LECTURE 3

Plan:

- Permutations and combinations with repetitions allowed
- Distributing objects into boxes

Reference:

Sections 6.5 of book:

DISCRETE MATHEMATICS AND ITS APPLICATIONS (7th edition)

by Kenneth Rosen

Example 1: How many strings of length 'r' can be formed from the uppercase letters of the English alphabet?

PAUSE

Example 1: How many strings of length 'r' can be formed from the uppercase letters of the English alphabet?

Solution:

Using product rule:

$$\underbrace{\boxed{1}\,\boxed{2}\,\boxed{\ldots\ldots}\,\boxed{r}}_{r}$$

$$26 \times 26 \times \cdots \times 26$$

$$= 26^r$$

PERMUTATIONS WITH REPETITIONS:

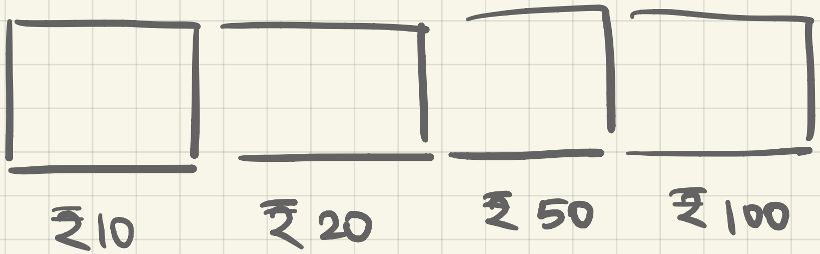
No. of r- permutations of a set of n distinct elements when repetitions are allowed is:

$$n^r$$

Example 2: A box contains some ₹10, ₹20, ₹50 and ₹100 notes. Assume that there are at least 5 notes of each value. In how many ways can 5 notes be selected? Notes of same denomination are indistinguishable.

PAUSE

Solution:



Example of selection:

* * | * | * | *   two 10, one 20, 50, 100

| | * * * | * *   three 50, two 100

Each selection can be denoted using a picture that arranges 5 * and 3 |

* | * * | * * * | 1 10, 2 20, 3 50's.

* * | * | * | * two 10, one 20, 50, 100

| | * * * | * * three 50, two 100

Sub-question: Find the no. of strings of length 8 over | and * that contain exactly five *.

8 position. To choose 3 positions for the |

This can be done in $8C_3$ ways.

$= 8C_5$

- Why 4^5 is incorrect?

5 notes

$\underset{1}{\sqcup}$ $\rightarrow$ Notes are not ordered.

There is no first note, second note, etc.

- 10 20

- 3 notes

$(10, 10, 10)$ $\ast\ \ast\ \ast \mid$

$(10, 10, 20)$ $\ast\ \ast \mid \ast$

$(10, 20, 20)$ $\ast \mid \ast\ \ast$

$(20, 20, 20)$ $\mid \ast\ \ast\ \ast$

COMBINATIONS WITH REPETITIONS:

1 * | 2 *** | 3 | . . . | n

No. of r- combinations of n (distinct) objects where repetitions are allowed is the same as

The number of strings over $(n-1)$ | (bars) and r * (stars)

$$(n-1+r)\,C_r$$

Example 3: A shop has 5 different kinds of cookies. How many different ways can 3 cookies be chosen?

PAUSE

Example 3: A shop has 5 different kinds of cookies. How many different ways can 3 cookies be chosen? Multiple cookies of the same type can be selected.

Solution:

Cookie 1 | Cookie 2 | Cookie 3 | Cookie 4 | Cookie 5

Choose three of them: * * *

Arrange 4 | and 3 *

$^{7}C_{3}$

Example 4: How many solutions does

$$x_1 + x_2 + x_3 = 15$$ have

where x_1, x_2, x_3 are non-negative integers?

PAUSE

Example 4: How many solutions does

$$x_1 + x_2 + x_3 = 15$$ have

where x_1, x_2, x_3 are non-negative integers?

Solution:

15 $*$ and 2 $|$

Eg:

$* * * * * \;|\; * * \;|\; * * * * * * * *$

$x_1 = 5 \qquad x_2 = 2 \qquad x_3 = 8$

$$^{17}C_{15}$$

Permutations and Combinations:

	Repetition allowed?	
r-permutations	No	nP_r
r- permutations	Yes	n^r
r-combinations	No	nC_r
r-combinations	Yes	$(n-1+r)C_r$

Example 5: How many different strings can be made by reordering letters of the word SUCCESS?

PAUSE

Example 5: How many different strings can be made by reordering letters of the word SUCCESS?

Solution:

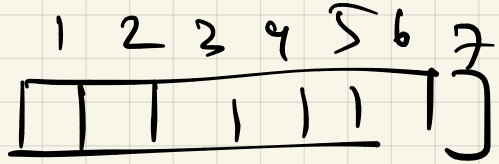
SUCCESS:

3 S

1 U

2 C

1 E



- First choose the positions for S: $7C_3$

4

Next choose position for U: $4C_1$

3

C : $3C_2$

E : $1\,C_1$

$$7C_3 \cdot 4C_1 \; 3C_2 \cdot 1 = \frac{7!}{\cancel{4!}\,3!} \cdot \frac{\cancel{4!}}{3!\,1!} \; \frac{2!}{2!} \cdot \frac{1!}{1!} = \frac{7!}{3!\,1!\,2!\,1!}$$

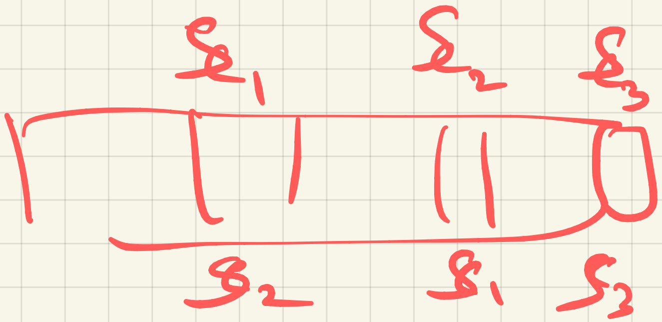
- First choose the positions for S : $^7C_3$ | 4

Next choose position for U : $^4C_1$ 3

C : $^3C_2$

E : $^1C_1$

$$^7C_3 \cdot {}^4C_1 \cdot {}^3C_2 \cdot 1 = \frac{7!}{\cancel{4!}\,3!} \cdot \frac{\cancel{4!}}{\cancel{3!}\,1!} \cdot \frac{\cancel{3!}}{2!\,1!} \cdot \frac{1!}{1!} = \frac{7!}{3!\,1!\,2!\,1!}$$



Permutations with indistinguishable objects:

Permutations of n objects where:

n_1 objects are of 1 kind

n_2 ———————

n_3

:

n_k are of 1 kind.

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

Distributing objects to boxes:

Example 6: There are 10 distinct objects to be placed in 4 distinct boxes such that

Box 1 contains 3 objects

Box 2 contains 4 objects

Box 3 contains 1

Box 4 contains 2.

In how many ways can this be done?

PAUSE

Solution:

2	3	1	2	1	2	1	2	4	4
Object 1	Object 2	Obj. 3	Obj. 4	Obj. 5	Obj. 6	Obj. 7	Obj 8	Ob. 9	Ob. 10

A string of length 10 containing 3

$$\frac{10!}{3!\,4!\,1!\,2!} = {}^{10}C_3 \cdot {}^{7}C_4 \cdot {}^{3}C_1 \cdot {}^{2}C_2$$

Distributing distinguishable objects to distinguishable boxes:

No. of ways to:

Distribute n different objects to k different boxes

so that Box i contains n_i objects

$(n_1 + n_2 + \dots + n_k = n)$

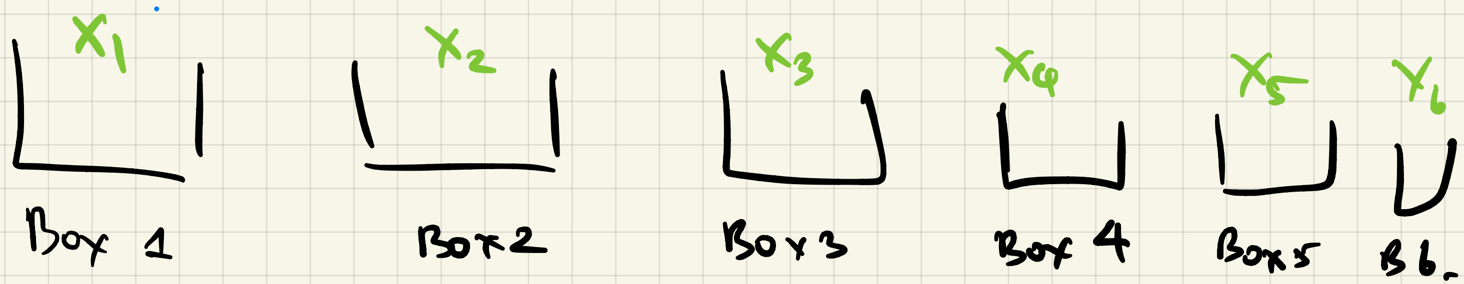
$$= \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Example 7: How many ways are there to place 10 identical balls into 6 different boxes?

PAUSE

Example 7: How many ways are there to place 10 identical balls into 6 different boxes?

Solution:



3 2 1 4 0 0

*** | ** | * | **** | |

We want to find no. of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$

$^{15}C_5$

Doubt

Why 6^{10} is incorrect?

Ball	Box
1 $\longrightarrow$	6
2 $\longrightarrow$
3
.
.

1 $\rightarrow$ 1

2 1

3 2

1 — 1

2 — 2

3 1

↳ counted differently when balls are distinguishable.

3 Balls.

2 Boxes.

(3, 0)

(2, 1)

(1, 2)

(0, 3)

Distributing indistinguishable objects into distinguishable boxes

No. of ways to

distribute n indistinguishable objects into k distinguishable boxes

$$= {}^{(n+k-1)}C_{k-1}$$

same as arranging $k-1$ bars and n stars.

Remark: Distributing objects into indistinguishable boxes is more difficult.

Please read the book if you're interested.

Summary:

- Permutations with repetitions
- Combinations with repetitions
- Permutations with indistinguishable objects.
- Distributing objects (distinguishable / indistinguishable) into distinguishable boxes