A #SAT Algorithm for Small Constant-Depth Circuits with PTF gates.

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Joint work with

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Boolean circuits

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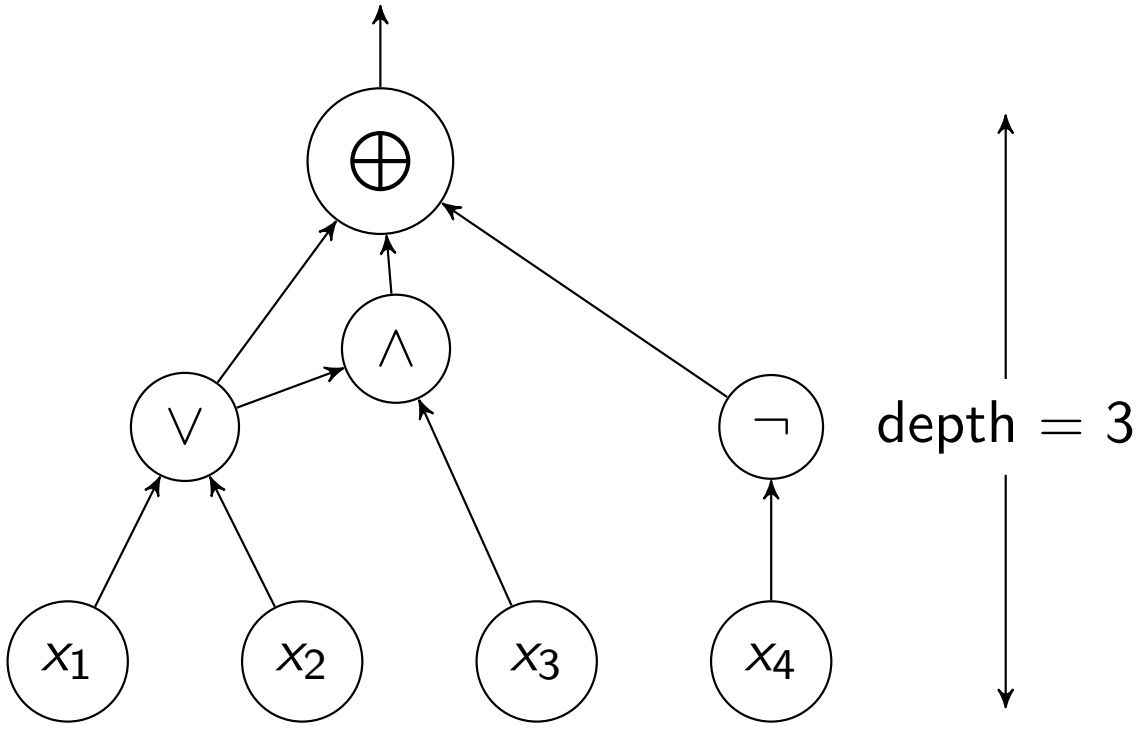

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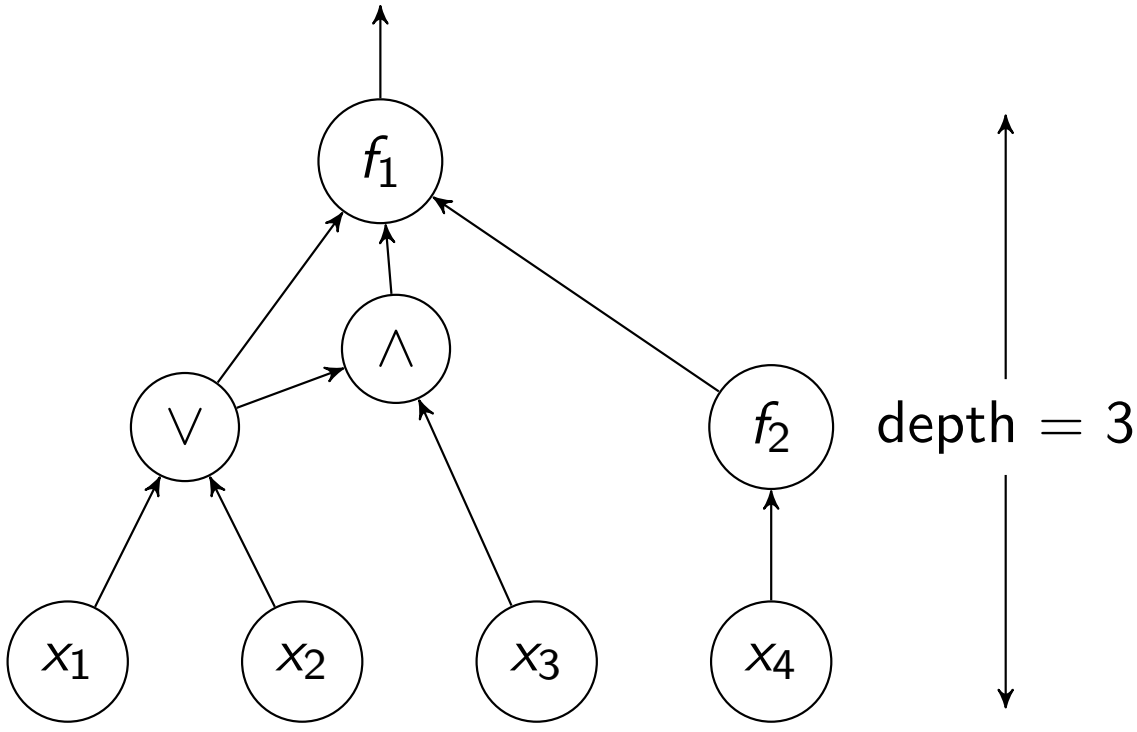

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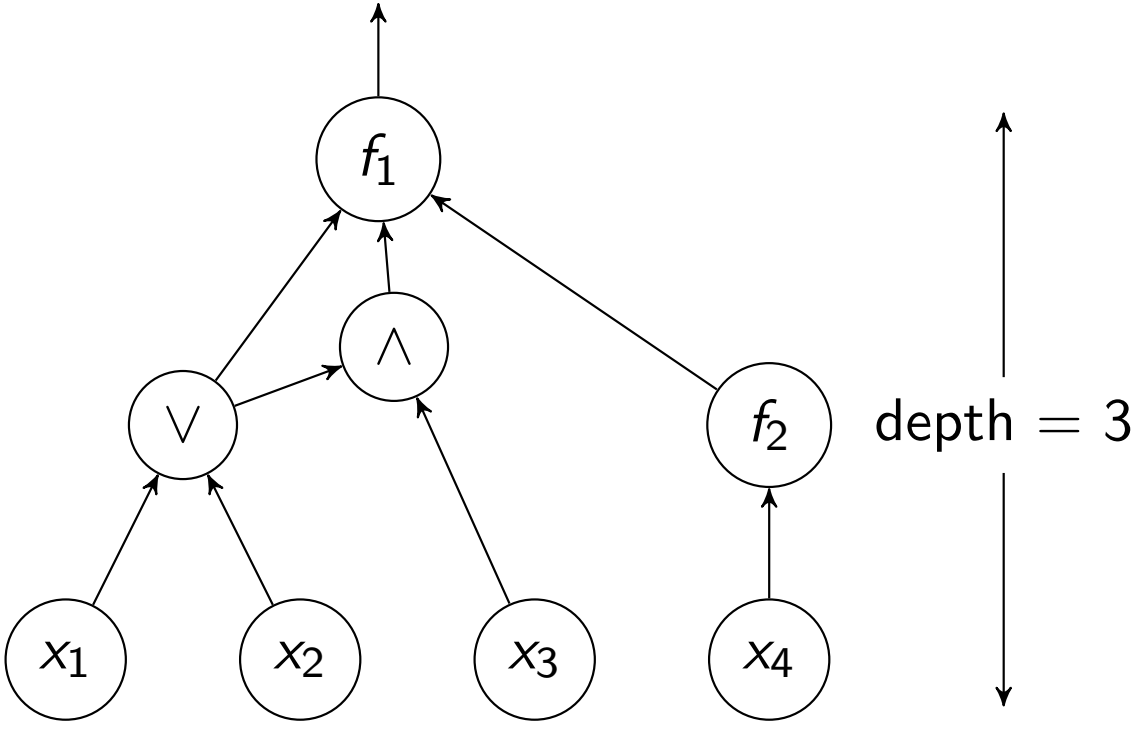

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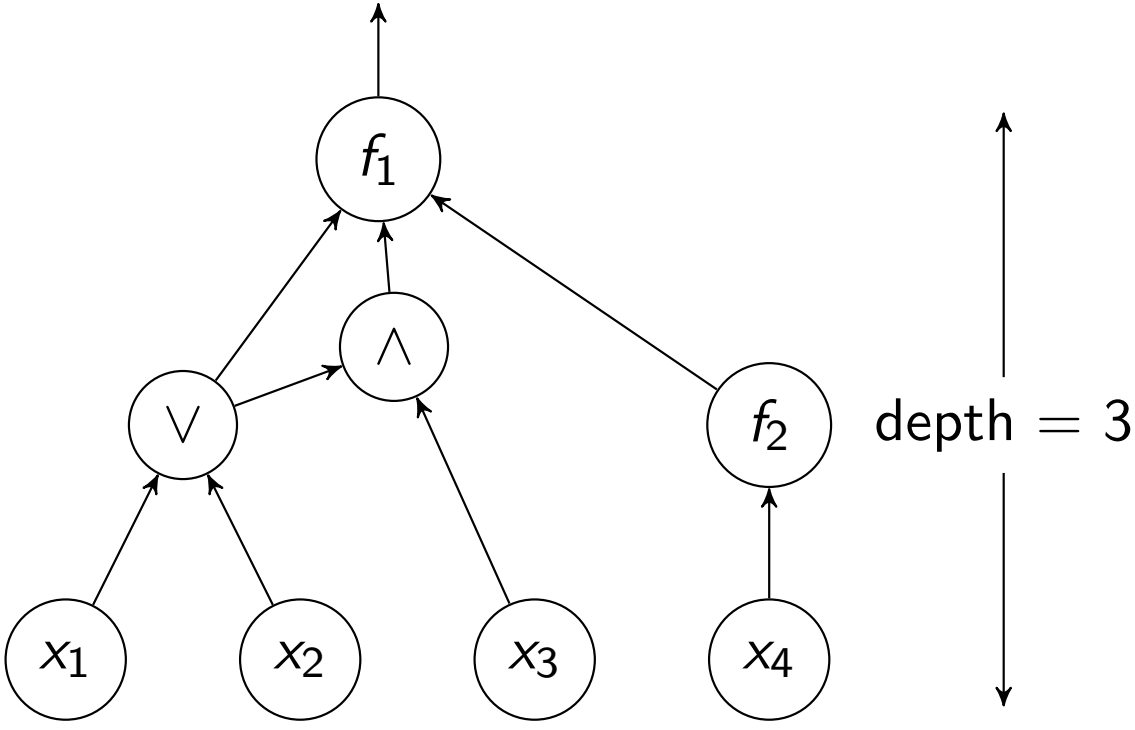

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Constant-depth circuits: d is independent of n.

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Can we design an algorithm that takes time $2^n/n^{\omega(1)}$ when $|C|$ is small, say $\mathrm{poly}(n)$?

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This intuitive connection has been formalised to derive lowerbounds for various interesting classes of circuits.

[Paturi, Pudlák, Zane 1997],[Paturi, Pudlák, Saks, Zane 2005], [Williams 2010], [Williams 2011].

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Polynomial Threshold circuits are a natural generalization of TC^0.

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For constant depth PTF circuits of size $n^{1+\epsilon_d}$, where d depends on the depth of the circuit, and sparsity $n^{2-\Omega(1)}$.

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Theorem (#SAT for a single k-PTF)

*Fix any constant k, there is a zero-error radomized algorithm that solves the #SAT problem for a single k-*PTF *in time* $\mathrm{poly}(n, M) \cdot 2^{n-s}$, *where* $s = \tilde{\Omega}(n^{1/k+1})$.

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Some comments on zero-error randomized algorithms.

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Look-up $\mathcal{T}$ and figure out whether it is satisfiable.

If look-up can be done in $\mathrm{poly}(|C|)$ time, then this step takes time $O(2^{n-m} \cdot \mathrm{poly}(|C|)) \ll 2^n$.

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Quick Look-up?

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Expensive to compute

Even for LTFs computing Chow parameters is known to be NP-hard. [O'Donnell, Servedio 2011].

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Our approach:

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Using Linear Descision Tree

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The tree can be constructed in time $2^{O(\Delta)} = \exp(m^{k+1})$ time.

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Compute k-PTF P'_σ obtained from P after restricting the last $n - m$ variables.
Time: $\mathrm{poly}(n, M)$.

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Our approach

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Develop strategies that work for these two steps even when the k-PTF gates are not sparse.

Satisfiability algorithms vs. SAT solving

Circuit satisfiability algorithms	SAT solving
worst case guarantees	guarantees for a subset of instances

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Some techniques in this talk could be of general interest!

Conclusion and open problems

Better than brute-force algorithms for

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Open problems

- Can the techniques here give better-than brute-force algorithms in the subquadratic regime?
- Other connections from learning algorithms to satisfiability algorithms?
- Can this technique inspire any SAT solving heuristic?

Thank You!