Lower Bound Techniques for QBF Proof Systems

Meena Mahajan

The Institute of Mathematical Sciences, HBNI, Chennai.

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Credits

My work on QBF proof complexity –

- partially supported by the EU Marie Curie IRSES grant CORCON.
- joint work with

Olaf Beyersdorff	Firedrich-Schiller Univ, Jena, Germany
Leroy Chew	Univ of Leeds, UK
Anil Shukla	formerly at IMSc, Chennai now at IIT Ropar

QBF Proof systems

- What are they?
- Why do we study them?

Satisfiability

- SAT: Satisfiability.
  eg. Is there an assignment to x, y, z satisfying all the clauses
  $(x \vee y \vee z), (x \vee \neg\, y \vee \neg\, z), (\neg\, x \vee y \vee \neg\, z), (\neg\, x \vee \neg\, y \vee z)$?
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  In practice – a solved problem! Many good SAT solvers around.
- Ambitious programs to design good solvers for problems harder than SAT.

QBF solvers

- QBF: Quantified Boolean Formula
  Subsumes SAT. eg. Is this QBF true?

  $$\exists x \exists y \exists z (x \vee y \vee z), (x \vee \neg\, y \vee \neg\, z), (\neg\, x \vee y \vee \neg\, z), (\neg\, x \vee \neg\, y \vee z)$$

- PSPACE-complete, so much more expressive than SAT.
  eg. Is this formula true?

  $$\exists e\ \forall u\ \exists c\ \exists d \quad (\neg\, e \vee c)(e \vee d)(\neg\, u \vee c)(u \vee d)(\neg\, c \vee \neg\, d)$$

- Quite a few QBF solvers developed in the last couple of decades.

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- Lower bounds in formal proof system
  (no short proof of unsat/falsity)
  $$\Downarrow$$
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  no short runs.
- Proving lower bounds – back to theory!

The Resolution Proof System for UNSAT

$\mathcal{C}$: bag of clauses. $\tilde{a}$: assignment to the variables.

If $\tilde{a}$ satisfies **Then** $\tilde{a}$ satisfies

$$\mathcal{C} = \boxed{\begin{array}{c} \vdots \\ A \vee x \\ B \vee \neg\, x \\ \vdots \end{array}} \qquad \mathcal{C}' = \boxed{\begin{array}{c} \vdots \\ A \vee x \\ B \vee \neg\, x \\ \vdots \\ A \vee B \end{array}}$$

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$$\mathcal{C}_0 \in \mathrm{SAT} \implies \mathcal{C}_1 \in \mathrm{SAT} \implies \ldots \implies \mathcal{C}_{t-1} \in \mathrm{SAT} \implies \mathcal{C}_t \in \mathrm{SAT}$$

$$\mathcal{C}_0 \notin \mathrm{SAT} \Leftarrow \ldots \Leftarrow \mathcal{C}_i \notin \mathrm{SAT} \Leftarrow \ldots \Leftarrow \mathcal{C}_t \notin \mathrm{SAT} \Leftarrow \Box \in \mathcal{C}_t$$

Extending Resolution to QBFs

QBFs: Quantified Boolean Formulas

- W.l.o.g., QBF in prenex CNF: $Q\vec{x} \cdot F(\vec{x})$; F a set of clauses.
- Resolution is sound: If $Q\vec{x} \cdot F(x)$ is true, and we add a clause C to F through resolution to get F', then $Q\vec{x} \cdot F'(x)$ is also true.

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- Resolution is sound: If $Q\vec{x} \cdot F(x)$ is true, and we add a clause C to F through resolution to get F', then $Q\vec{x} \cdot F'(x)$ is also true.
- But Resolution alone is not enough. Consider

$$\exists x \ \ \forall u \quad (x \vee \neg\, u) \ \ (\neg\, x \vee u).$$

  Resolution can add $(x \vee \neg\, x)$ or $(u \vee \neg\, u)$. Useless.
- Universal variable u has to be handled differently.
- Two ways to proceed, modelling
  - CDCL-based solvers
  - expansion-based solvers

The Evaluation Game on QBFs

- QBF $Q\vec{x} \cdot F(x)$
- Two players, Red and Blue, step through quantifier prefix left-to-right. Red picks values for $\exists$ variables, Blue for $\forall$ variables.

  Assignment constructed: $\tilde{a}$.

  Red wins a run of the game if $F(\tilde{a})$ true. Otherwise Blue wins.

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Red: $x = 1$, Blue: $u = 0$: Blue wins
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  Blue can always win: set $u \neq x$.
- $Q\vec{x} \cdot F(x)$ false if and only if Blue has a winning strategy.
- Use this to extend Resolution.

The $\forall$ reduction rule

Consider this scenario:

- $Q\vec{x} \cdot F(x)$ is true. So Red has a winning strategy.
- $F(x)$ has a clause C in which the rightmost variable (as per $Q\vec{x}$) is a universal variable u.
  i.e. $C = A \vee \ell$; $\ell \in \{u, \neg\, u\}$; all variables in A are left of u.

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Then, by the time Blue has to fix u, Red's strategy must ensure that sub-clause A is already satisfied.

That is, Red has a winning strategy on $Q\vec{x} \cdot [F(x) \wedge A]$.
So $Q\vec{x} \cdot [F(x) \wedge A]$ is also true.

The proof system QU-Res = Res+∀Red

$$Q\vec{x} \cdot \mathcal{C}$$

Grow the bag of clauses $\mathcal{C}$ using

- Resolution: If $A \vee x$ and $B \vee \neg x$ are in the bag, can add $A \vee B$ (provided not a tautology),
- ∀-Reduction: If $A \vee \ell(u)$ in the bag, and all variables in A left of u, can add A,

until the empty clause $\square$ is added.

The proof system QU-Res (cont'd)

- Sound: A derivation of $\Box$ reveals a winning strategy for Blue. [vanGelder 2012]
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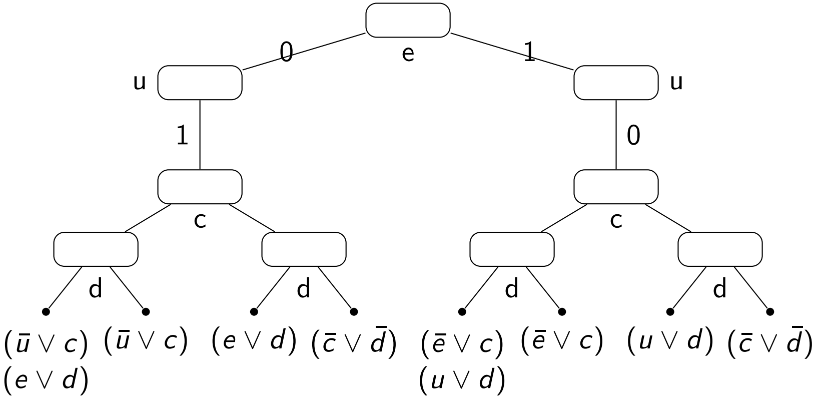
- Sound: A derivation of $\Box$ reveals a winning strategy for Blue. [vanGelder 2012]
- Complete: Use a winning strategy of Blue to decide which clauses to derive.
  - Suffices to resolve with existential pivots only (Q-Res, [KleineBüningKarpinskiFlögel 1995])
  - Suffices to eliminate variables in right-to-left order of quantification blocks (Level-ordered Q-Res)

A derivation in Q-Res

$\exists e \ \forall u \ \exists c \ \exists d \ [(\neg\ e \vee c), (\neg\ u \vee c), (e \vee d), (u \vee d), (\neg\ c \vee \neg\ d)]$



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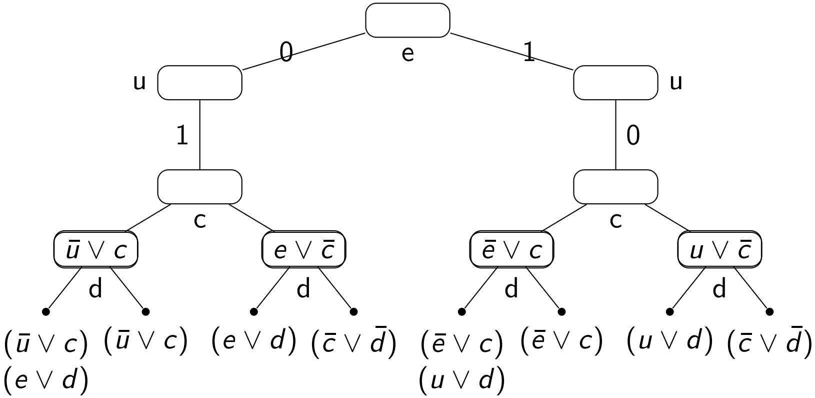

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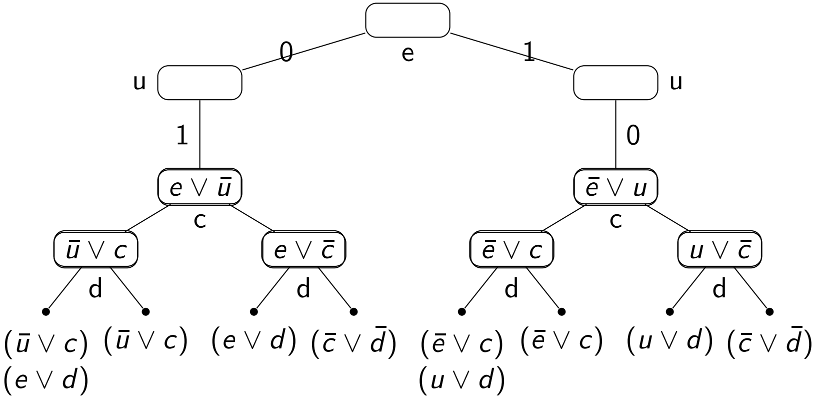

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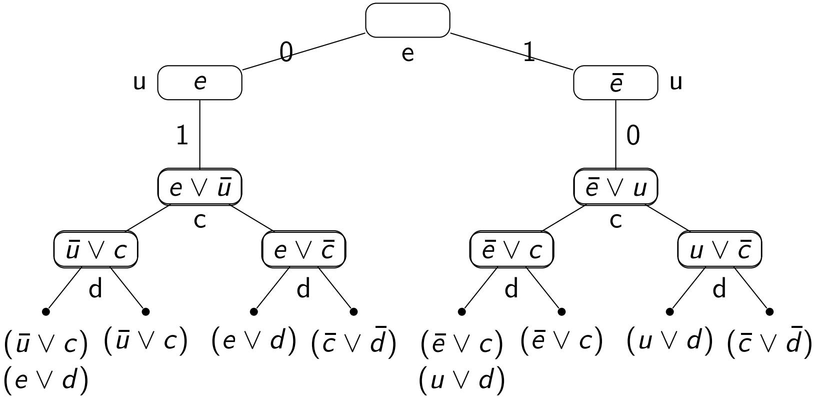

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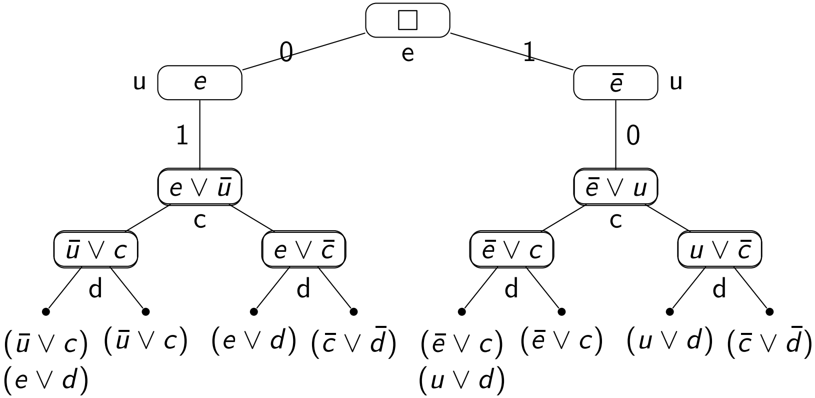

Adding the $\forall$-reduction rule

[Beyersdorff,Bonacina,Chew ITCS 2016]

P: Any sound and complete line-based proof system for UNSAT
eg Cutting Planes, Polynomial Calculus, Frege,
restrictions of Frege (AC^0-Frege, $AC^0[p]$-Frege, TC^0-Frege ...)

$$\Downarrow$$

P+$\forall$Red: a sound and complete proof system for QBF

The CP+∀Red proof system

- CP+∀Red: Cutting Planes + ∀ Reduction.

  [Beyersdorff,Chew,M,Shukla FSTTCS 2016]

- Cutting Planes: Encode clauses as integer inequalities.

$$\begin{array}{lcl} x \vee y \vee z & \rightarrow & x + y + z \geq 1 \\ x \vee \neg\, y \vee z & \rightarrow & x + (1 - y) + z \geq 1 \\ & & (x - y + z \geq 0) \\ x \vee \neg\, y \vee \neg\, z & \rightarrow & x + (1 - y) + (1 - z) \geq 1 \\ & & (x - y - z \geq -1) \end{array}$$

- Bags of inequalities, not clauses.
- Evaluation game: Red tries to satisfy all inequalities.
  Blue tries to falsify some inequality.

The rules in Cutting Planes

If Red ($\exists$) can win

$$\begin{array}{rcl} & \cdots & \\ \langle a \cdot x \rangle & \geq & A \\ \langle b \cdot x \rangle & \geq & B \\ \langle kc \cdot x \rangle & \geq & C \\ & \cdots & \end{array}$$

(for $k \in \mathbb{Z}^{>0}$)

($\langle a \cdot x \rangle$ means $a_1x_1 + a_2x_2 + \ldots + a_nx_n$.)

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$$\boxed{\langle (a+b) \cdot x \rangle \quad \geq \quad A + B}$$

the $+$ rule

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the $\times$ rule

$$\langle c \cdot x \rangle \geq \left\lceil \frac{C}{k} \right\rceil$$

the $\div$ rule

$\forall$ reduction in CP+$\forall$Red

- If Red can win with $\mathcal{I}$ containing $\langle a \cdot x \rangle \geq A$ where the rightmost non-zero coefficient in a is blue, a = a' b 00...0, (ie a universal variable, u)
- then Red can win with

$$\mathcal{I} \cup \ \{ \underset{\substack{\uparrow \\ (\text{Set } u = 1)}}{\langle a' 000\ldots00 \rangle} \cdot x \geq A - b \} \ \cup \ \{ \underset{\substack{\uparrow \\ (\text{Set } u = 0)}}{\langle a' 000\ldots0 \rangle} \cdot x \geq A \} \ .$$

- This Blue-elimination is the $\forall$-Reduction rule.

Proofs in the CP+$\forall$Red proof system

Keep using the $+$, $\times$, $\div$ and $\forall$Reduction rules.

$$\begin{array}{c}
\text{Red can win with } \mathcal{I} = \mathcal{I}_0 \\
\Downarrow \\
\text{Red can win with } \mathcal{I}_1 \\
\Downarrow \\
\text{Red can win with } \mathcal{I}_2 \\
\Downarrow \\
\vdots \\
\Downarrow \\
\text{Red can win with } \mathcal{I}_t.
\end{array}$$

If $\mathcal{I}_t$ contains $\mathbf{0} \geq \mathbf{1}$,
then Red can't win with $\mathcal{I}_t$, and so Red can't win with $\mathcal{I}$.

Expansion-Based Systems

$$\forall u Q\vec{x} \cdot F(u, \vec{x}) \text{ is true}$$

$$\Updownarrow$$

$$[Q\vec{x} \cdot F(0, \vec{x})] \wedge [Q\vec{x} \cdot F(1, \vec{x})] \text{ is true}$$

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$$Q\vec{x}^{u/0} Q\vec{x}^{u/1} \cdot \left[F(0, \vec{x}^{u/0}) \wedge F(1, \vec{x}^{u/1})\right] \text{ is true}$$

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- Expand the initial formula judiciously, on the fly.

  Then use standard resolution.
- Expansion-based systems:
  $\forall$Exp+Res [Janota,Marques-Silva 2015],
  IR [Beyersdorff,Chew,Janota 2014].

Part of a derivation in ∀Exp+Res

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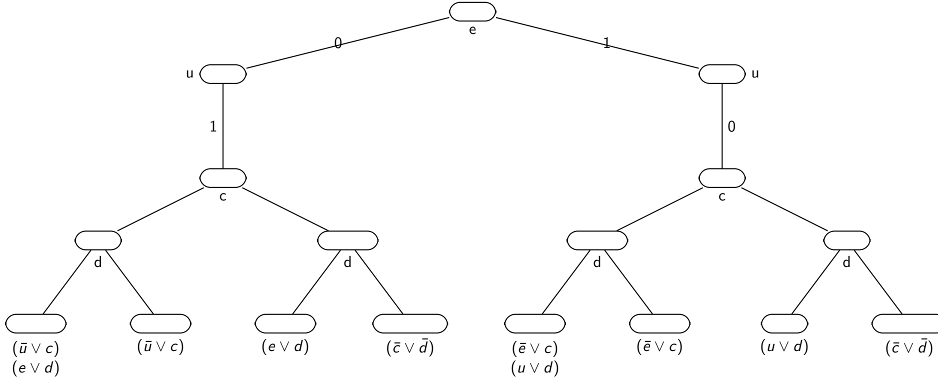

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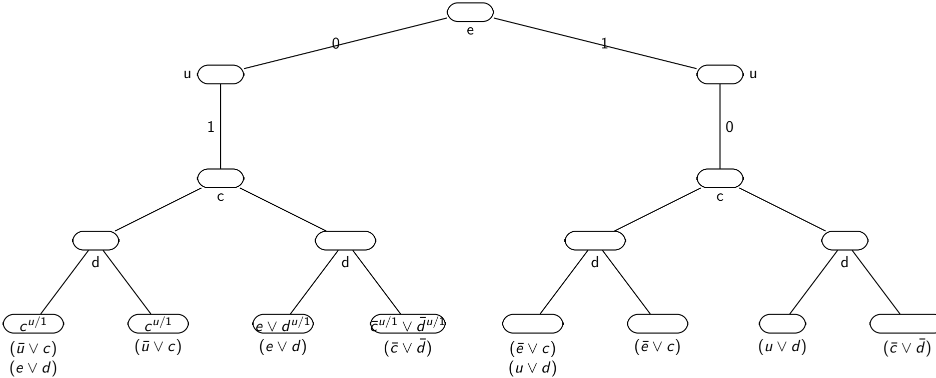

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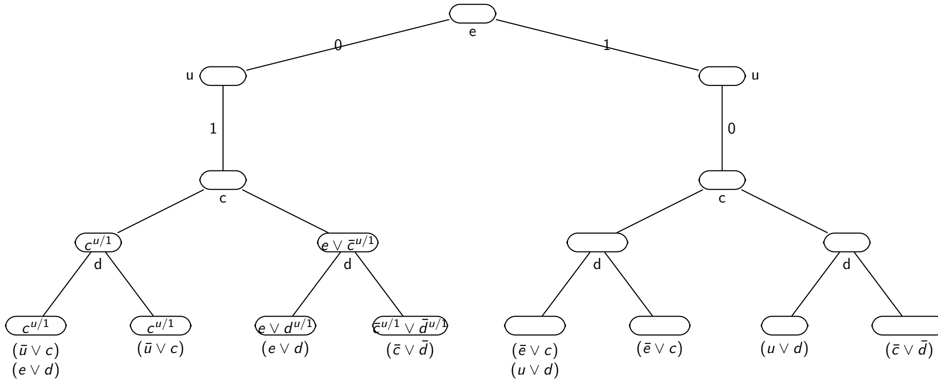

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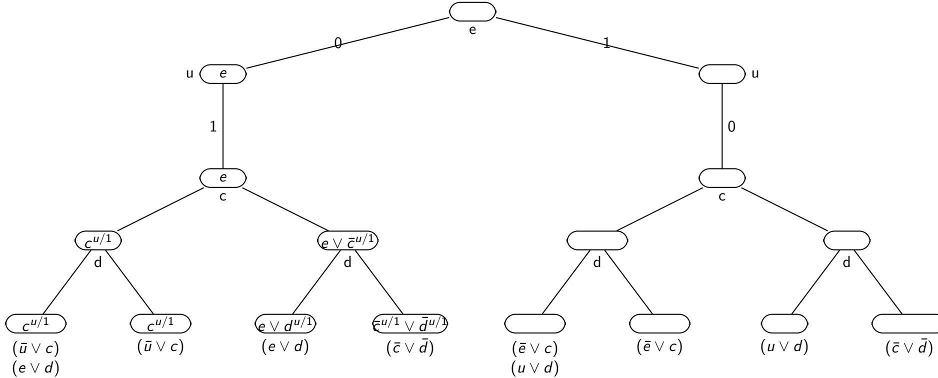

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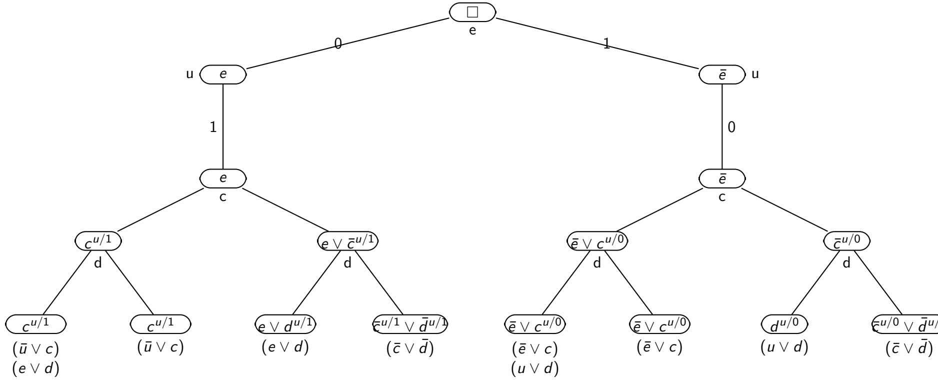

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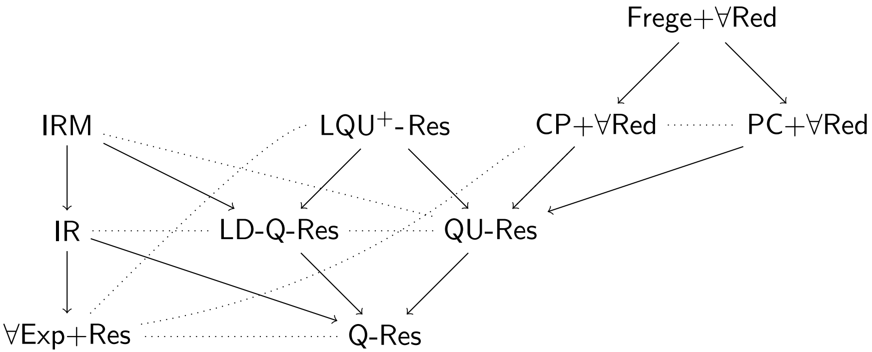
- Proof Systems that use merging:
  LD-Q-Res (Long-Distance QRes), [Balabanov,Jiang 2012]
  LQU^+-Res, [Balabonav,Widl,Jiang 2014]
  IRM (Instantiation, Resolution, Merge) [Beyersdorff,Chew,Janota 2014].

The relative power of some QBF proof systems:



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  not useful for understanding QBF solvers
- by adapting techniques for propositional hardness.
  let's review

The Size-Width Relation in Resolution

In Resolution, Short proofs are narrow.

(Size of proof: number of steps. Width of proof: max width of clause in proof.)

Theorem ([Ben-Sasson,Wigderson 2001])

For all unsatisfiable CNFs F in n variables:

$$S(\vdash_{Res_T} F) \geq \exp\left(w\left(\vdash_{Res} F\right) - w(F)\right) .$$

(tree-like proofs; no reusing clau

$$S(\vdash_{Res} F) = \exp\left(\Omega\left(\frac{\left(w\left(\vdash_{Res} F\right) - w(F)\right)^2}{n}\right)\right) .$$

The Size-Width relation in Q-Res

In Q-Res, this fails completely!

[Beyersdorff,Chew,M,Shukla STACS 2016, ACM Trans. Comp. Logic 2018]

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$$\forall\, u_1 u_2 \ldots u_n \;\; \exists\, e_0 e_1 \ldots e_n \;\; \left[\begin{array}{cc} (e_0) & \\ (\neg\, e_{i-1} \lor u_i \lor e_i) & \text{for} \;\; i \in [n] \\ (\neg\, e_n) & \end{array}\right]$$

- Using Resolution, derive $u_1 \lor \ldots \lor u_n$. ($n+1$ steps)
- Then using $\forall$Red, derive $\square$. (n steps)
- So proof of size $O(n)$. Even tree-like.
- We show: Any proof must derive $u_1 \lor \ldots \lor u_n$.
- So width of any proof $\Omega(n)$.

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Problem: accumulation of universal variables.

Possible fix: Redefine $\mathsf{Width}_{\exists}$. Count only existential variables.

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No!

Completion Principle: clausal encoding of

$$\exists\, X \in \{0,1\}^{n\times n} \;\; \forall\, z \;\; (z \vee \exists \text{ all-1s row}) \wedge (\neg\, z \vee \exists \text{all-0s column})$$

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No!

Completion Principle: clausal encoding of

$$\exists X \in \{0,1\}^{n\times n} \ \ \forall z \ \ (z \vee \exists \text{ all-1s row}) \wedge (\neg z \vee \exists \text{all-0s column})$$

Under appropriate short clausal encoding, proof of size $O(n^2)$.
Even tree-like proof: no reusing derived clauses.
We show: Any proof must have $\text{width}_\exists$ $\Omega(n)$.

Size-Width$_\exists$ relation for non-tree-like proofs

$$\exists e_1 \forall u_1 \exists c_1 \exists d_1 \quad \exists e_2 \forall u_2 \exists c_2 \exists d_2 \quad \dots \quad \exists e_n \forall u_n \exists c_n \exists d_n$$

for $i \in [n]$, $\quad (\neg\, e_i \vee c_i) \quad (e_i \vee d_i)$
$\quad (\neg\, u_i \vee c_i) \quad (u_i \vee d_i)$

$$\neg\, c_1 \vee \neg\, d_1 \vee \neg\, c_2 \vee \neg\, d_2 \vee \dots \vee \neg\, c_n \vee \neg\, d_n$$

Winning strategy for universal player: $u_i = \neg\, e_i$.

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Short proofs in Q-Res, size $n^{O(1)}$.

We show: Width$_\exists$ of any Q-Res proof $\Omega(n)$.

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Large width requirement does not give size lower bound.

Lower Bounds for QBF proof systems

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  not useful for understanding QBF solvers
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  interpolation?

Feasible Interpolation – the propositional case

$$F = A(\vec{p}, \vec{q}) \wedge B(\vec{p}, \vec{r}) \text{ in } \mathrm{UNSAT}$$

$$\Updownarrow$$

for all assignments $\vec{a}$ to $\vec{p}$, either $A(\vec{a}, \vec{q})$ or $B(\vec{a}, \vec{r})$ in UNSAT.

$$A(\vec{a}, \vec{q}) \text{ in SAT} \implies B(\vec{a}, \vec{r}) \text{ in UNSAT.}$$

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- We want an interpolant circuit C in $\vec{p}$ variables:

$$C(\vec{a}) = 0 \implies A(\vec{a}, \vec{q}) \text{ is in UNSAT, and}$$
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$$A(\vec{a}, \vec{q}) \text{ is in } \mathrm{SAT} \implies C(\vec{a}) = 1 \implies B(\vec{a}, \vec{r}) \text{ is in } \mathrm{UNSAT}.$$

Feasible Interpolation – the propositional case (cont'd)

Theorem ([Krajíček 1997],[Pudlák 1997])

- *Resolution proofs of size s give Boolean circuits of size $s^{O(1)}$ computing interpolants.*
- *Cutting Planes proofs of size s give real arithmetic circuits of size $s^{O(1)}$ computing interpolants.*

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- *If $\vec{p}$ variables appears only positively in $A(\vec{p},\vec{q})$ or only negatively in $B(\vec{p},\vec{r})$, then interpolant circuit is (real-) monotone.*
- *All resolution / cutting-plane proofs of the clique-colour formulas are of exponential size.*

(Clique-colour formulas: CNF encodings of
"$\exists$ a graph that is $(k-1)$-colourable and has a k-clique.")

Feasible Interpolation for QBFs

$$\exists\vec{p}\ Q\vec{q}\ Q\vec{r}\ [\ A(\vec{p},\vec{q}) \wedge B(\vec{p},\vec{r})\ ] \text{ is false}$$
$$\Updownarrow$$
$$\exists\vec{p}[Q\vec{q}A(\vec{p},\vec{q}) \wedge Q\vec{r}B(\vec{p},\vec{r})] \text{ is false}$$
$$\Downarrow$$
for all assignments $\vec{a}$ to $\vec{p}$, either $Q\vec{q}\ A(\vec{a},\vec{q})$ or $Q\vec{r}\ B(\vec{a},\vec{r})$ is false.

Interpolant circuit:

$$C(\vec{a}) = 0 \implies Q\vec{q}\ A(\vec{a},\vec{q}) \text{ is false, and}$$
$$C(\vec{a}) = 1 \implies Q\vec{r}\ B(\vec{a},\vec{r}) \text{ is false.}$$

Feasible Interpolation works for many QBF proof systems

The Clique-coClique formulas: CNF encodings of
$\exists$ an n-vertex graph G, $\forall$ u, u implies G has a k-clique,
$\neg u$ implies G has no k-clique.

(Note: To express no clique, universal quantifiers used.
Not succinctly expressible as UNSAT instance.)

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Theorem ([Beyersdorff,Chew,M,Shukla ICALP15, LMCS17, FSTTCS16])

All the resolution-based QBF proof systems
Q-Res, QU-Res, LD-Q-Res, LQU^+-Res, $\forall$Exp+Res, IR, IRM
as well as the proof system CP+$\forall$Red, admit feasible monotone interpolation.
All Clique-coClique formulas need exponential-sized proofs in all these proof systems.

Lower Bounds for QBFproof systems

- from propositional hardness.
  not useful for understanding QBF solvers
- by adapting techniques for propositional hardness. let's review:
  - Size lower bounds from Width lower bounds does not work with the simplest extension of Resolution, Q-Res.
  - Feasible Interpolation works for all Resolution based systems and for CP+$\forall$Red.

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  - Feasible Interpolation works for all Resolution based systems and for CP+$\forall$Red.
- from strategy extraction.
  all-new; specific to QBFs

The winning strategy of the universal player in the evaluation game leads to new lower bound techniques.

Strategy Extraction

- Main idea: A proof reveals information about a winning strategy.
- Examine a proof.
- Construct a circuit of a special type for computing the winning strategy.
- Circuit type: depends on the proof system
  Circuit size: depends on the proof size
- If the winning strategy is hard to compute in the relevant circuit model, then all proofs in the proof system must be large.

From Proof to Decision List for Winning Strategy

Blue has to choose the value of a variable u.
Blue knows values of all variables left of u; partial assignment $\vec{a}$.
Proof lines $L_1, L_2, \ldots, L_m$.
$\forall$Red on u at $(1 <)$ $i_1 < i_2 < \ldots < i_k$ $(\leq m)$.
L_{i_r}: eliminate u from L_{j_r}, $j_r < i_r$.

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if	$L_{i_1}(\vec{a})$ false	**then**	set u to make $L_{j_1}(\vec{a})$ false
elseif	$L_{i_2}(\vec{a})$ false	**then**	set u to make $L_{j_2}(\vec{a})$ false
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elseif	$L_{i_k}(\vec{a})$ false	**then**	set u to make $L_{j_k}(\vec{a})$ false
else			set $u = 0$.

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else			set $u = 0$.

[Beyersdorff,Chew,Janota 2015], [Beyersdorff,Bonacina,Chew 2016]:
This strategy is a winning strategy for Blue.

Strategy description: A Decision List for each universal variable.

Strategy Extraction from P+∀Red proofs

Proof with s ∀Reduction steps

$$\Downarrow$$

Winning strategy can be computed by a Decision List with s steps.

- QU-Res: Each condition is a clause.
  (is $a_1 \vee a_2 \vee \ldots \vee a_n$ true?)
- CP+∀Red: Each condition is a linear threshold function.
  (is $c_1 a_1 + c_2 a_2 + \ldots + c_n a_n \geq b$?)

Genuine QBF bounds for P+$\forall$Red proofs

Proof with s $\forall$Reduction steps

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Contrapositive gives lower bounds on number of $\forall$Reduction steps, not just on proof size.

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Winning strategy can be computed by a Decision List with s steps.

Contrapositive gives lower bounds on number of $\forall$Reduction steps, not just on proof size.

i.e. Proving the QBF false will require large size even with a SAT oracle (appropriately formalised).

Winning Strategies compute desired functions

Fix Boolean function f. Fix small circuit C computing f (size m).
Define QBF $Q_{f,C}$:

$$\exists x_1 x_2 \dots x_n \forall w \exists z_1 z_2 \dots z_m \quad \left[\begin{array}{c} (w \neq z_m) \\ (z_i = \text{value of } i\text{th gate of } C(x)) : \quad i \in [m] \end{array} \right]$$

(z_i clauses enforce $z_m = f(x)$.)

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(z_i clauses enforce $z_m = f(x)$.)

- Blue can choose $w = f(x)$ and force a win.
- No other way for Blue to win.
- $f(x)$ not hard to compute – it has a small circuit.
  If no small decision list, then no small proof.

Winning Strategies Hard for Decision Lists

- The PARITY function has an $O(n)$ size circuit.
  The PARITY function requires exponentially long decision lists of clauses. ([Håstad]: $\oplus \notin AC^0$.) Hence

Theorem ([Beyersdorff,Chew,Janota 2015])

Any Q-Res or QU-Res proof for Q-PARITY must be of exponential size.

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Any Q-Res or QU-Res proof for Q-$\textsc{Parity}$ must be of exponential size.

- The Inner Product function has an $O(n)$ size circuit.
  $\mathrm{IP}(x, y) \triangleq \langle x \cdot y \rangle \mod 2$.
  The IP function needs $> 2^{n/2} - 1$ steps in a decision list of linear threshold functions. ([Turán,Vatan 1997]) Hence

Theorem ([Beyersdorff,Chew,M,Shukla 2016])

Any CP+$\forall$Red proof for Q-IP must be of exponential size.

Other Sources of Hardness

- [Beyersdorff,Pich LICS 2016]
  Every lower bound in Frege+$\forall$Red stems from
  - either propositional hardness,
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  No other source of hardness.

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  No other source of hardness.
- Not true for weaker systems.

Other Sources of Hardness (cont'd)

$$
\begin{array}{cc}
\exists x_1 \cdots x_n \forall u_1 \cdots u_n \exists t_1 \cdots t_n & \\
(x_i \vee u_i \vee t_i) & i \in [n] \\
(\neg\, x_i \vee \neg\, u_i \vee t_i) & i \in [n] \\
(\neg\, t_1 \vee \cdots \vee \neg\, t_n) &
\end{array}
$$

- Blue has a trivial winning strategy: $u_i = x_i$.
- But the formula is still hard to prove false in QU-Res. Why?

Winning Strategies needing Varied Responses (cont'd)

- Many responses needed in winning strategy.
  2^n possible values for $u_1 \cdots u_n$, all necessary.
  cost of formula high.
- Each line can contribute only so much: capacity small.
- Hence proof size must be large.

Theorem ([Beyersdorff,Blinkhorn,Hinde ITCS2018])

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For any proof π of a QBF ϕ in a system $P+\forall Red$,

$$\mathit{Size}(\pi) \times \mathit{Capacity}(\pi) \geq \mathit{Cost}(\phi)$$

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Similar result for expansion-based systems; [Beyersdorff,Blinkhorn STACS 18].

To conclude ...

QBF Proof systems

- What are they? Formal systems for proving false QBFs false.
- Why do we study them?

  Lower bounds can help guide development of better solvers.
  (Also, strong lower bounds will separate complexity classes.)

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  Extracting strategies from proofs leads to lower bounds.

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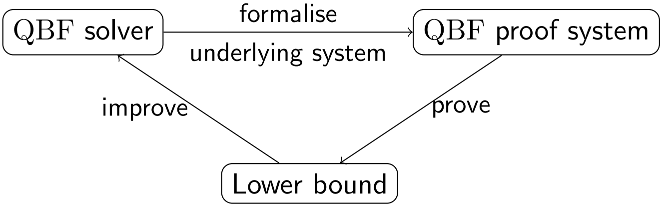
  (Also, strong lower bounds will separate complexity classes.)
- What do we know?

  Extracting strategies from proofs leads to lower bounds.
- What next? Continue the cycle



Thank you