Problem Set 4
Discrete Mathematics 2019

CHENNAI MATHEMATICAL INSTITUTE

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> *The purpose of computation is insight, not numbers.*
>
> Richard Hamming

Problem 1. For each course at a university, there may be one or more other courses that are its prerequisites. How can a graph be used to model these courses and which courses are pre-requisites for which courses? Should edges be directed or undirected? Looking at the graph model, how can we find courses that do not have any prerequisites and how can we find courses that are not the prerequisite for any other courses?

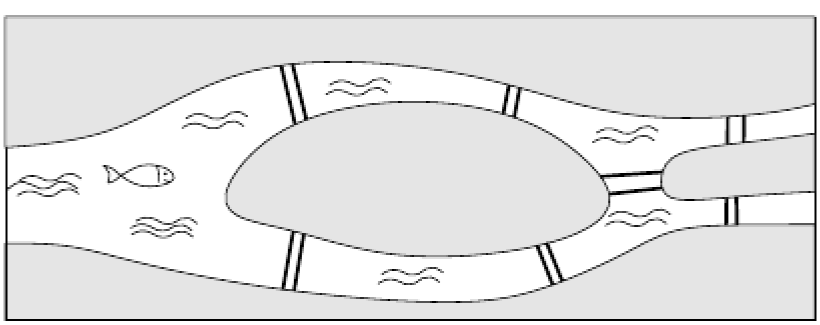
Problem 2. Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

(a) Use a bipartite graph to model the four employees and their qualifications.

(b) Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.

(c) If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

Problem 3. The following sketch of a city plan depicts 7 bridges:



> *Mathematics is the art of giving the same name to different things.*
>
> Henri Poincare

(a) Show that one cannot start walking from some place, cross each of the bridges exactly once, and come back to the starting place (no swimming please). Can one cross each bridge exactly once if it is not required to return to the starting position?

This is a historical motivation for the notion of the Eulerian graphs. The scheme (loosely) corresponds to a part of the city of Königsberg, Královec, Królewiec, or Kaliningrad—that's what it was variously called during its colorful history—and the problem was solved by Euler in 1736.

(b) How many bridges need to be added (and where) so that a closed tour exists?

Problem 4. Can a simple graph exist with 15 vertices each of degree five?

Problem 5. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

Problem 6. For which values of n are these graphs bipartite?
a) K_n b) C_n c) W_n
where K_n, C_n and W_n respectively mean the complete graph, cycle graph, and wheel graph on n vertices.

Problem 7. What is the maximum possible number of edges of a graph with n vertices and k components?

Problem 8. Prove that the complement of a disconnected graph G is connected.
(The complement of a graph $G = (V, E)$ is the graph $(V, \{\{v_1, v_2\} : \{v_1, v_2\} \notin E\})$.)

Problem 9. Prove that a graph is bipartite if and only if it contains no cycle of odd length.

Problem 10. Prove that any graph $G = (V, E)$ having no cycles and satisfying $|V| = |E| + 1$ is a tree.

Problem 11. Prove that a graph on n vertices with c components has at least $n - c$ edges.

Problem 12. Prove that any connected undirected graph G with at least two vertices contains a vertex v such that deleting v from G results in a connected graph (note that the graph need not be a tree).

> *In theory, there is no difference between theory and practice. But, in practice, there is.*
>
> Jan L.A. van de Snepscheut

Problem 13. For a simple undirected graph G, for each vertex v let $f(v)$ be the sum of degrees of its neighbours. Prove that $\sum f(v) = \sum(deg(v)^2)$ where both the sums range over all vertices of G (the RHS is the sum of squares of degrees of the vertices).

Problem 14. For a graph G, let $L(G)$ denote the so-called line graph of G, given by $L(G) = (E, \{\{e, e'\} : e, e' \in E(G), e \cap e' \neq \phi\})$. Decide whether the following is true for every graph G:

(a) G is connected if and only if $L(G)$ is connected.

(b) G is Eulerian if and only if $L(G)$ has a Hamiltonian cycle.
(A Hamiltonian cycle is a cycle which visits each vertex exactly once.)

Bonus Problem 15. Define the "diameter" and "radius" of a graph (in analogy with the intuitive meaning of these notions).

Compiled from various sources, by Ashwani Anand (M.Sc. CS, Ist year) and Ekanshdeep Gupta (B.Sc. Math and CS, IIIrd year)