = { ______ ______ starting from position 3 }

Required answer = $|A_1 \cup A_2 \cup A_3| = 4 + 4 + 4 - 2 - 1 - 2 + 1$

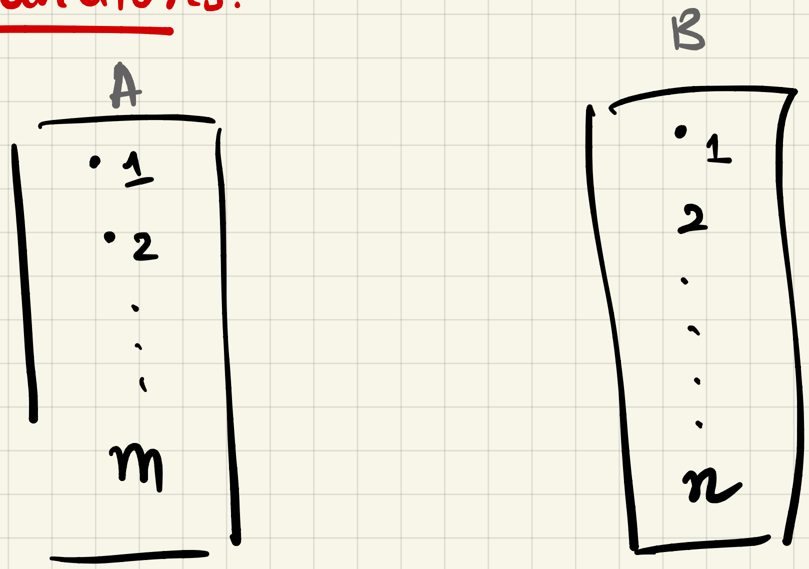
$|A_1| = 4$ $\quad |A_1 \cap A_2| = 2$ $\quad |A_1 \cap A_2 \cap A_3| = 1$

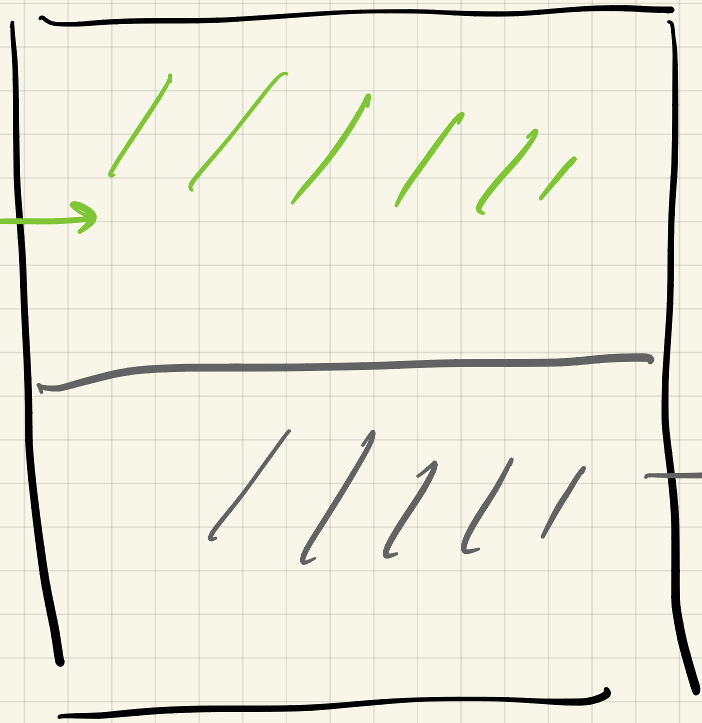
$|A_2| = 4$ $\quad |A_1 \cap A_3| = 1$

$|A_3| = 4$ $\quad |A_2 \cap A_3| = 2$

$= 8$

Number of onto functions:





set of all functions

onto functions.

At least one element of B has no pre-image

C_i = { functions where element i has no pre-image }

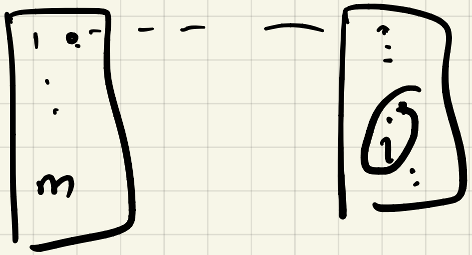
$$|C_1 \cup C_2 \cup \cdots \cup C_n| = \sum_i |C_i| \qquad \binom{n}{1}(n-1)^m$$

$$- \sum |C_i \cap C_j| \qquad \binom{n}{2}(n-2)^m$$

$$+ \sum |C_i \cap C_j \cap C_k| \qquad \binom{n}{3}(n-3)^m$$

$$\vdots \qquad \binom{n}{n-1}(n-(n-1))^m$$

$$+ (-1)^{n+1} |C_1 \cap C_2 \cap \cdots \cap C_n| \qquad 0$$

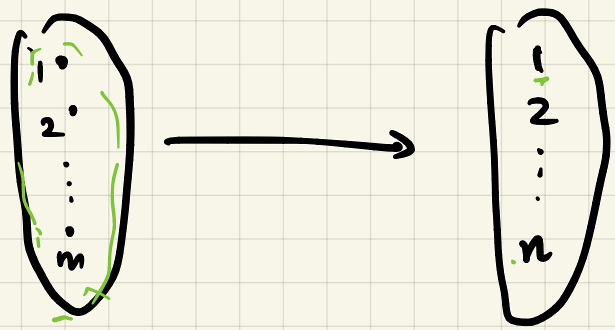


$$|C_i| = (n-1)^m$$

$$|C_i \cap C_j| = (n-2)^m$$

$$|C_i \cap C_j \cap C_k| = (n-3)^m$$

No. of onto functions:



$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \dots + (-1)^{n-1}\binom{n}{n-1}(n-(n-1))^m$$

Derangements:

A permutation of objects so that no object is in its original position.

Examples:

1 2 3

↓

$\boxed{1}$ 3 2

2 1 $\boxed{3}$

2 3 1 — derangements.

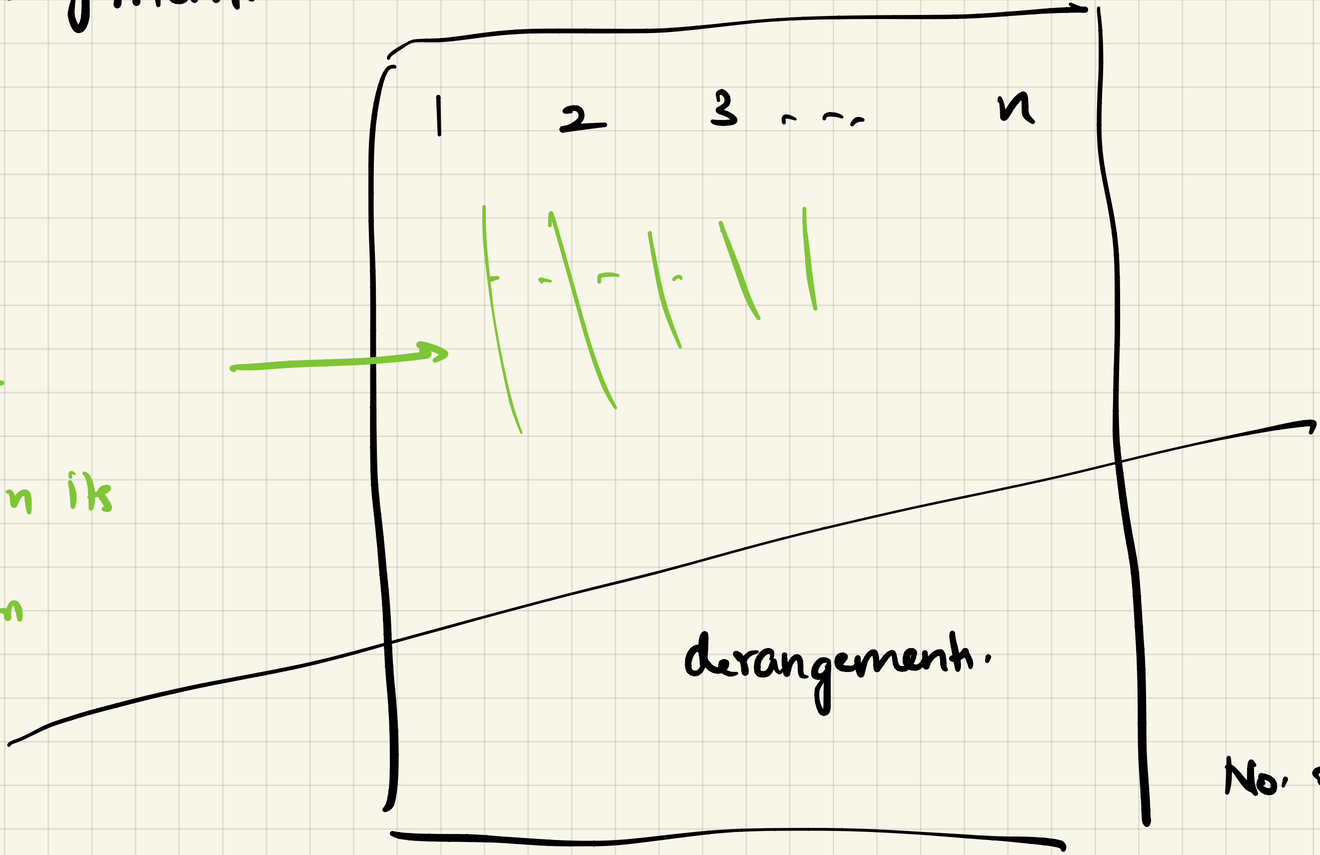
3 1 2

3 $\boxed{2}$ 1

Question: What is the number of derangements of a set with n elements?

No. of derangements:

at least 1 element is in its original position



No. of permutations $= n!$

A_i = set of permutations where i is in its original position (i^{th})

$|A_1 \cup A_2 \cdots \cup A_i| =$ green part.

$$|A_i| = (n-1)!$$

$$|A_i \cap A_j| = (n-2)!$$

$$\vdots$$

$$|A_1 \cup A_2 \cdots \cup A_n| = \binom{n}{1}(n-1)!$$

$$- \binom{n}{2}(n-2)!$$

$$+ \binom{n}{3}(n-3)!$$

$$\vdots$$

$$+ (-1)^{n+1}\, 1!$$

derangements $= n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$

No. of derangements of a set with n elements is:

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + (-1)^n \binom{n}{n} 0!$$

Example: find the no. of non-negative integral solutions to $x_1 + x_2 + x_3 = 15$

s.t. $x_1 \leq 4$ and $x_2 \leq 8$ and $x_3 \leq 7$

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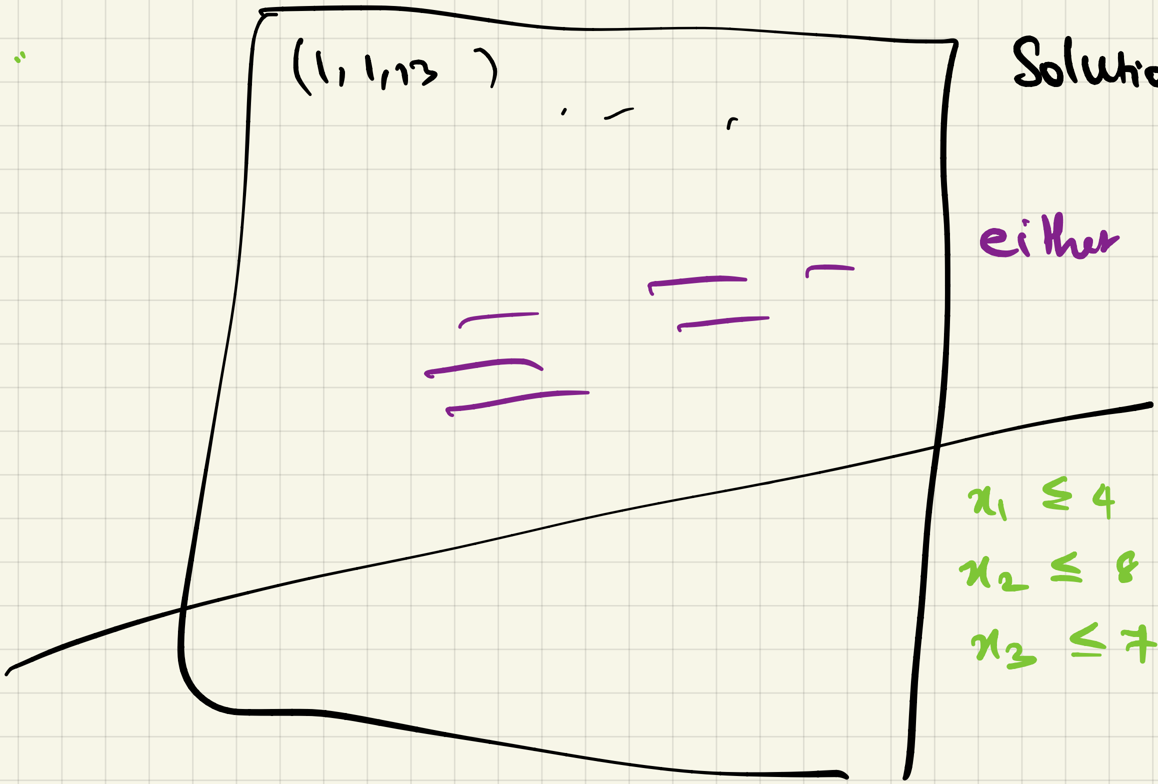

Solutions to $x_1 + x_2 + x_3 = 15$

either $x_1 > 4$ or $x_2 > 8$ or $x_3 > 7$

$x_1 \leq 4$

$x_2 \leq 8$

$x_3 \leq 7$

Reqd. answer = Total no. of solutions $- |A_1 \cup A_2 \cup A_3|$

A_1 = Set of solutions with $x_1 > 4$

A_2 = ——— $x_2 > 8$

A_3 = ——— $x_3 > 7$

$x_1 + x_2 + x_3 = 15$

total no. of solution: 2 bars and 15 stars

$= {}^{17}C_2$

solutions with $x_1 > 4$.

$|A_1|$

$x_1 = (\underbrace{5 + x_1'}_{})$ $\quad 5 + x_1' + x_2 + x_3 = 15 \quad = {}^{12}C_2$

$x_1' + x_2 + x_3 = 10$

solutions with $x_2 > 8$ $\quad x_1 + x_2' + x_3 = 6 \quad = {}^{8}C_2 \quad |A_2|$

solutions with $x_3 > 7$ $\quad x_1 + x_2 + x_3' = 7 \quad = {}^{9}C_2 \quad |A_3|$

$|A_1 \cap A_2|$ ... .. - (continue the calculation)

Summary:

- Principle of Inclusion Exclusion
- # of onto functions
- # of derangements.