DISCRETE MATHEMATICS

Course Outline:

Module 1: Counting

Module 2: Logic

Module 3: Graph theory

Module 4: Regular expressions

LECTURE 1

Plan: Basics of counting

- 1. Product rule
- 2. Sum rule
- 3. Subtraction rule
- 4. Division rule
- 5. Pigeon hole Principle

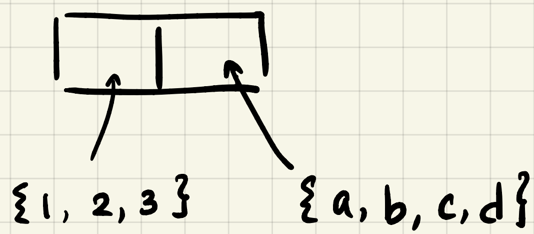
Reference:

Sections 6.1 and 6.2 of book:

DISCRETE MATHEMATICS AND ITS APPLICATIONS
(7th edition)

by Kenneth Rosen

Part 1: Product rule



1	a	2	a	3	a
	b		b		b
	c		c		c
	d		d		d

$$= 4 + 4 + 4$$

$$= 3 \times 4 = 12$$

- Suppose a procedure can be broken down into a sequence of two tasks.
- Suppose there are n_1 ways of doing the first task.
- Suppose there are n_2 ways of doing the second task.
- Then there are $n_1 \times n_2$ ways of doing the procedure

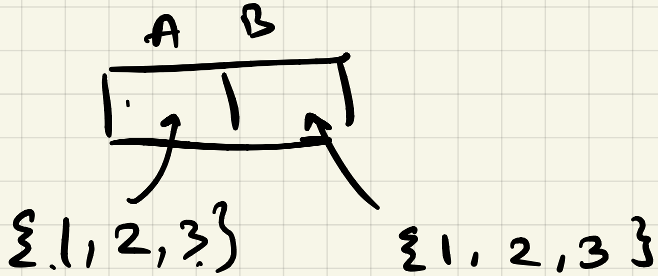
Example 1: Two new employees A and B. There are 3 offices. In how many ways can offices be assigned to the 2 employees? Both the employees can be allotted the same office.

Please pause and try the problem before seeing the solution.

Example 1: Two new employees A and B. There are 3 offices. In how many ways can offices be assigned to the 2 employees? Both the employees can be placed in the same office.

Solution:

Procedure :



$$\left.\begin{array}{l}\text{Task 1: Write a letter from } \{1,2,3\} \text{ in Cell A} \\ \text{Task 2: Write a number from } \{1,2,3\} \text{ in Cell B}\end{array}\right\}$$

Task 1: 3 ways Task 2: 3 ways

Total := $3 \times 3 = 9$ ways

Example 2: Count the number of bit strings of length 7.

bit string: a string containing only 0's and 1's

000 010

PAUSE

Example 2: Count the number of bit strings of length 7.

Solution:

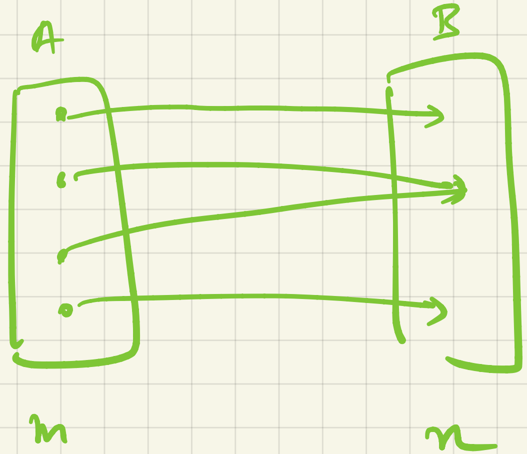
$$N_7 = 2 \times N_6$$

$$= 2 \times 2 \times N_5$$

$$\vdots$$

$$= 2^7$$

Example 3: How many functions are there from a set of 'm' elements to a set of 'n' elements?



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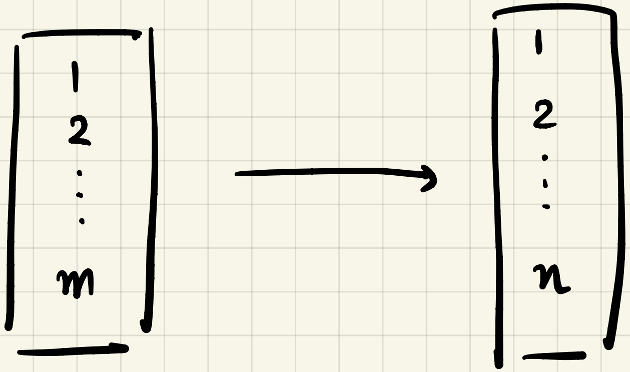
Example 3: How many functions are there from a set of 'm' elements to a set of 'n' elements?

Solution:



$$n \times n \times \dots \times n$$

$$= n^m$$

Example 4: How many one-to-one functions are there from a set of 'm' elements to a set of 'n' elements? $(n \geq m)$

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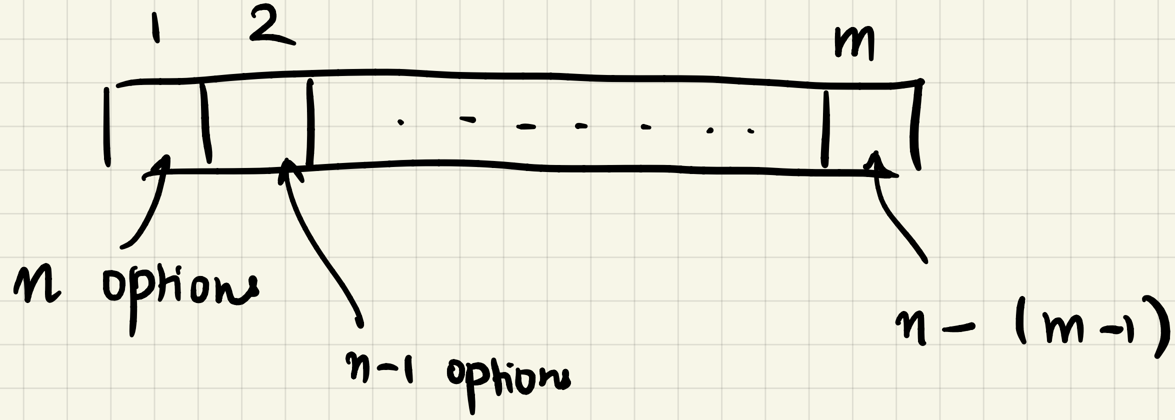
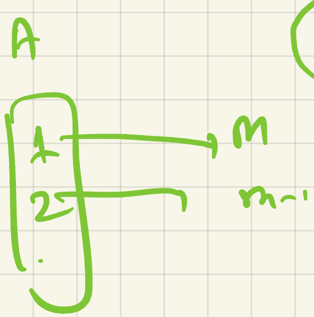
Solution:



$$= n \times (n-1) \times \cdots \times (n-m+1)$$

$\{1, 2, \ldots, n\}$

$\{1, 2, \ldots, m\}$ $m!$



Example 5: Suppose $T = \{1, 2, 3\}$

Subsets of T are $\{\ \}$

$\{1\}, \{2\}, \{3\}$

$\{1, 2\}, \{1, 3\}, \{2, 3\}$

$\{1, 2, 3\}$

Given a set $S = \{1, 2, \ldots, n\}$. What is the total number of subsets of S? How do we use product rule?

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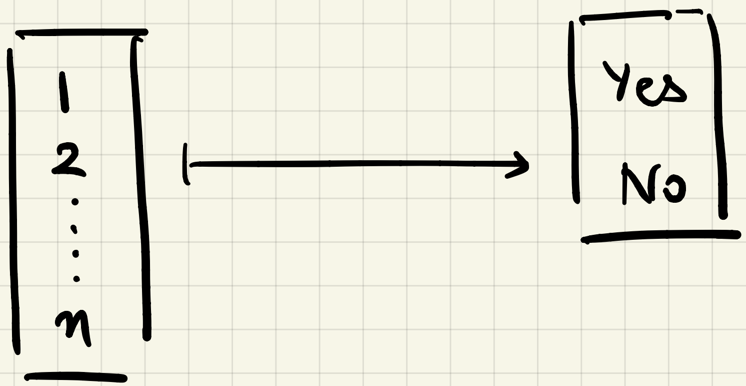
Example 5:

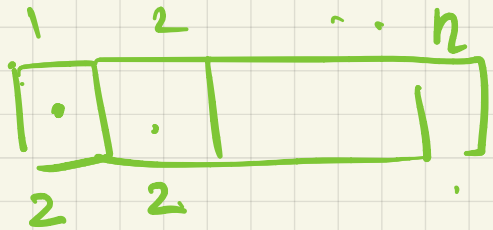
Given a set $S = \{1, 2, \ldots, n\}$. What is the total number of subsets of S? How do we use product rule?

Solution:



Each subset of S is a function from S to $\{\text{Yes}, \text{No}\}$

There are $2^{|S|}$ such functions. Hence $2^{|S|}$ subsets

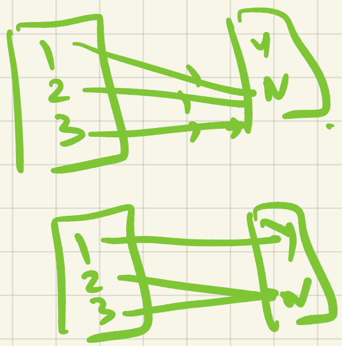


$S = \{1, 2, 3\}$

$\{\ \}$,

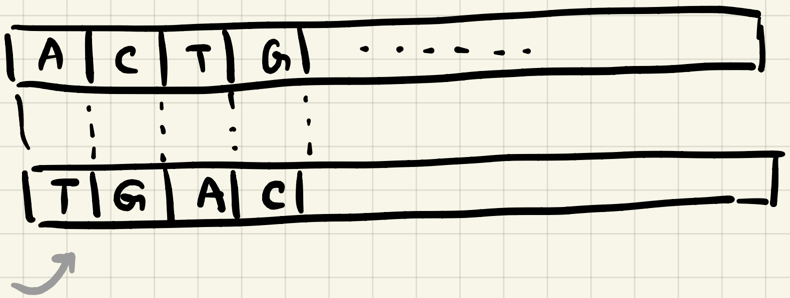
$\{1\}$ $\{2\}$ $\{3\}$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$



Example 6: Gene sequencing

DNA molecule:



$\{A, C, T, G\}$

a link

Gene: a segment of a DNA molecule

Genome: a set of genes

Insect: $\sim 10^8$ links in a DNA molecule

By product rule, $\sim 4^{10^8}$ possible DNA molecules

Explains the variation in behaviour

Part 2: Sum rule

Tile $\square \longrightarrow$ Either filled with $\{1, 2, 3\}$ or $\{A, B, C, D\}$

How many different tiles are present? $4 + 3 = 7$

- Suppose there is a task that can be done either in n_1 or n_2 ways
- There is no intersection / overlap between the n_1 and n_2 ways
- Then the task can be done in $n_1 + n_2$ ways

Example 7: A student can choose a project from either

CS / Math / DS.

CS has 4 options
Math has 3 options
DS has 7 options

How many project options does the student have?

Solution: $4 + 3 + 7 = 14$

Example 8: How many bit strings of length between 1 and 4 are present ?

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Example 8: How many bit strings of length between 1 and 4 are present?

Solution: # bit strings of length 1 = 2

2 = 2^2

3 = 2^3

4 = 2^4

Total $= 2 + 2^2 + 2^3 + 2^4$

Solution uses a combination of product and sum rules

Example 9: A password can have between 6 to 8 characters.

Each character can be a letter $\{A \ldots\ldots Z\}$ or a digit $\{0 \ldots 9\}$.

What is the total number of passwords?

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Example 9: A password can have between 6 to 8 characters. Each character can be a letter $\{A \ldots z\}$ or a digit $\{0 \ldots 9\}$. What is the total number of passwords?

Solution:

$$\# \text{ passwords of length } 6 = (26+10)^6$$

$$7 = (26+10)^7$$

$$8 = (26+10)^8$$

$$\text{Total} = 36^6 + 36^7 + 36^8$$

Example 10: A password should be exactly 6 characters. Atleast one of the characters should be a digit. How many passwords are available?

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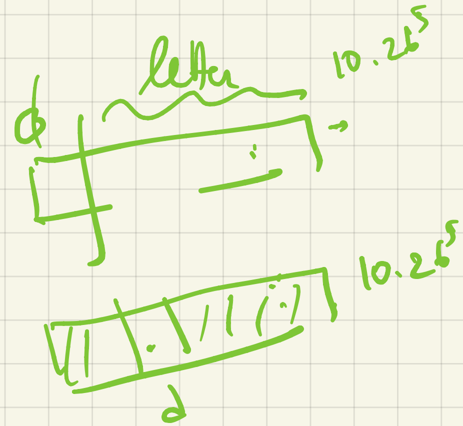
An incorrect approach:



$$
\begin{aligned}
\#\text{ passwords with digit in first cell} &= 10 \cdot 36^5 \\
\text{in } 2^{nd} \text{ cell} &= 10 \cdot 36^5 \\
&\vdots \\
\text{in } 6^{th} \text{ cell} &= 10 \cdot 36^5 \\
\text{Total} &= 6 \times 10 \times 36^5
\end{aligned}
$$

Problem: double counting 5 a 3 b c d is counted in first as well as third category.

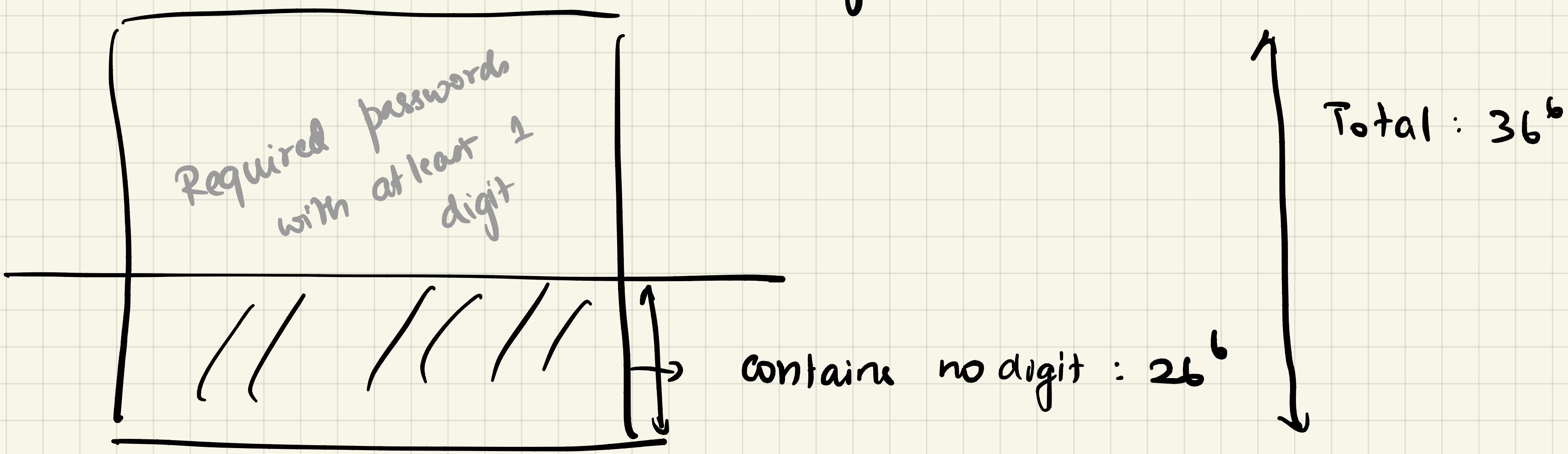
Example 10: A password should be exactly 6 characters. Atleast one of the characters should be a digit. How many passwords are available?

Solution:

Total no. of 6 letter words $= 36^6$

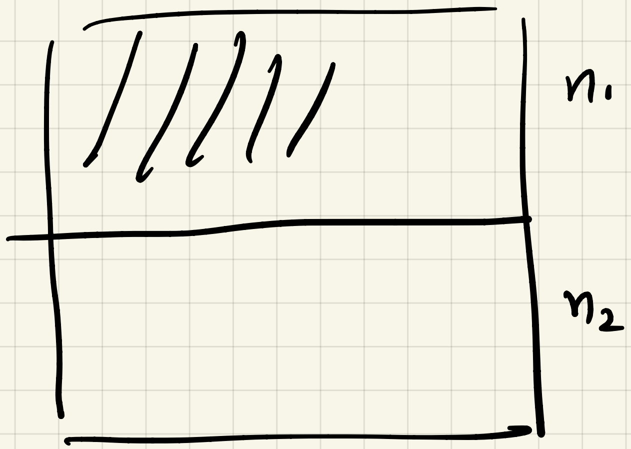
words containing no digits $= 26^6$

$\therefore$ # words containing at least 1 digit $= 36^6 - 26^6$



Part 3: Subtraction rule

In the sum rule, we consider tasks that can be done either in n_1 ways or n_2 ways where there is no overlap between the n_1 and n_2 ways



Total $= n_1 + n_2$

What if there are overlaps ?

Example 11: A company receives 350 applications

- 220 majored in CS
- 147 ——— in business
- 51 in both

How many applicants majored neither in CS nor in business?

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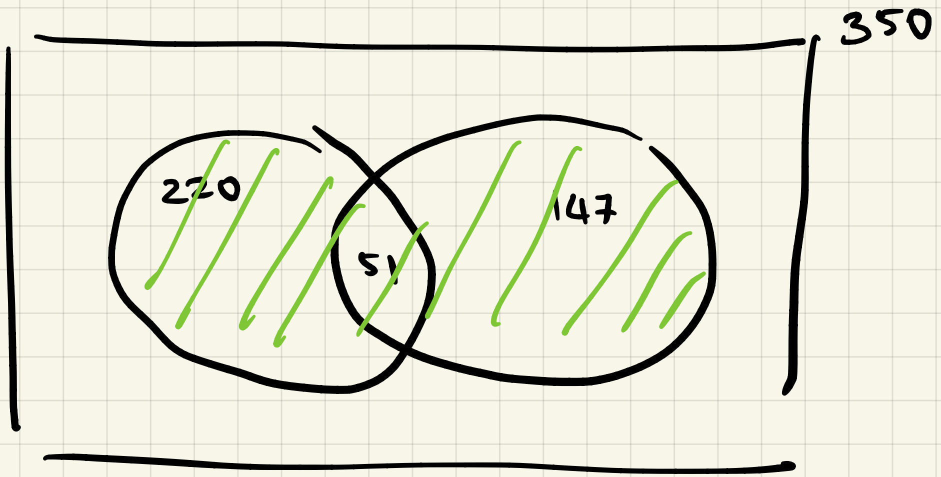
- 220 majored in CS
- 147 majored in business
- 51 in both

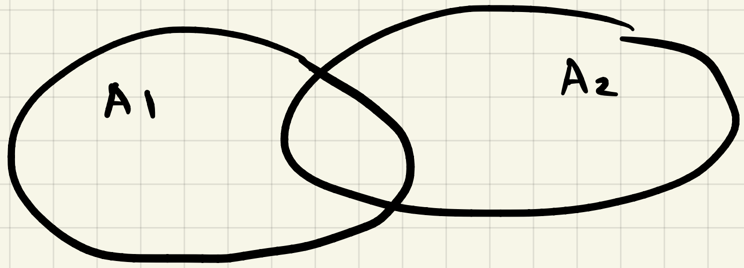
How many applicants majored neither in CS nor in business?

Solution:



$$350 - (\underbrace{220 + 147 - 51}_{\text{\# green part}})$$

Subtraction rule: (Inclusion exclusion for two sets)



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example 12: Among bit strings of length 4, how many start with 1 or end with 0?

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Example 12: Among bit strings of length 4, how many start with 1 or end with 0?

Solution:

A_1 : bit strings starting with 1

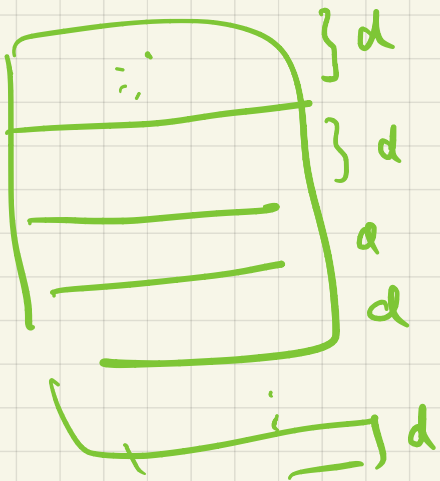
A_2 : bit strings ending with 0

$|A_1| = 2^3 \qquad |A_2| = 2^3 \qquad |A_1 \cap A_2| = 2^2$

$$|A_1 \cup A_2| = 2^3 + 2^3 - 2^2$$

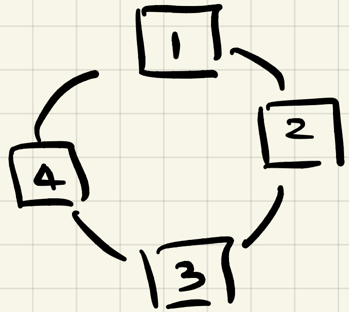
Part 4: Division rule

- A task can be done using a procedure that can be implemented in n ways.
- For each way 'w', there are 'd' ways (including w) that are considered equivalent.
- There are totally n/d ways to perform the task.



n/d

Example 13: There is a circular table



- Two seatings are considered same if left and right neighbours are identical
- How many different seatings are possible?

For eg.

A

D B

C

$\equiv$

C

B D

A

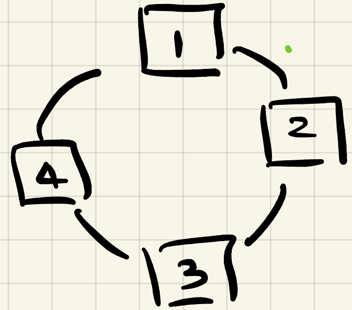
$\not\equiv$

A

B C

D

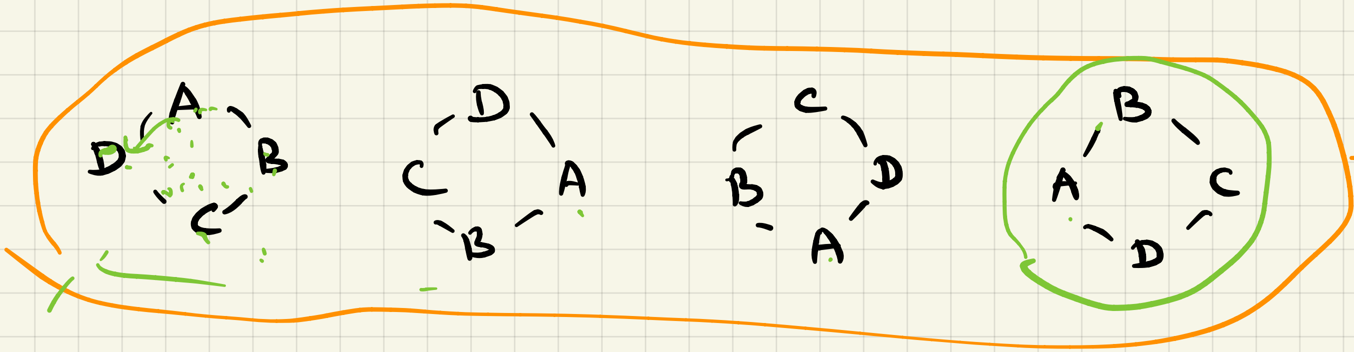
Example 13: There is a circular table



- Two seatings are considered same if left and right neighbours are identical
- How many different seatings are possible?

Solution: Total no. of arrangements $= 4 \times 3 \times 2 \times 1 = 24$

Cyclic shifts of an arrangement are considered same



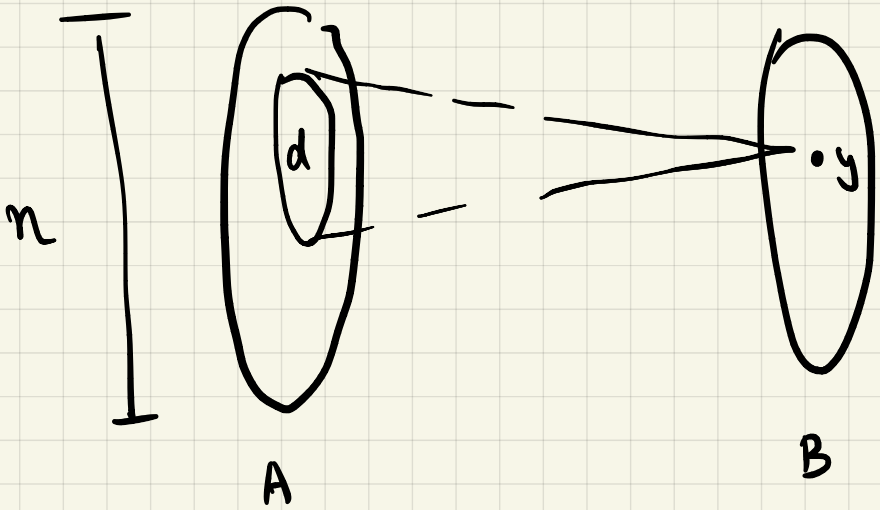
→ 1 seating

24 arrangements can be divided into $24/4 = 6$ ways

Example 14: f is a function from A to B

- where A and B are finite sets
- for every $y \in B$ there are d elements of A mapped to y
- A has n elements

How many elements are present in B?



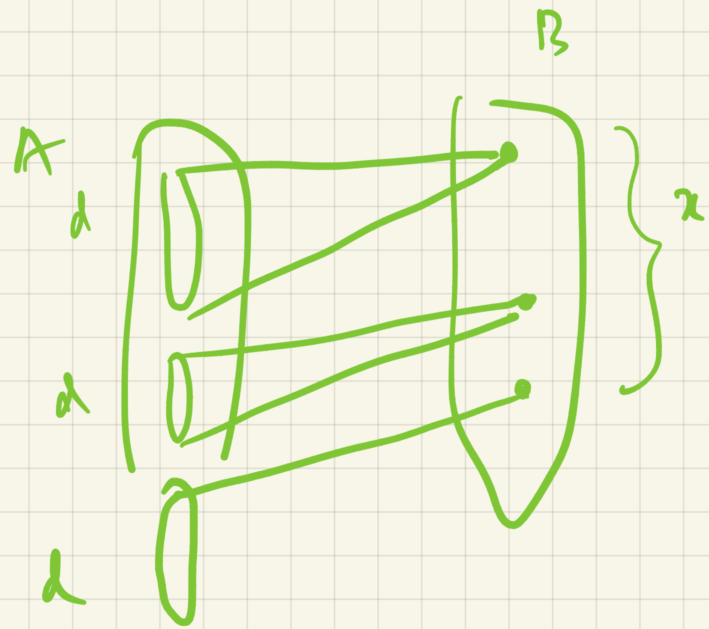
Example 14: f is a function from A to B

- where A and B are finite sets
- for every $y \in B$ there are d elements of A mapped to y
- A has n elements

How many elements are present in B?

Solution:

$$n/d$$

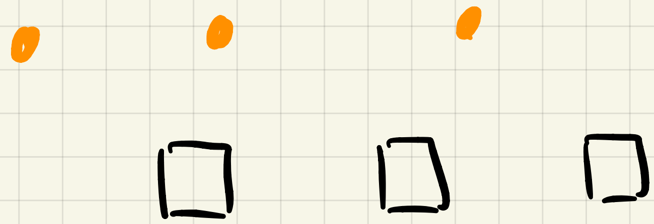


$$x \cdot d = n \qquad x = n/d$$

Part 5: Pigeon-hole Principle

$k+1$ objects are to be placed in k boxes

- at least one box contains atleast 2 objects.



(objects

(boxes

at least 1 box should contain ≥ 2 objects

Example 15: Some examples using PHP

- In any group of 27 English words, there must be two that start with the same letter.
- A function from a set of $k+1$ elements to k elements cannot be one-to-one.

1)

Word 1

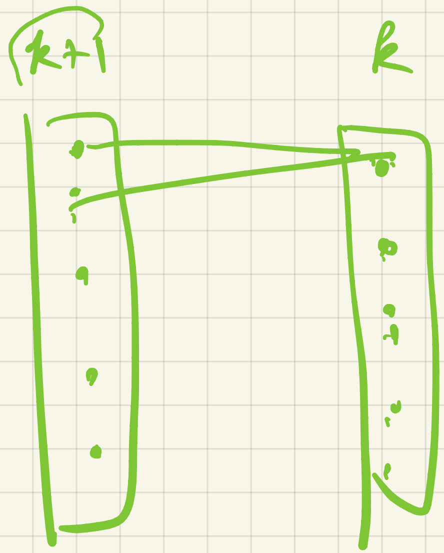
Word 2

⋮

Word 27

26 letters

2)



Example 16: For every integer $n \geq 1$,

every n consecutive numbers has a multiple of n.

$n = 3$

106	107	108
1	2	0

122	123	124
2	0	1

$(2+1) \bmod 3$

PAUSE

Example 16: For every integer $n \geq 1$,

every n consecutive numbers has a multiple of n.

Solution: Consider no.s $a, a+1, a+2, \ldots, a+(n-1)$

Suppose 'a' gives remainder 'r' when divided by 'n'.

ie., $a = k \cdot n + r$

Note that $0 \leq r \leq n-1$

Then $a+1$ gives remainder $(r+1) \bmod n$

$a+2$ - - - - $(r+2) \bmod n$

$\vdots$

$a+n$ - - - - - $(r+n) \bmod n$

The no.s $r \bmod n$, $(r+1) \bmod n$, ... $(r+n) \bmod n$ are all different and between 0 and $n-1$. Hence at least one of them is 0.

Hence there is at least one multiple of 'n'.

Example 17: Show that for every integer 'n' there is a multiple of n that contains only 0's and 1's in its decimal expansion.

$n=2$ | 2, 4, 6, 8, $\boxed{10}$

$n=3$ | 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33 .... 99, ... $\boxed{111}$

$n=4$ = $\boxed{100}$

See text book for an answer

LECTURE 1 : Summary

Plan: Basics of counting

- 1. Product rule
- 2. Sum rule
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