THE COIN PROBLEM & $AC^{\circ}[\oplus]$

SRIKANTH SRINIVASAN
IITB MATH.

JOINT WITH:

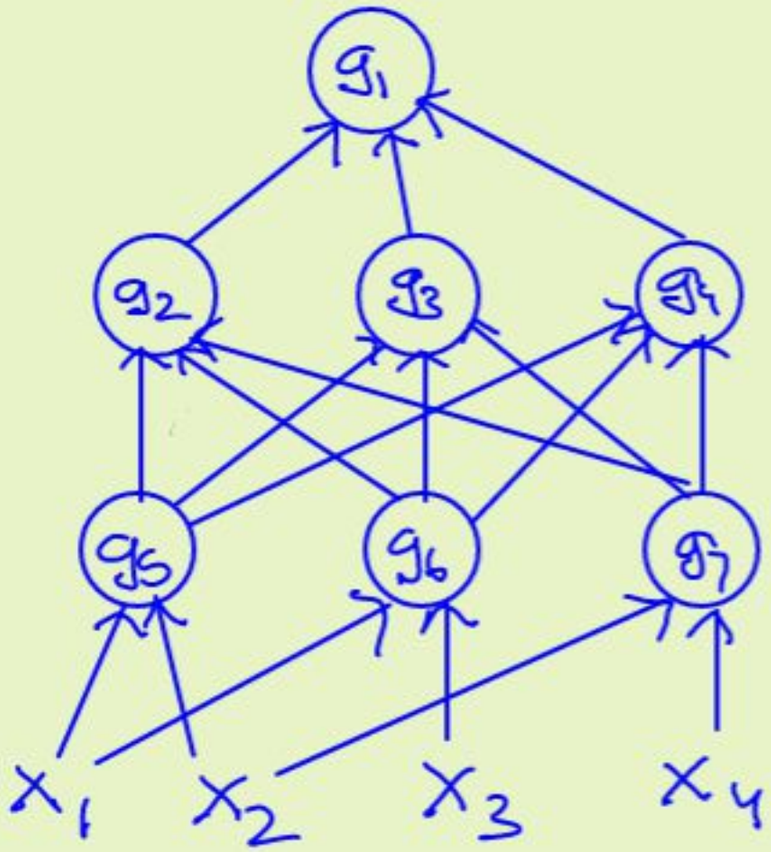
NUTAN LIMAYE (IITB CSE)

KARTEEK SREENIVASAIAH (IITH CSE)

UTKARSH TRIPATHI (IITB MATH)

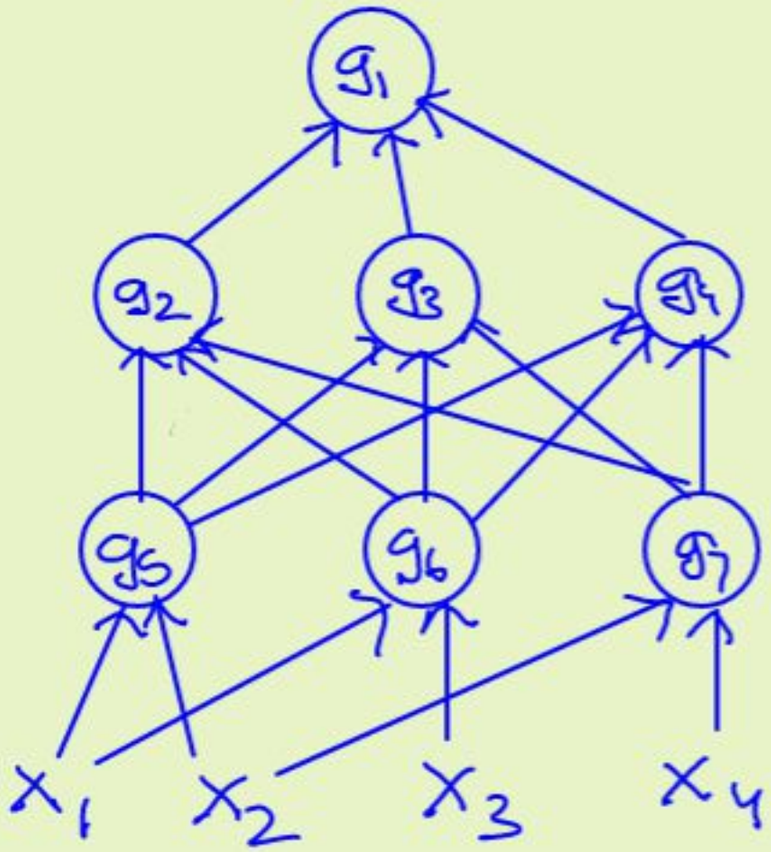
S VENKITESH (IITB MATH)

Boolean Circuits



→ Directed Acyclic Graph

Boolean Circuits



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↪ $g_i \in$ "Simple Ops"

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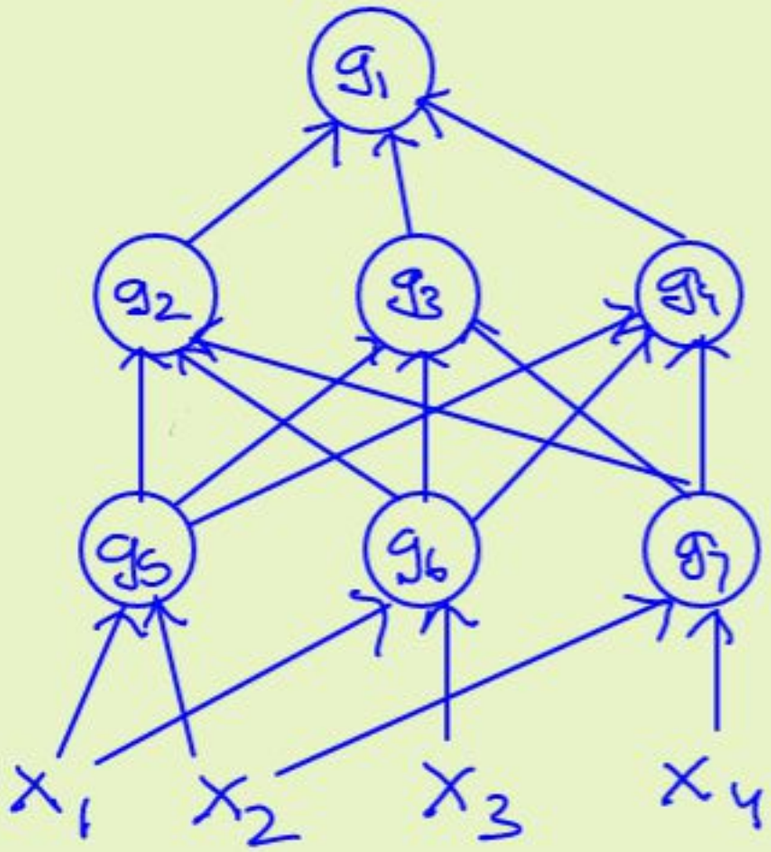

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$f: \{0,1\}^n \to \{0,1\}$

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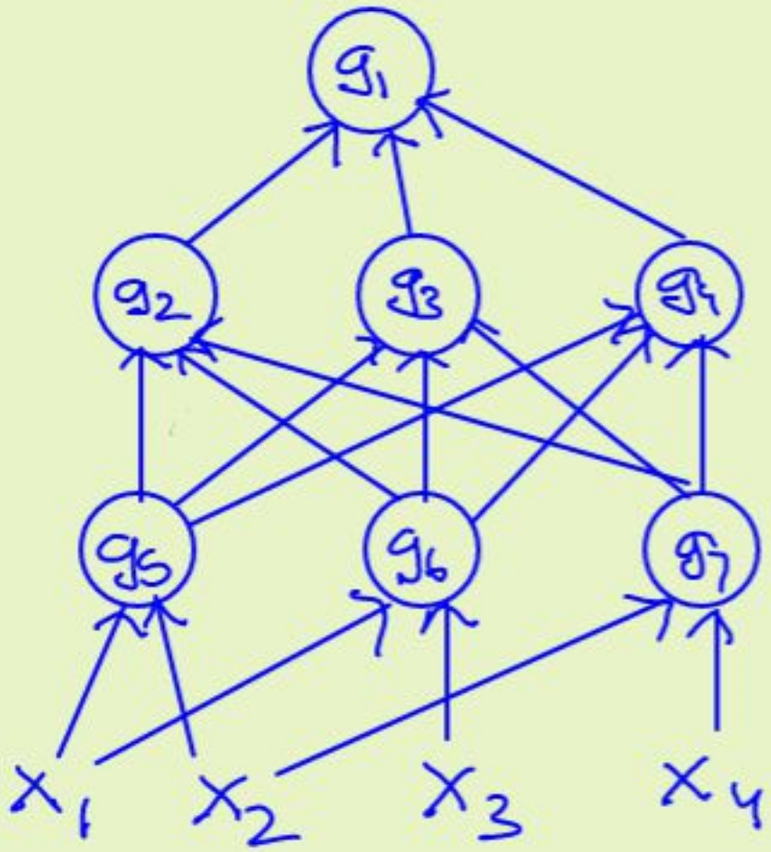

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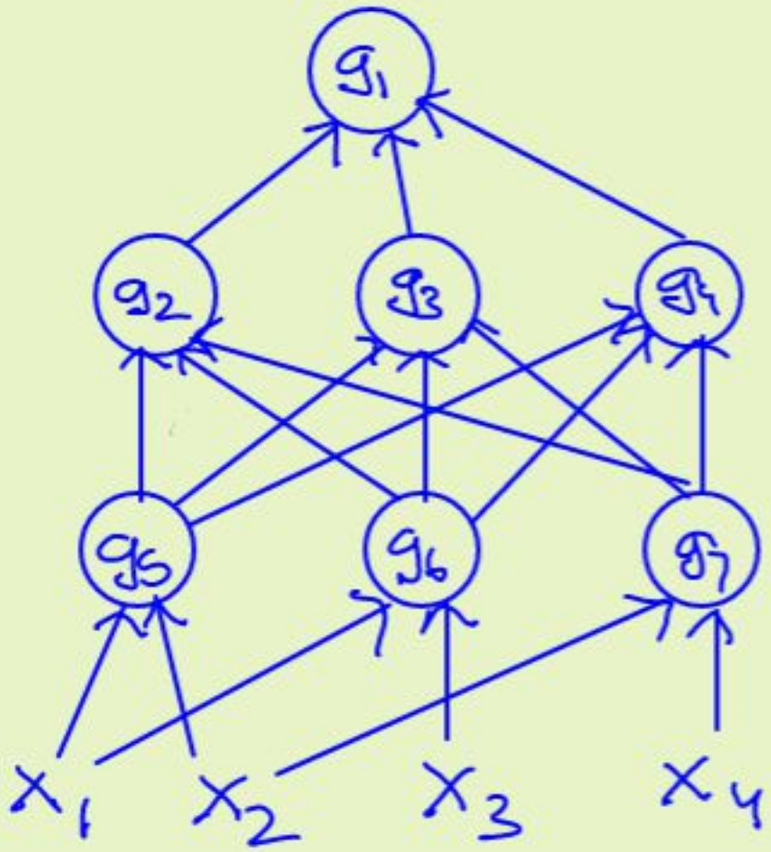

→ Directed Acyclic Graph

⟶ $g_i \in$ "Simple Ops"

⟶ Computes $f: \{0,1\}^n \to \{0,1\}$

⟶ Formula : Tree

Size = # of ops = 4

Depth = length of longest path = 3

The Big Questions

→ $\mathcal{E}$ - a class of ckts

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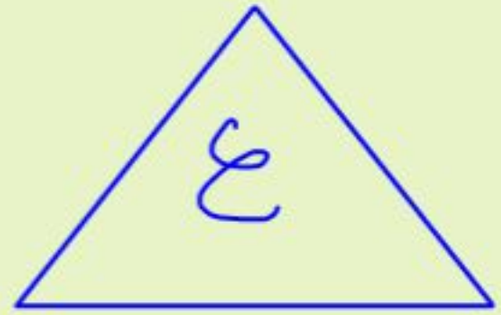
→ $\mathcal{C}$ - a class of ckts

→ Lower bounds



Explicit $f: \{0,1\}^n \to \{0,1\} \notin \mathcal{C}$

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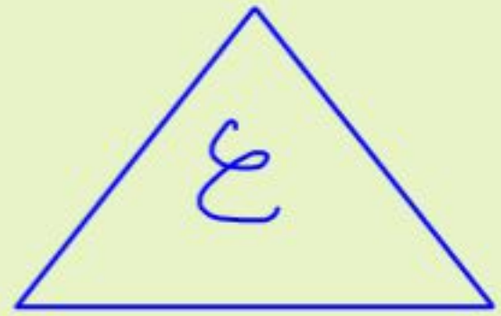

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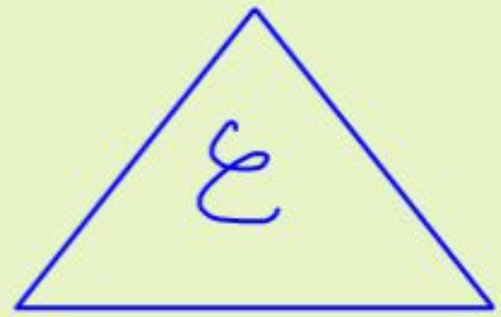

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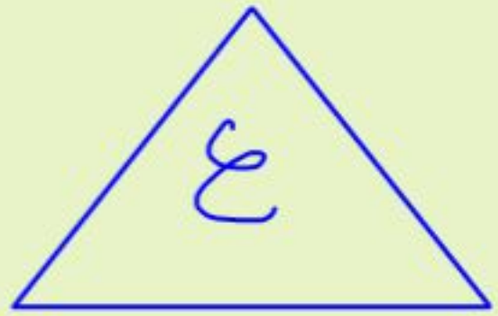

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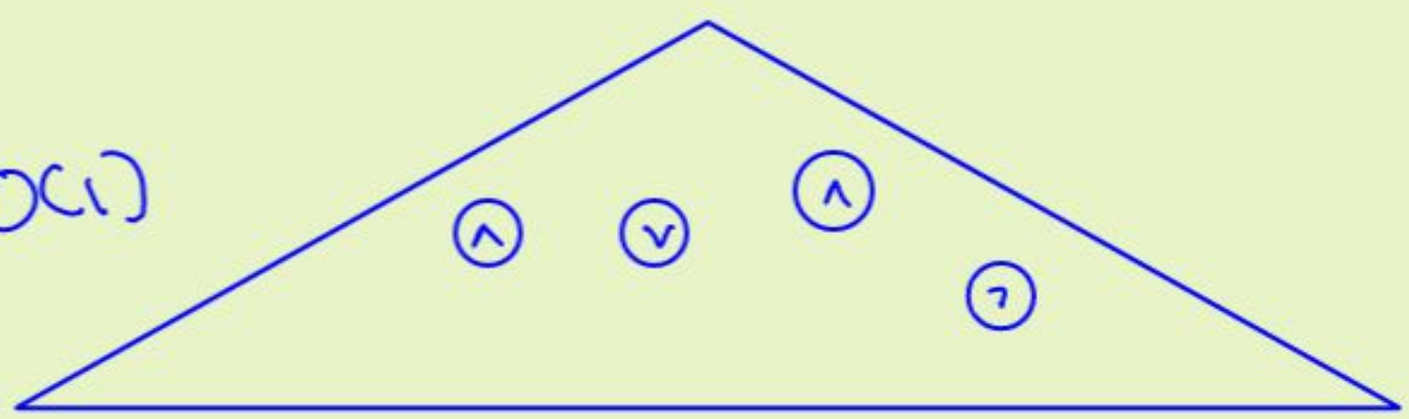
→ Hierarchy theorems

AC^0 & $AC^0[\oplus]$

AC^0:

$\wedge, \vee, \neg$ gates

$d = O(1)$



$AC^0[\oplus]$:

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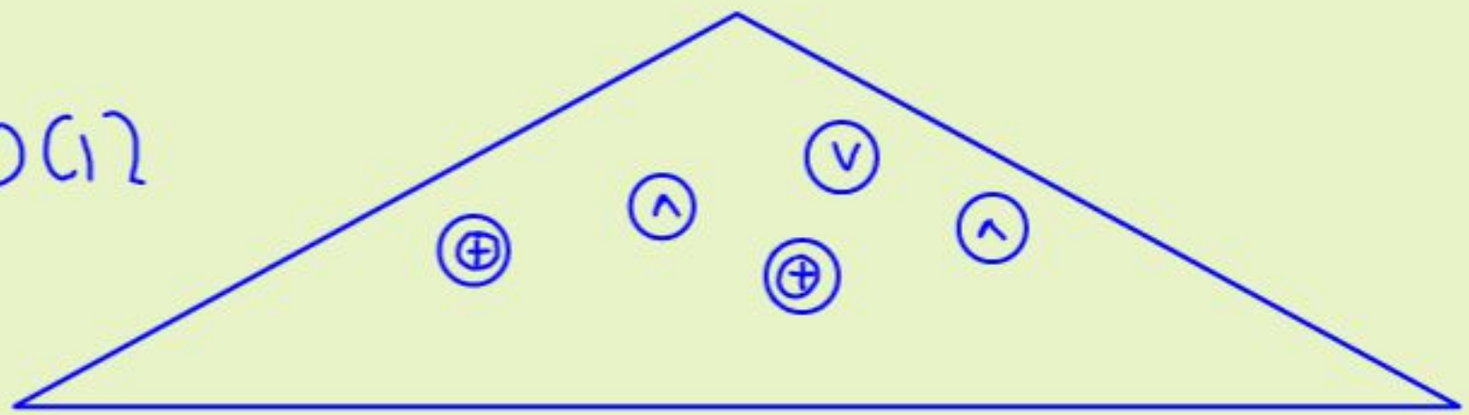

AC^0 & $AC^0[\oplus]$

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Strong PRGs [N]	Weak PRGs [FSUV]
⋮	⋮

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→ Explicit separations?

AC^0

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$$AC^0_3\text{-SIZE}[s] \subsetneq AC^0_3\text{-SIZE}[s^{O(1)}]$$

if $s < \exp(o(\sqrt{n}))$

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Fixed-Depth Size Hierarchy thm

Thm [Rossman; Amano] $\forall s = n^{O(1)}$

$AC^0\text{-SIZE}[s] \subsetneq AC^0\text{-SIZE}[s^{1+\varepsilon}]$

$\exists F \in AC^0_2\text{-SIZE}[n^k]$ s.t.

$\forall d \quad F \notin AC^0_d\text{-SIZE}[n^{k-\varepsilon}]$

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$AC^0_3[\oplus]\text{-SIZE}[s] \subsetneq AC^0_3[\oplus]\text{-SIZE}[s^{\log s}]$

Quasipoly. Size Hierarchy theorem

Our result

Thm: $\forall s \leq \exp(n^{o(1)})$, $d = O(1)$

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explicitly.

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Explicit hard functions chosen to solve the Coin Problem.

δ-Coin Problem

Coin with bias $\frac{1}{2}+\delta$ or $\frac{1}{2}-\delta$.

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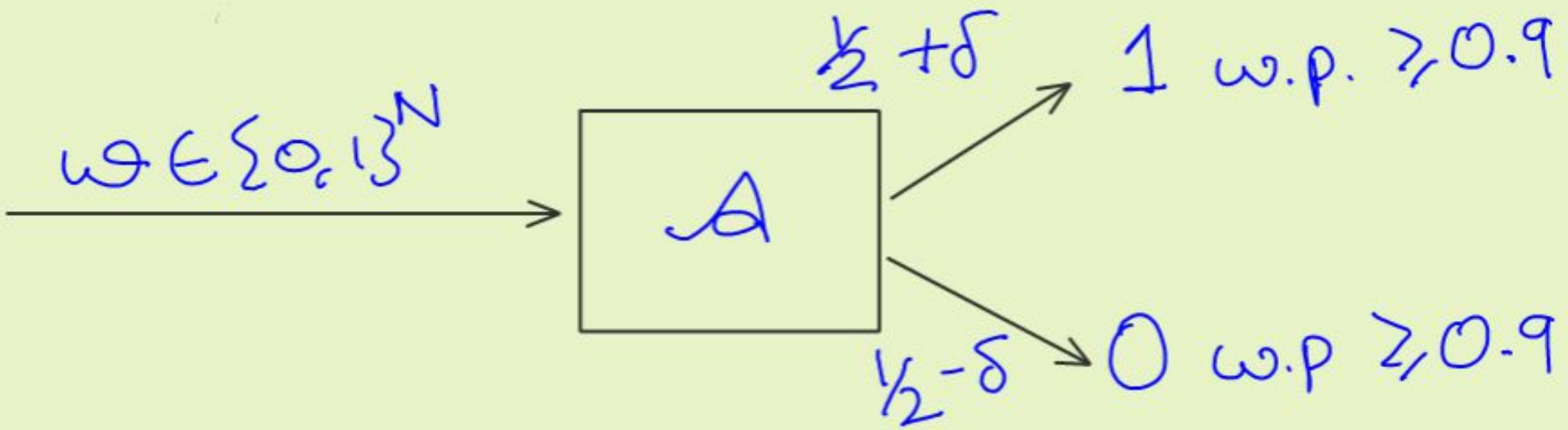

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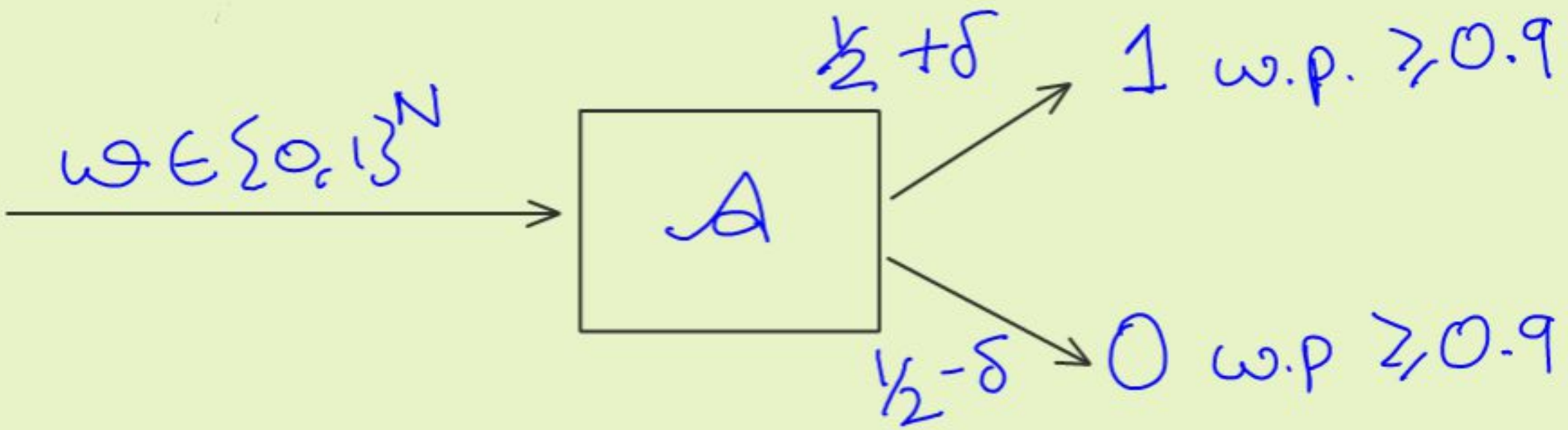

$N =:$ Sample complexity of $\mathcal{A}$

Solving the δ-Coin Problem

→ Natural problem. Closely related to Majority.

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→ Circuit complexity?

[ABO, Val., Bop., SV, OW, Ama., Aar., BV, CGR, LV.]

ABO: 80's

Aar.: formal defn. [2010]

LV.: 2017.

$AC^0[\oplus]$ & δ- Coin Problem

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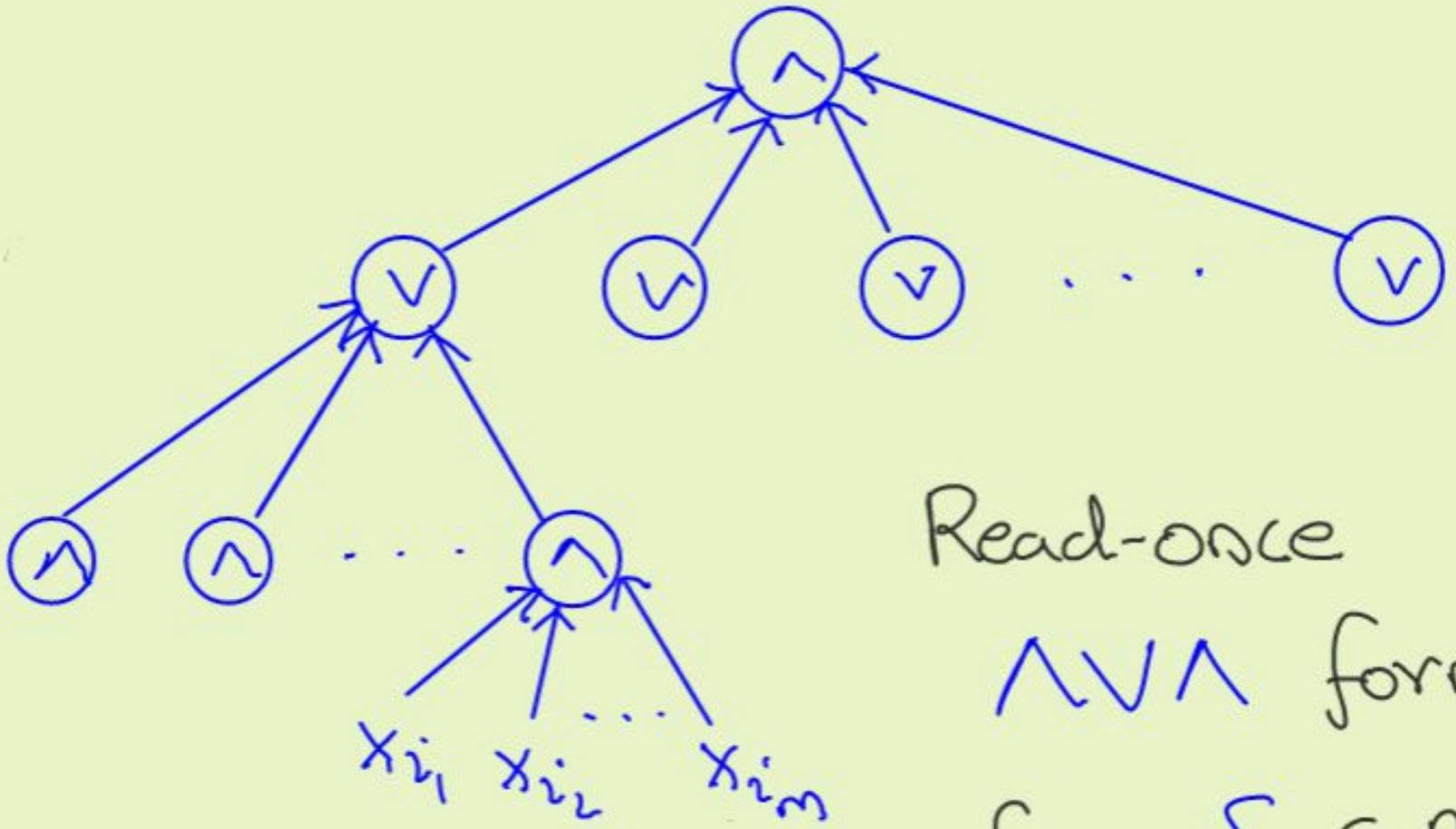
→ [O'Donnell-Wimmer, Amano]:

AC^0_3 ckt. of size $\exp(\sqrt{1/\delta})$.

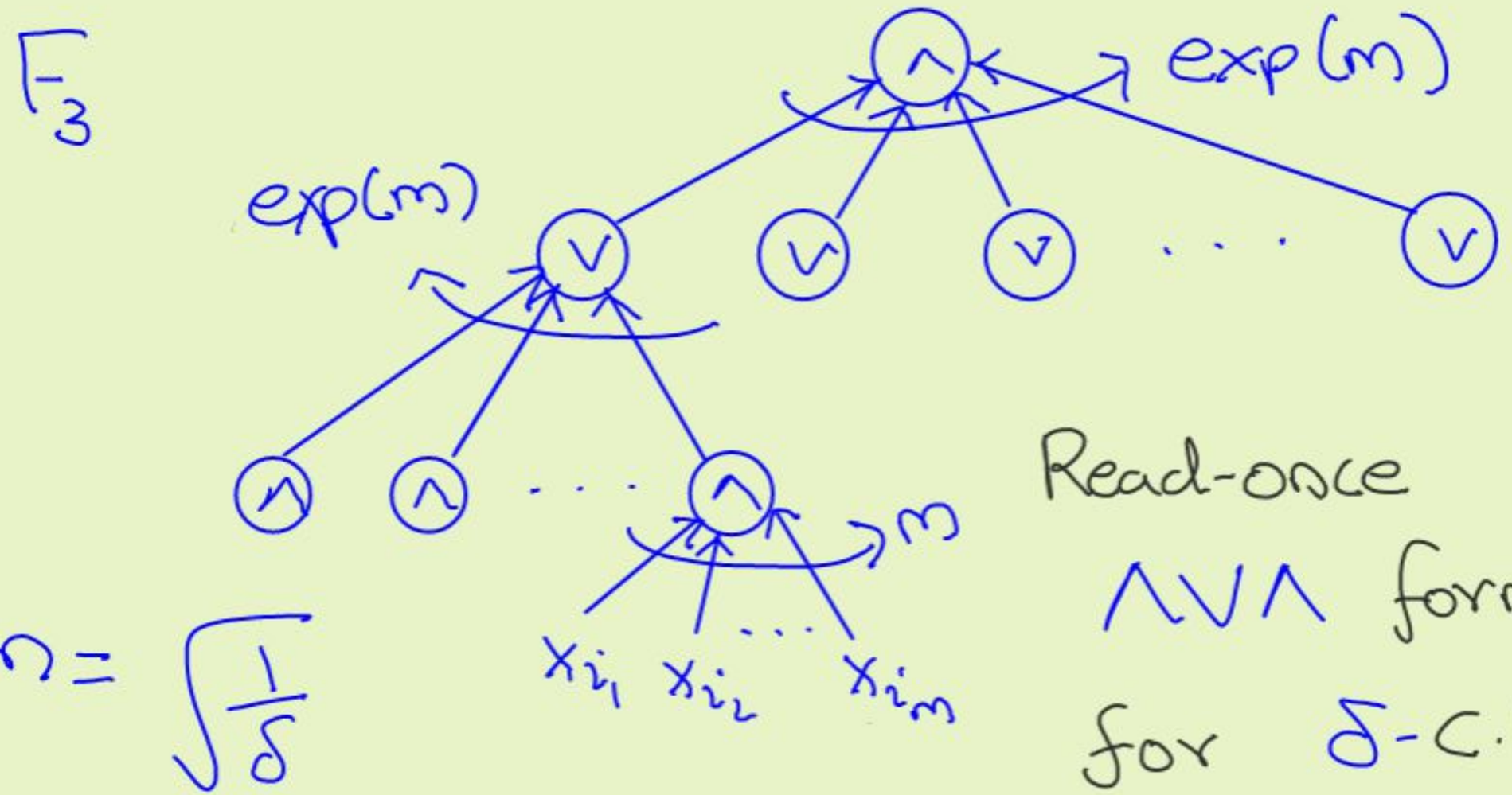
The O'Donnell-Wimmer-Amano construction

F_3



Read-once $\wedge\vee\wedge$ formula for δ-C.P.

The O'Donnell-Wimmer-Amano construction



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Back to Size hierarchy thms

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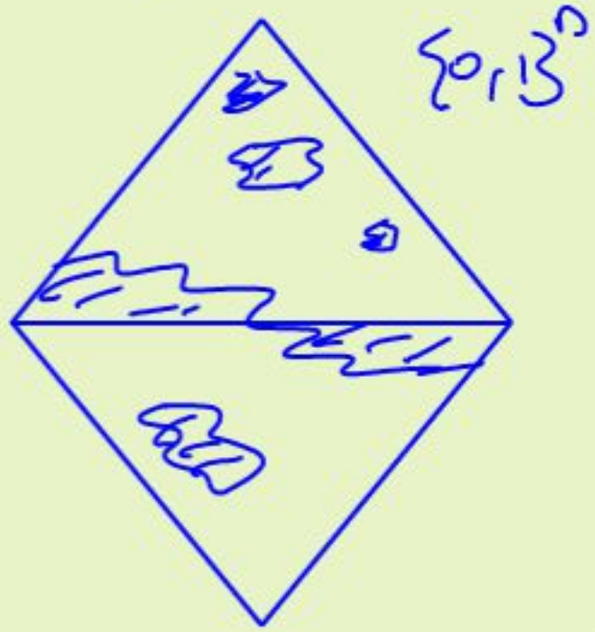
$$f: \{0,1\}^n \to \{0,1\}$$

$$\Pr_x [f(x) = Maj_n(x)] \geq 0.9$$



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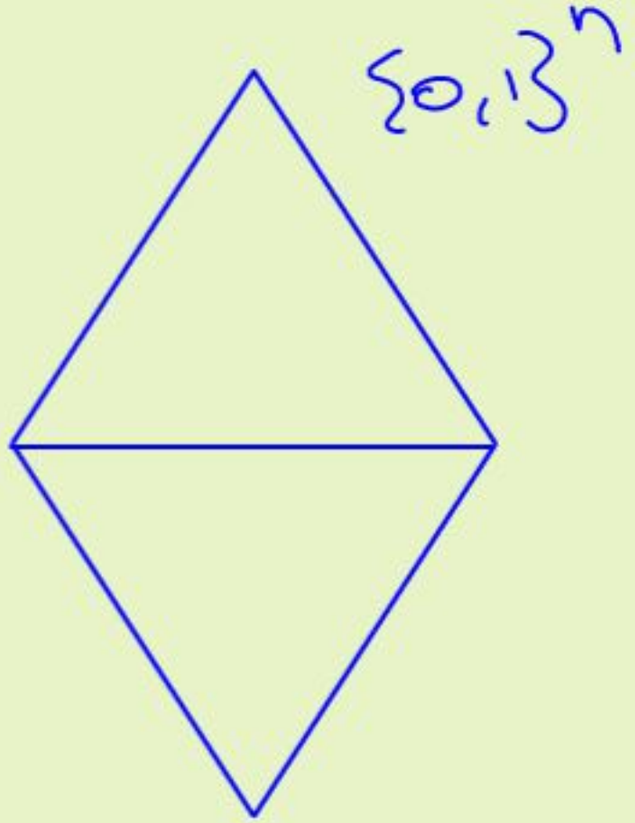
Constructing AM's in AC^0

Want: AM $f: \{0,1\}^n \to \{0,1\}$



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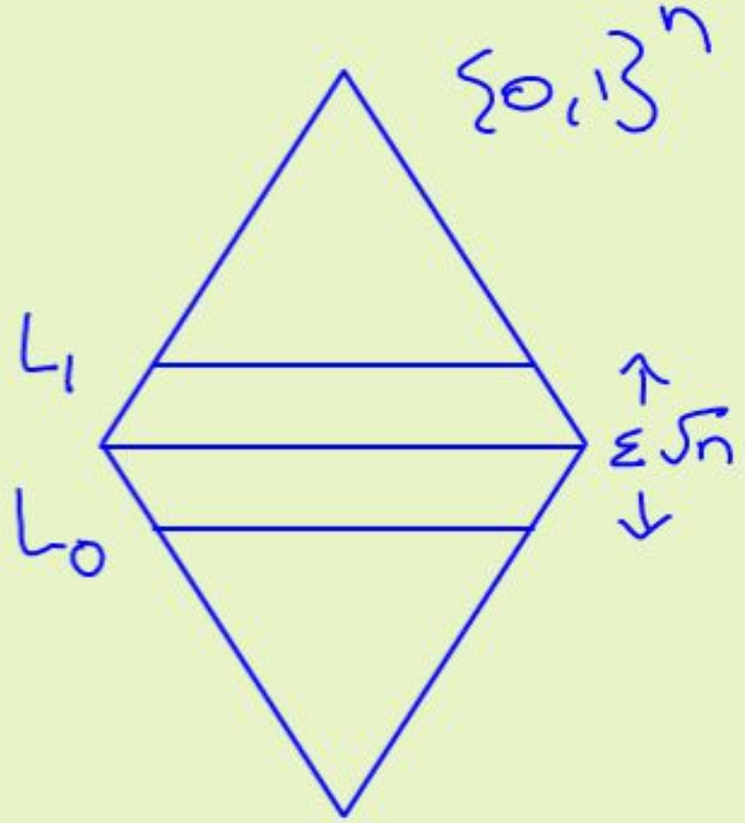
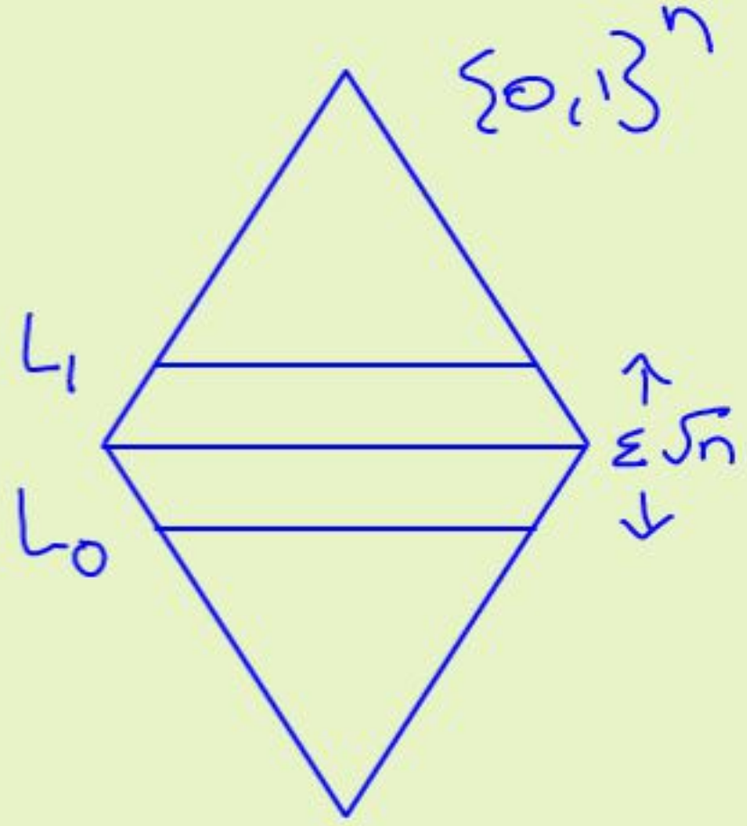


Constructing AM's in AC^0

Want: AM $f: \{0,1\}^n \to \{0,1\}$

$x \in L_1$: $\Pr_{i \in [n]}[x_i = 1] = \frac{1}{2} + \frac{\varepsilon}{\sqrt{n}}$

$x \in L_0$: $\Pr_{i \in [n]}[x_i = 1] = \frac{1}{2} - \frac{\varepsilon}{\sqrt{n}}$



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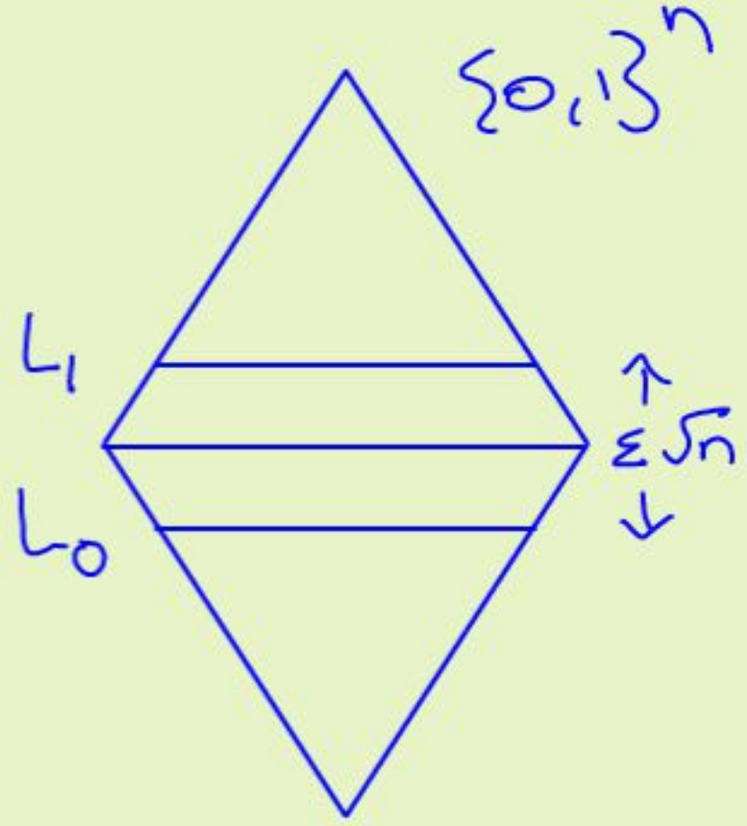

Obs: $C: \{0,1\}^N \to \{0,1\}$ solves $\delta = \varepsilon/\sqrt{n}$ - C.P.

$\Rightarrow$ $\Pr_{i_1,\ldots,i_N \in [n]}[C(x_{i_1},\ldots,x_{i_N}) \neq Maj_n(x)] < 0.1.$

Constructing AM's in AC^0

$C(y_1, \ldots, y_N)$ solves δ-C.P. size $\exp\left(\frac{1}{\sqrt{\delta}}\right)$

$\Downarrow$ $\delta \approx 1/\sqrt{n}$

$C(x_{i_1}, \ldots, x_{i_N})$ computes AM size $\exp(n^{1/4})$

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$\rightarrow$ Not explicit.

Formulas solving the coin problem

→ Computes

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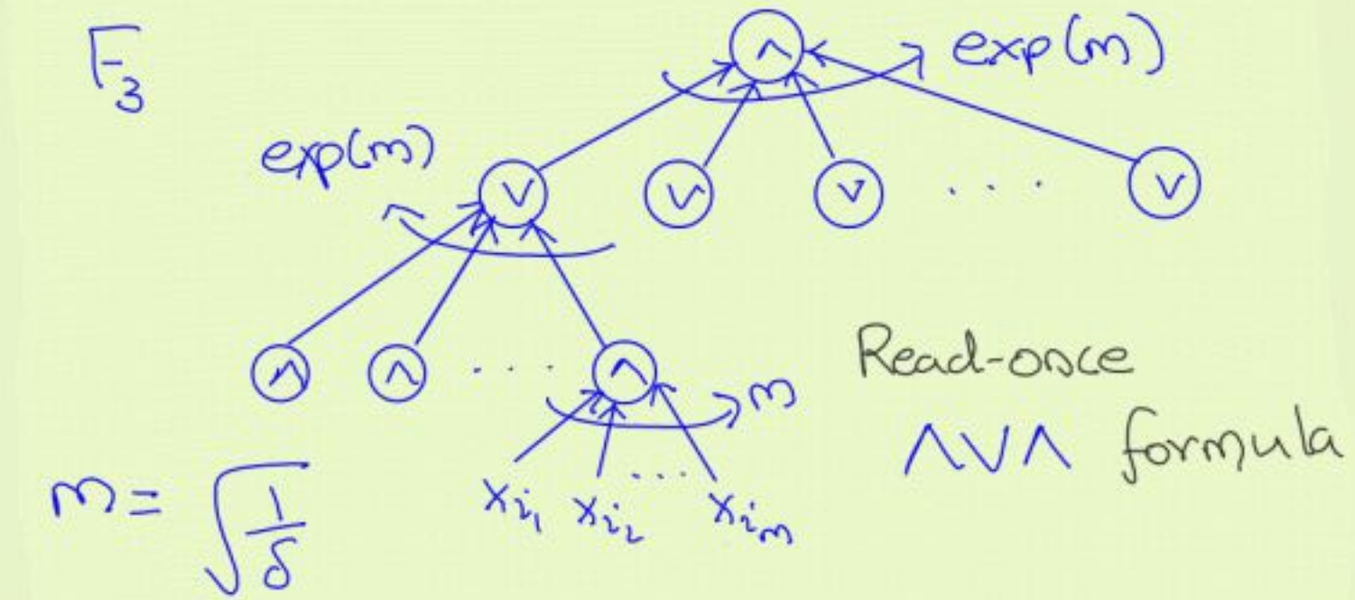

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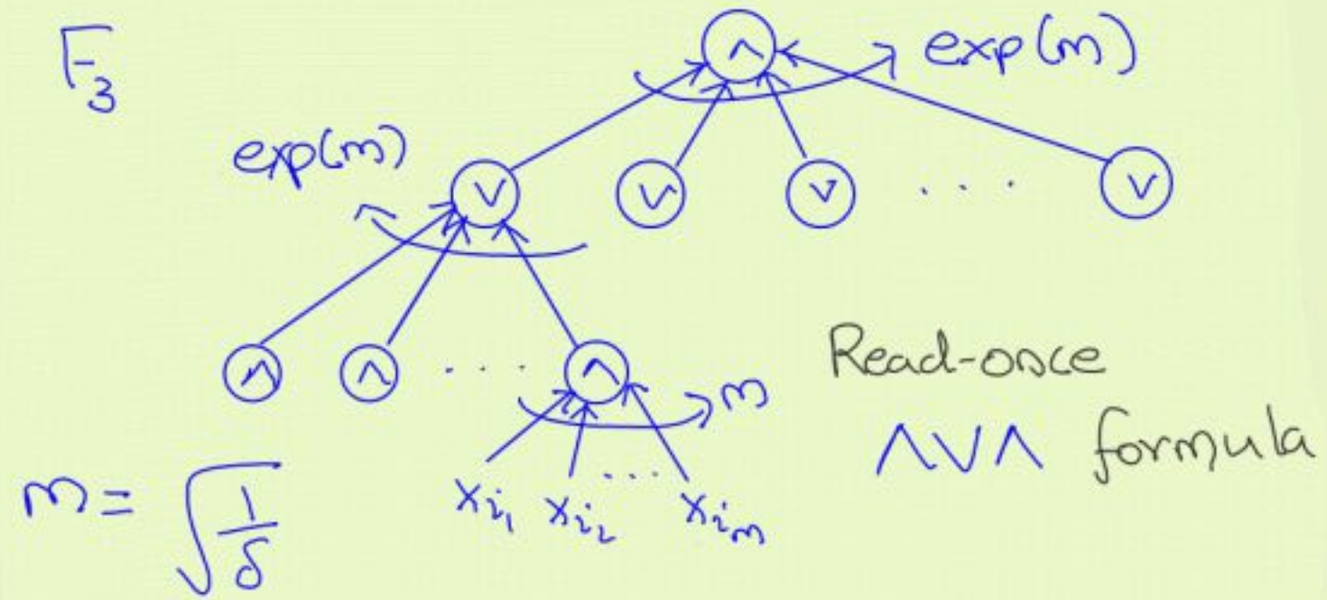
→ Size-optimal!



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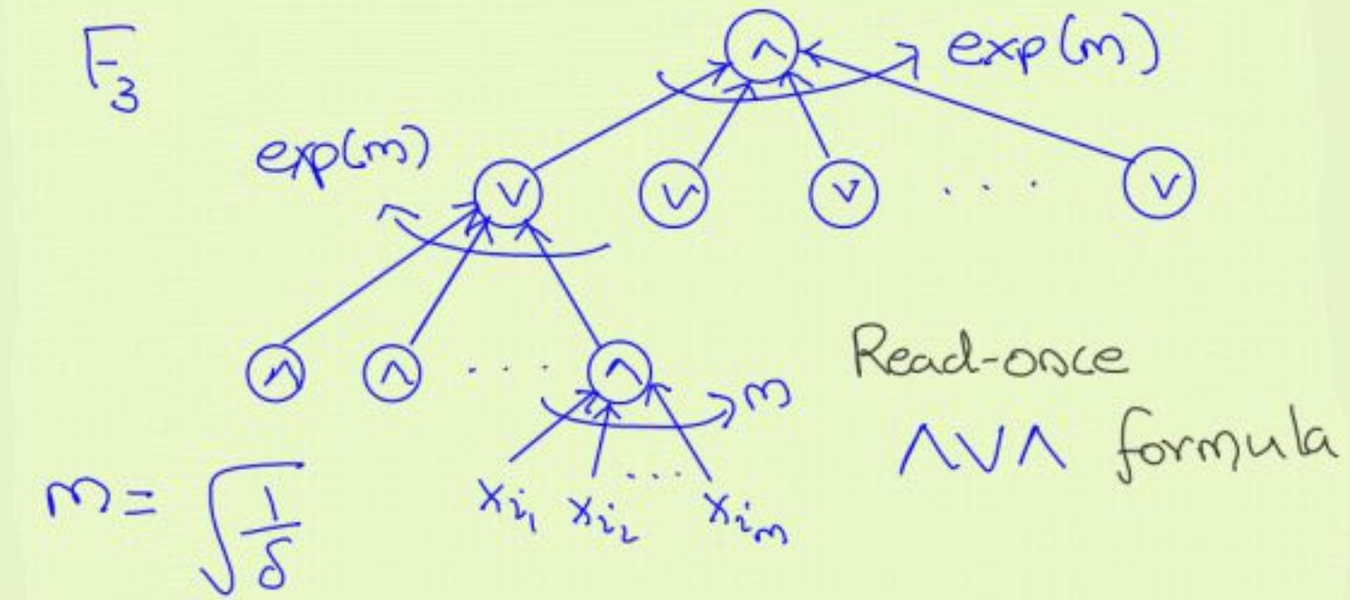

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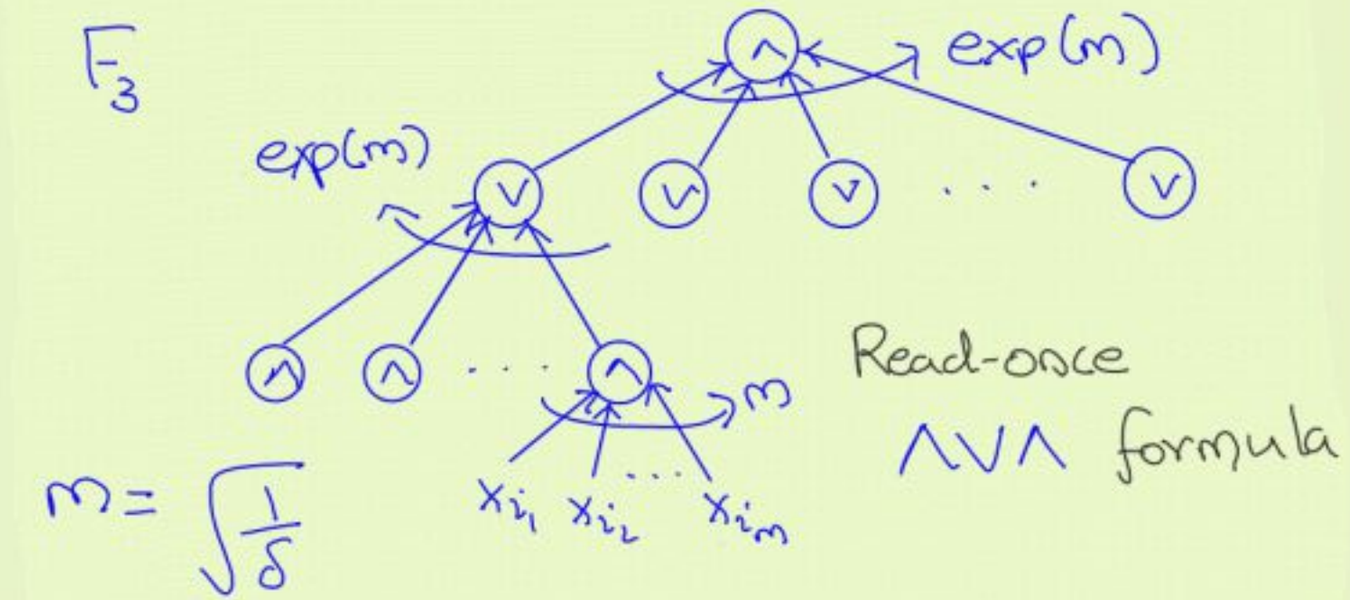

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→ Idea: Reduce N while fixing size.

Main Result

Thm 1: δ-C.P. solved by explicit $\wedge\vee\wedge$ formulas of size $\exp(1/\delta)$ & sample complexity $\text{poly}(1/\delta)$.

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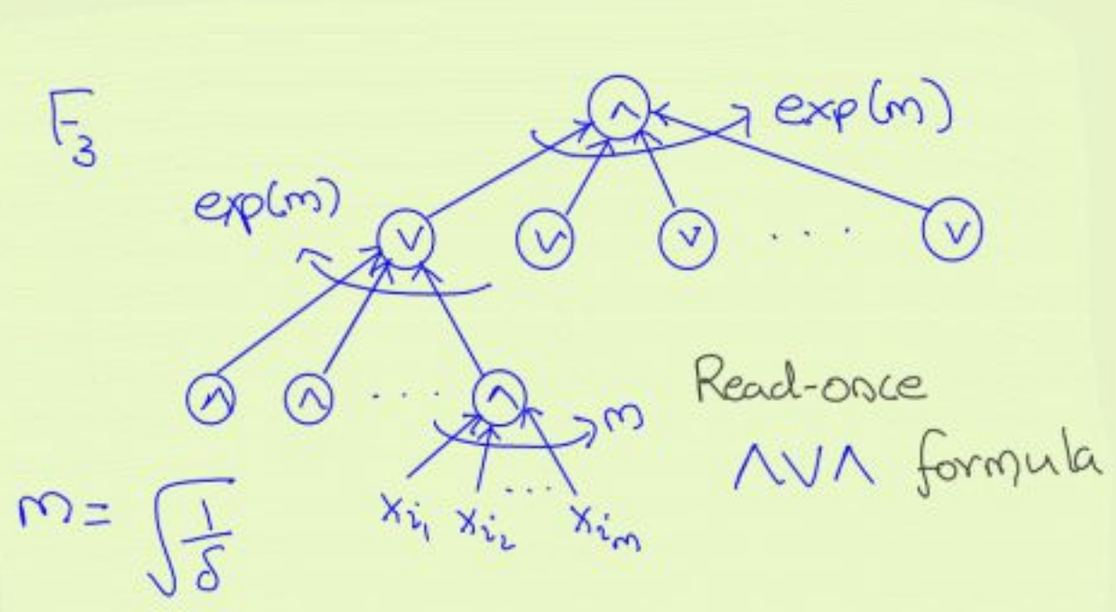
Thm 2: Any $AC^0_3[\oplus]$ ckt. for δ-C.P. must have size $\exp(1/s\delta)$.

$\Updownarrow$

Level-1 Fourier coeff. bound of CHLT'18

Proof of Thm 1

$$P_i^{(1)} := \Pr_{(1/2+\delta)}[\text{ht. } i \text{ formula accepts}]$$



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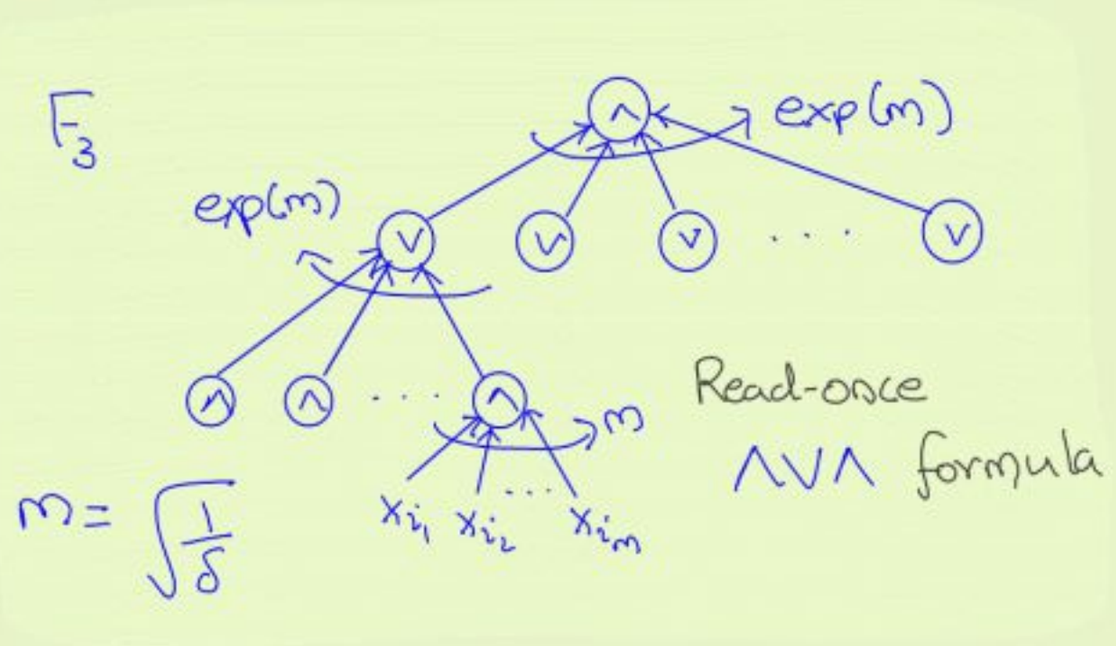
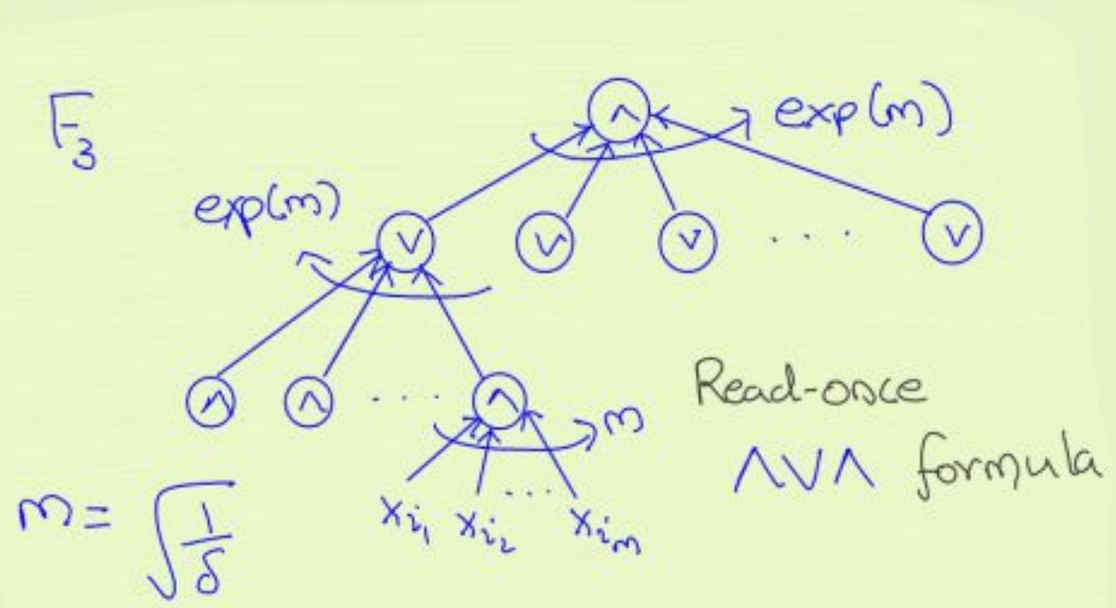
$$P_0^{(1)} = \frac{1}{2}+\delta$$

$$P_1^{(1)} = \left(\frac{1}{2}+\delta\right)^m \approx \frac{1}{2^m}(1+\delta m)$$



Proof of Thm 1

$$P_i^{(1)} := \Pr_{(\frac{1}{2}+\delta)}[\text{ht. } i \text{ formula accepts}]$$



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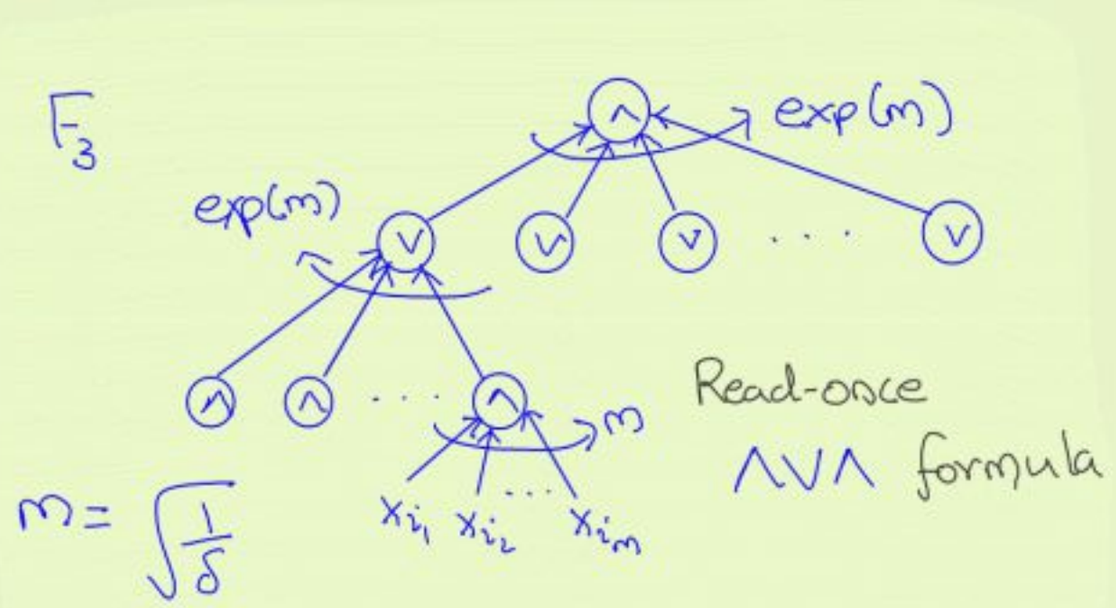

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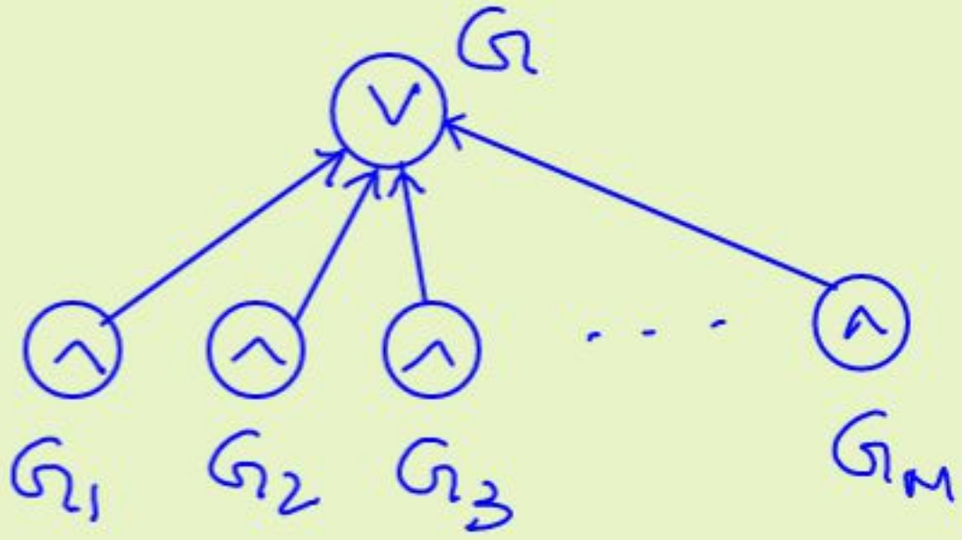
Key quantity: $\dfrac{\min\{p_i^{(1)}, q_i^{(1)}\}}{\min\{p_i^{(0)}, q_i^{(0)}\}}$

Janson's Inequality

G - monotone formula



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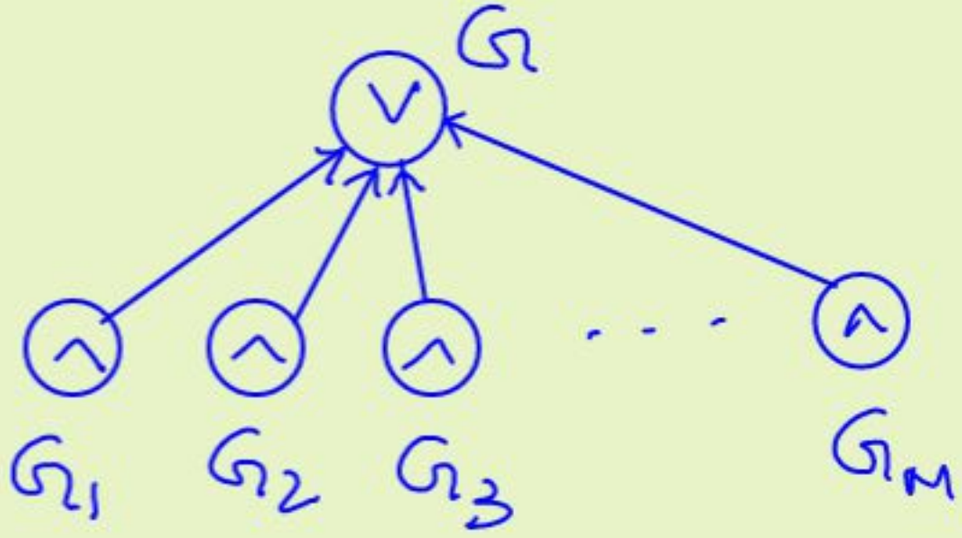

$$\prod_i \Pr[G_i = 0] \leq \Pr[G = 0]$$

Janson's Inequality

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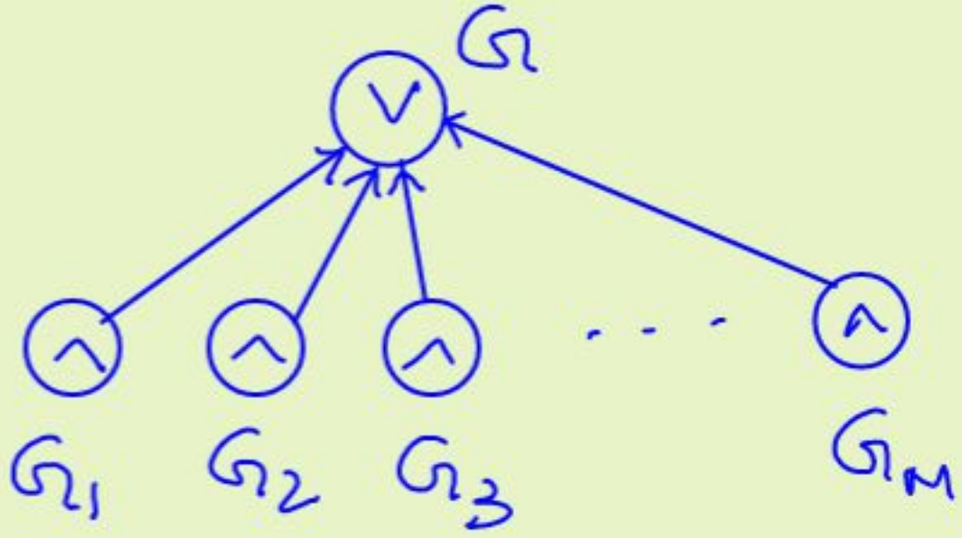

$$\prod_i \Pr[G_i = 0] \le \Pr[G = 0] \le \prod_i \Pr[G_i = 0] \cdot (1+\alpha)$$

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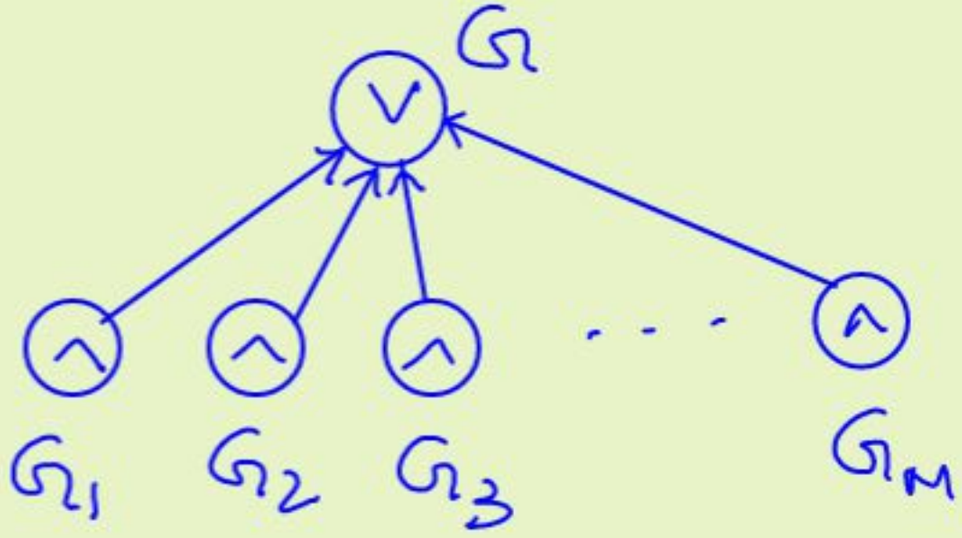

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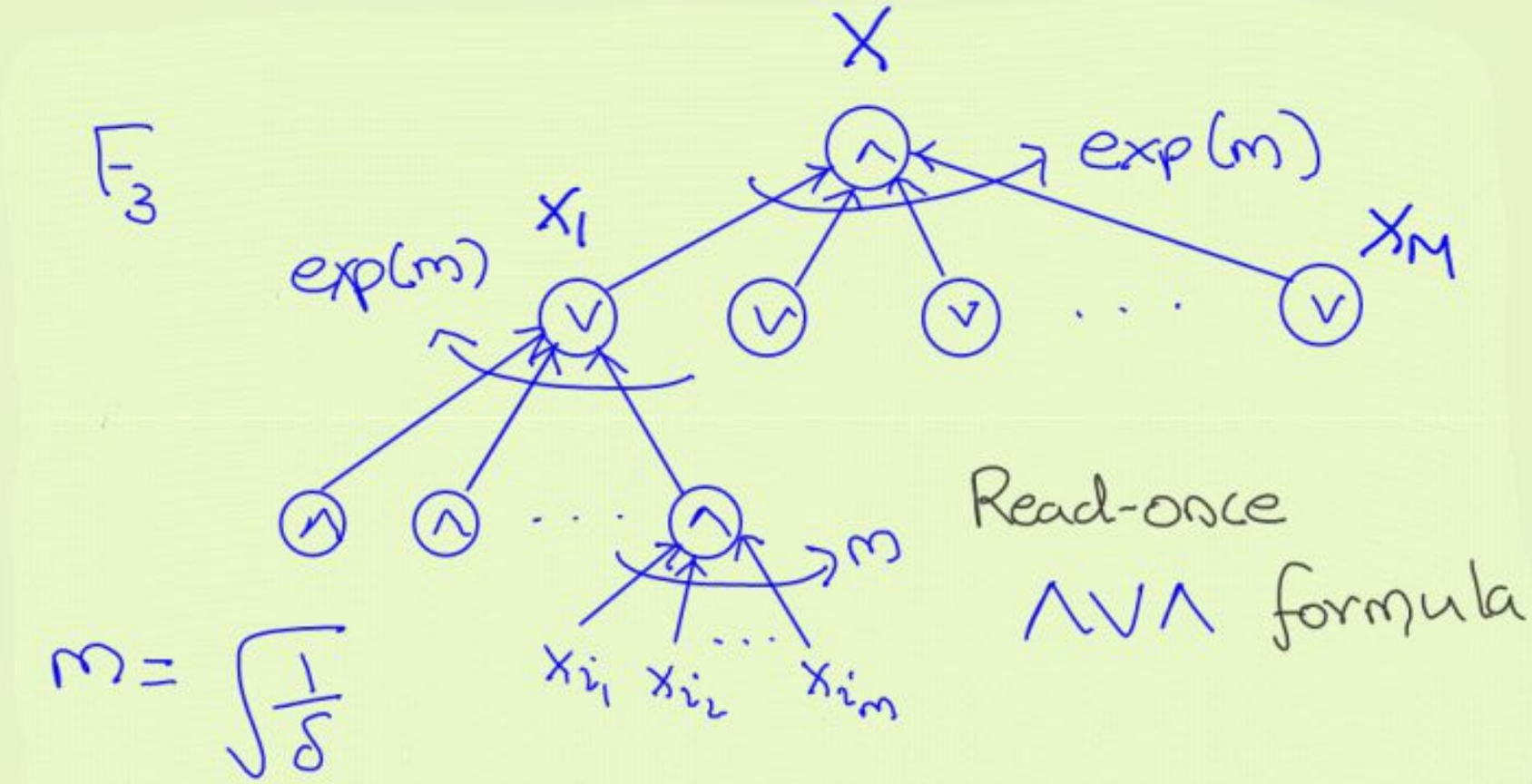
↓ Janson

α small if most pairs G_i, G_j variable disjoint

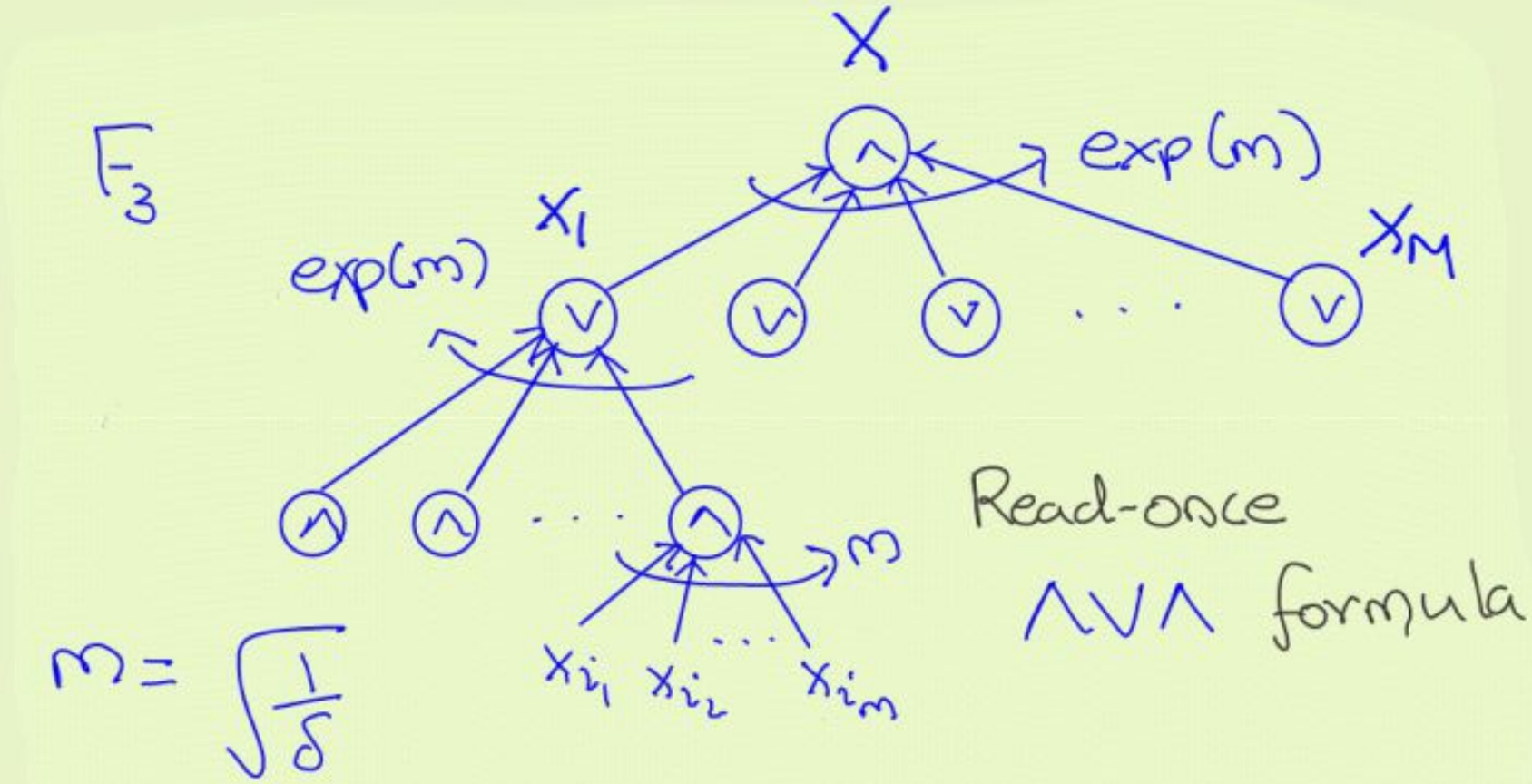
Formula construction



Read-once

$\wedge\vee\wedge$ formula

Formula construction



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$X \longmapsto X_1 \cup \dots \cup X_M$: most $X_i \cap X_j = \emptyset$

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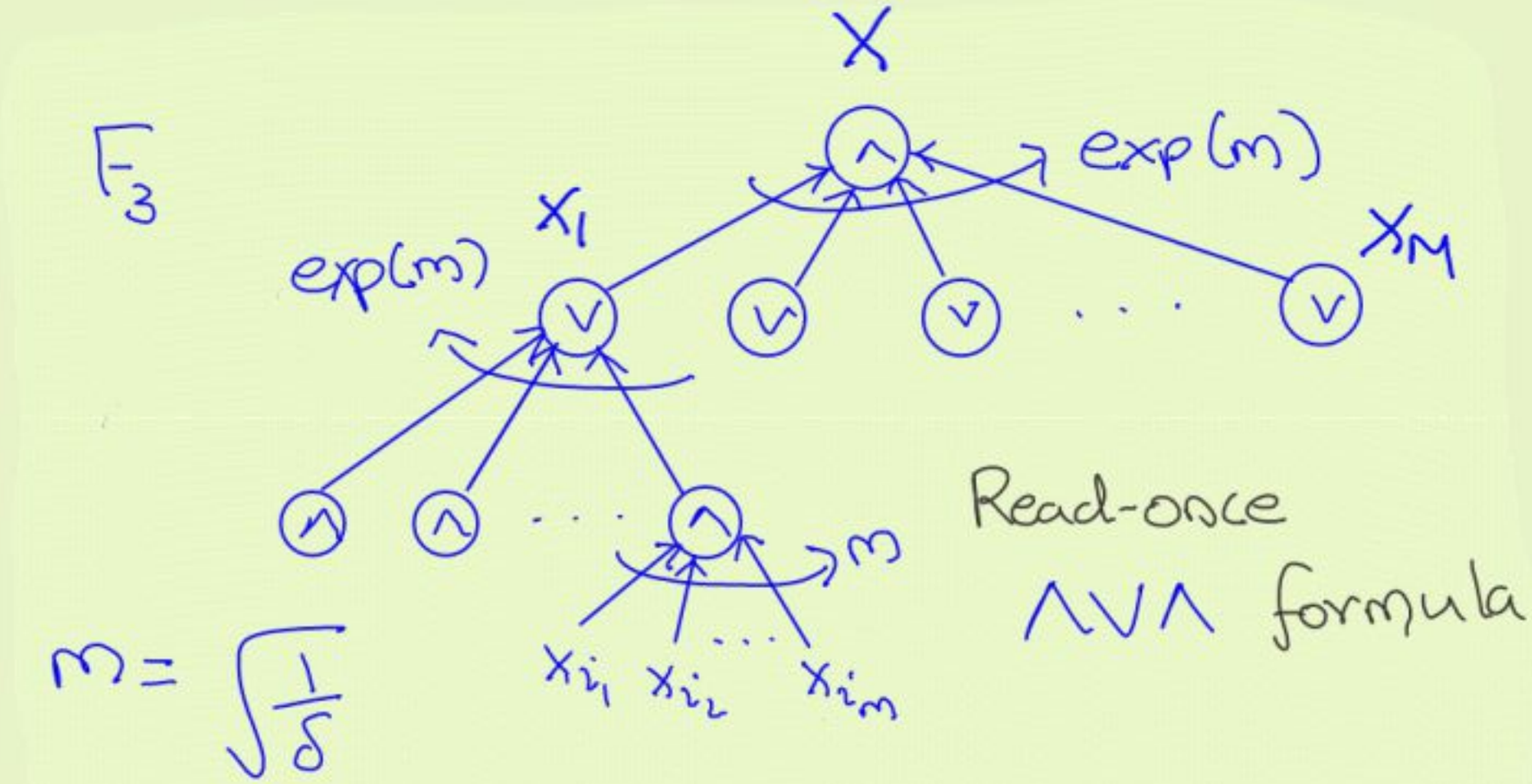

Read-once

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$X \longmapsto X_1 \vee \cdots \vee X_M$: most $X_i \cap X_j = \emptyset$

Use an NW-design!

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$\rightarrow \mathcal{F} \subseteq \binom{[N_1]}{N_2}$

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→ Standard designs are better:

Most $X \cap Y = \emptyset$.

→ $X = [N_1]$

$\mathcal{F} = \{X_1, \ldots, X_M\}$

Questions

1. Tight $AC^0[\oplus]$ lower bounds for MOD_3 or MAJ ?

2. Optimal sample complexity $1/\delta^2$ for δ-C.P. ?

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Thanks !