UNIVERSAL TREES

QUASI-POLYNOMIAL

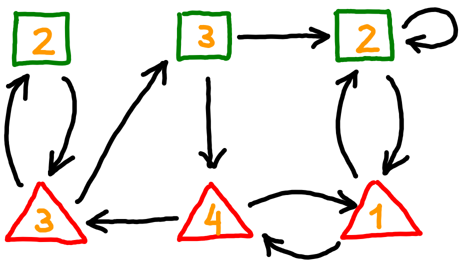
PARITY GAMES

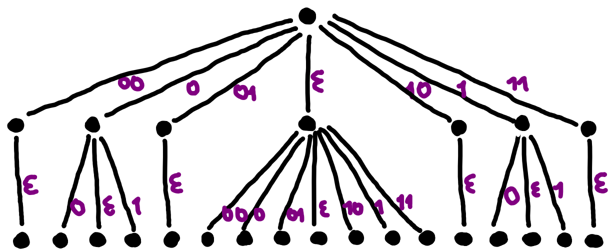
SEPARATING AUTOMATA

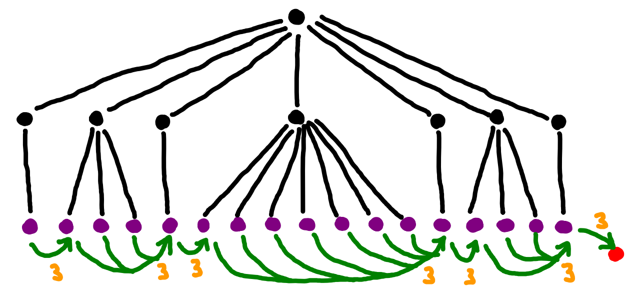
$$(c_{d-1},\ldots,c_{p+2},c_p,c_{p-2},\ldots,c_1) \xrightarrow{p} \text{reject} \quad \text{if } c_p = n$$

$$(c_{d-1},\ldots,c_{p+2},c_p,c_{p-2},\ldots,c_1) \xrightarrow{p} (c_{d-1},\ldots,c_{p+2},c_p+1,0,\ldots,0)$$

$$(c_{d-1},\ldots,c_{p+2},c_p,c_{p-2},\ldots,c_1) \xrightarrow{p-1} (c_{d-1},\ldots,c_{p+2},c_p,0,\ldots,0)$$





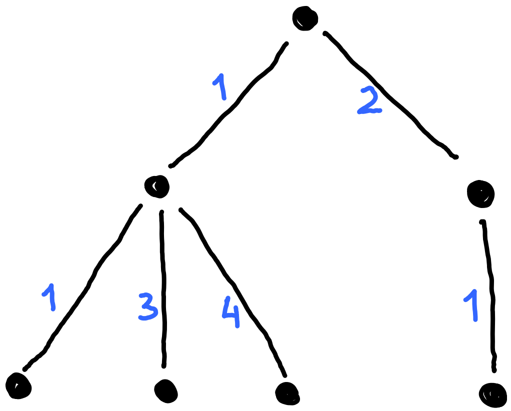


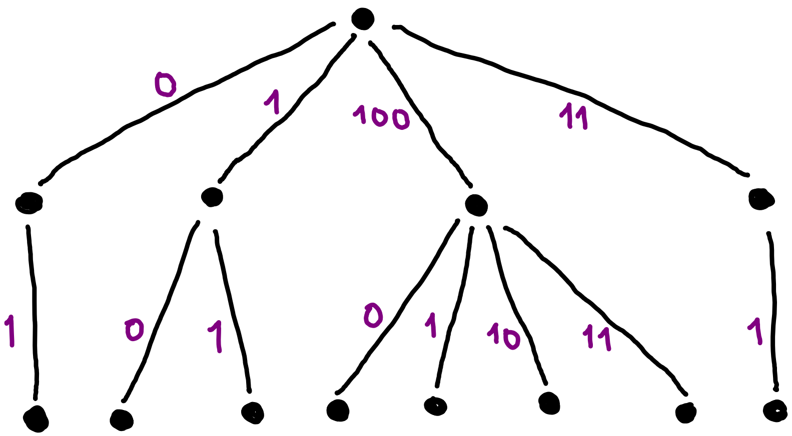
MARCIN JURDZIŃSKI

UNIVERSITY OF WARWICK

COLLABORATORS: W. CZERWIŃSKI, L. DAVIAUD, N. FIJALKOW, R. LAZIĆ, K. LEHTINEN, P. PARYS

ORDERED TREES

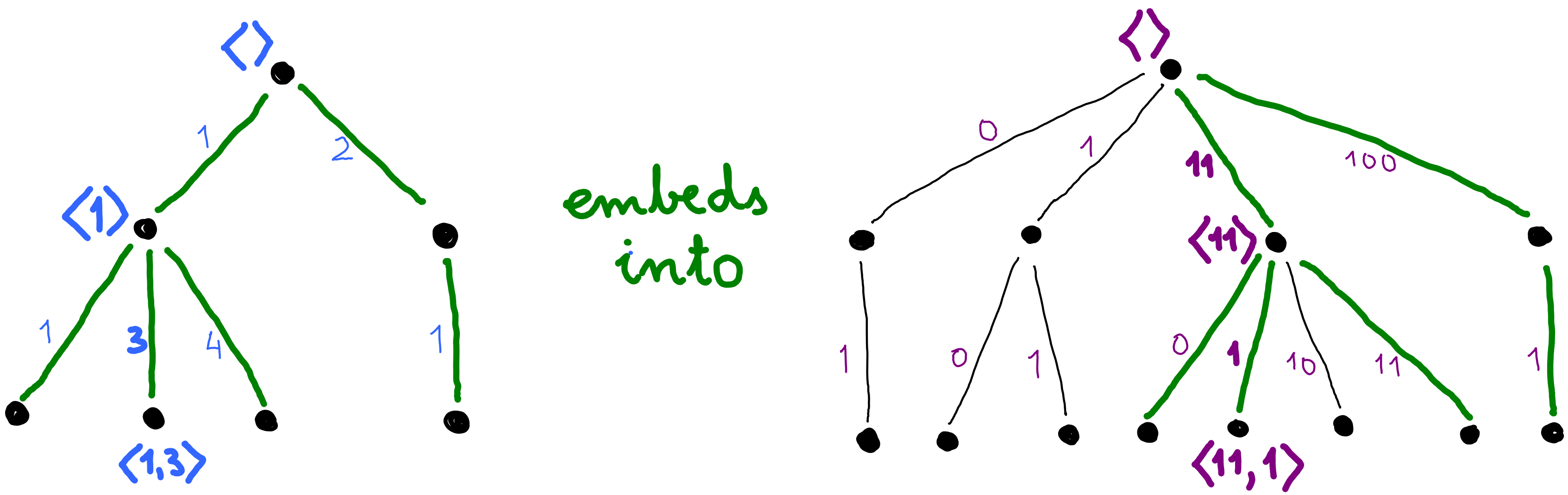




$(\mathbb{N}, <)$

$(\mathbb{B}^*, <_{lex})$

ORDERED TREES



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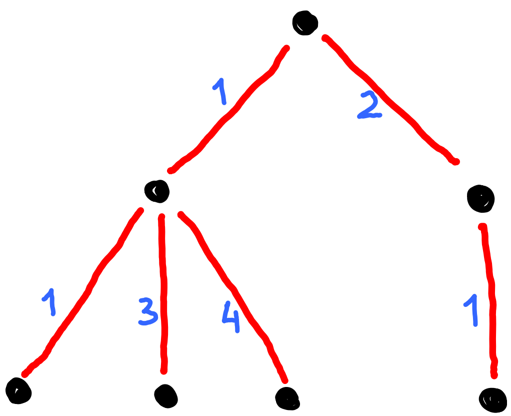
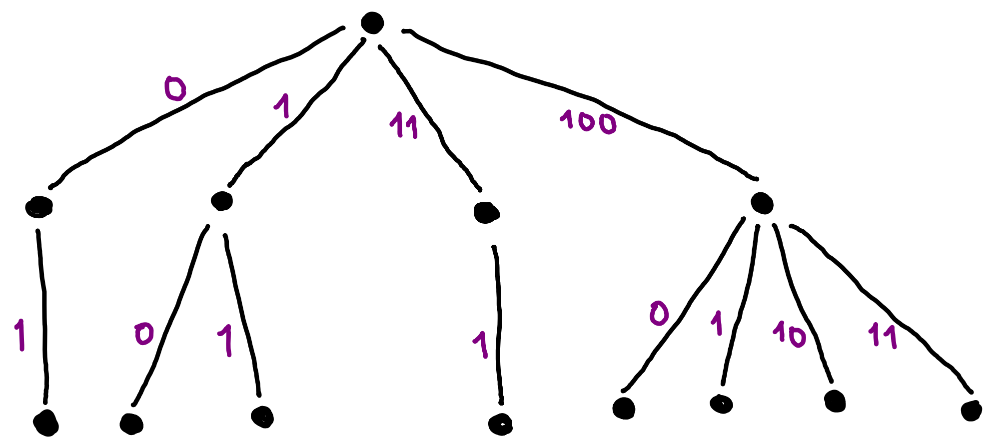
ORDERED TREES



does not embed into



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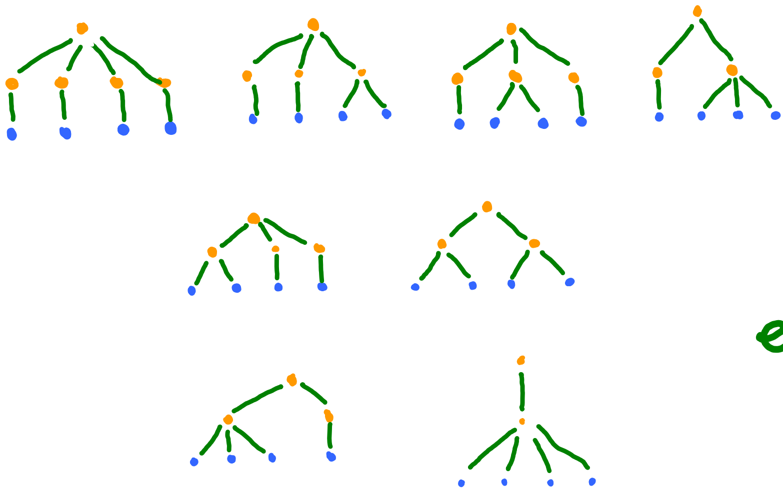
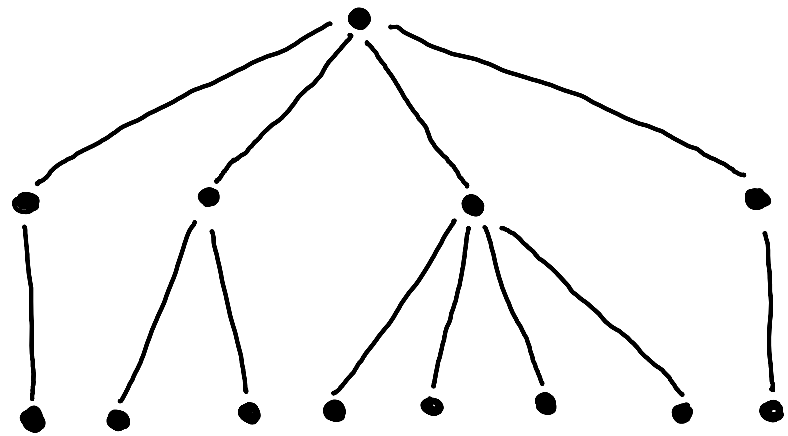
UNIVERSAL TREES



all embed into



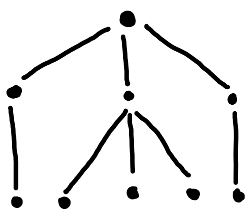
(4, 2)- universal tree

leaves: 4

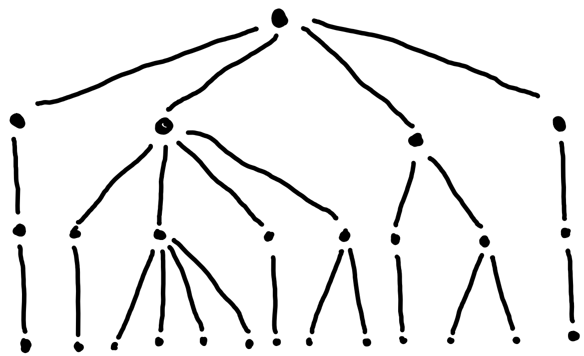
height: 2

UNIVERSAL TREES

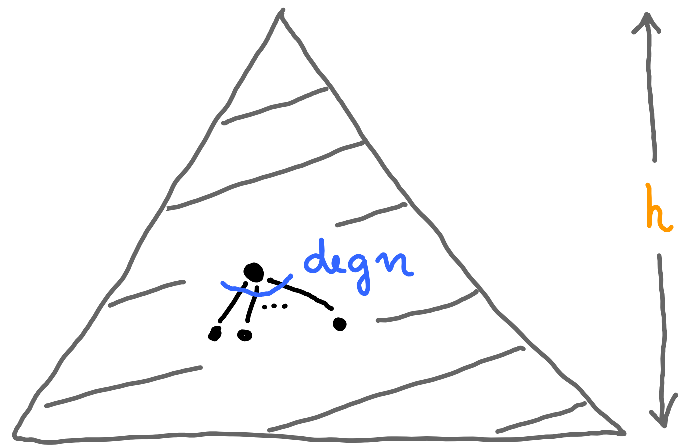
DEFINITION T is an (n, h)-universal tree if every tree with n leaves and of height h embeds into T



$(3,2)$-universal



$(4,3)$-universal



(n,h)-universal

$\Theta(n^h)$ size

QUASI-POLYNOMIAL UNIVERSAL TREES

THEOREM [J., Lazić 2017]

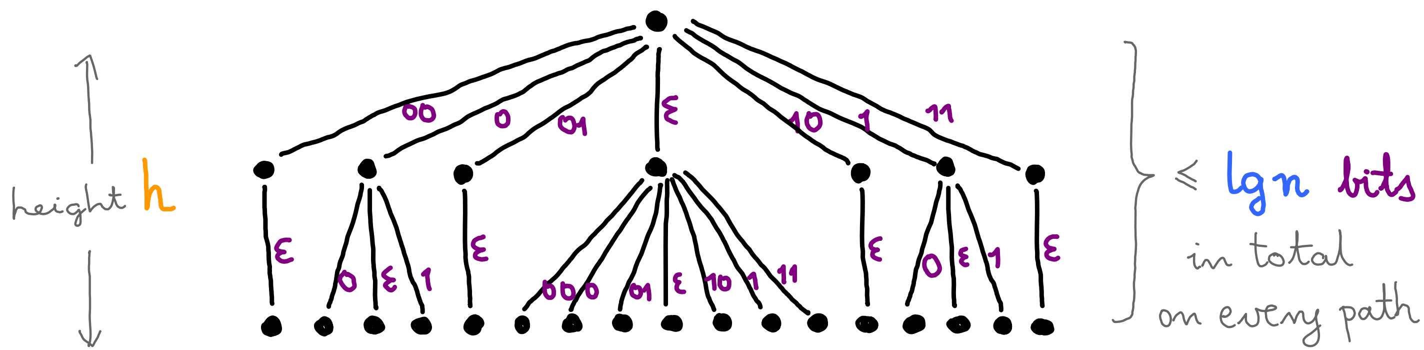
There is an (n, h)-universal tree of size $n \cdot \binom{\lg n + h}{\lg n}$

- $\mathrm{poly}(n)$ if $h = O(\log n)$
- $n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$ if $h = \omega(\log n)$

UNIVERSAL TREES: A QUASI-POLYNOMIAL UPPER BOUND

THEOREM [J., Lazić 2017]

There is an (efficiently navigable) (n,h)-universal ordered tree of size $n \cdot \binom{\lg n + h}{\lg n} = n^{\lg\left(\frac{h}{\lg n}\right) + O(1)}$



$(4,2)$-universal ordered tree

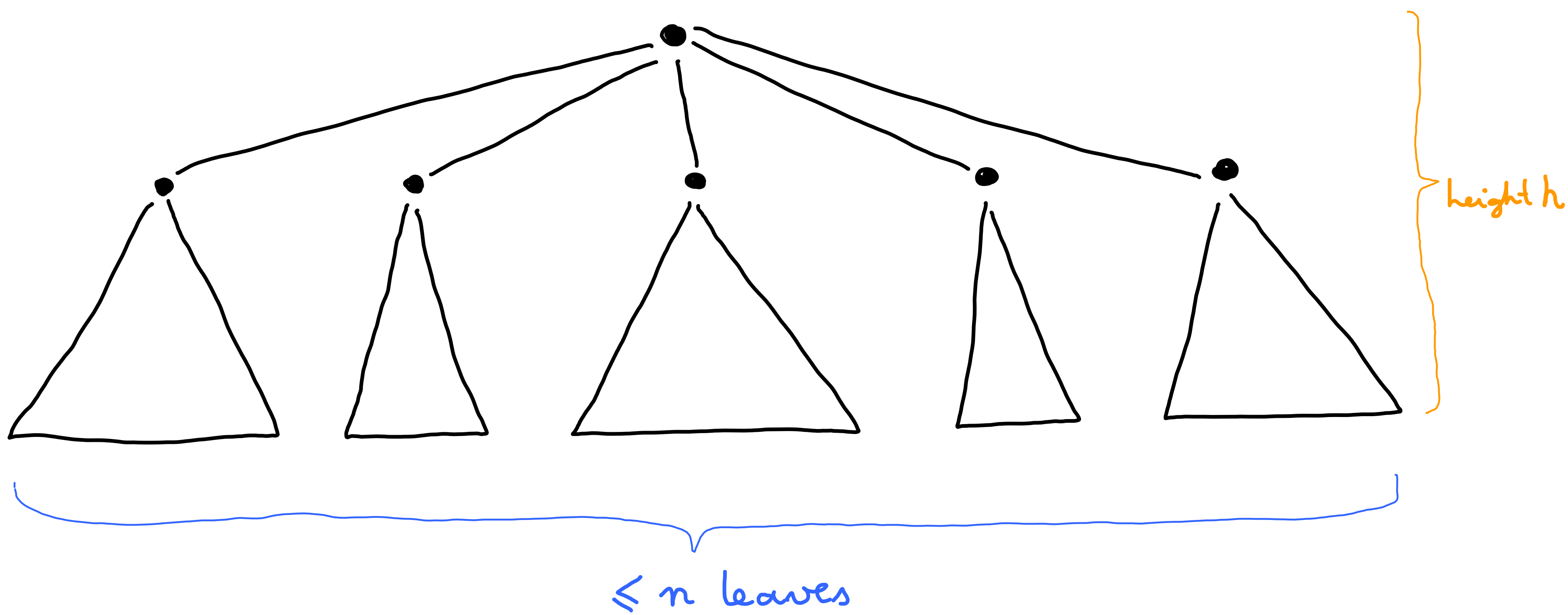
$$\boxed{00 \prec 0 \prec 01 \prec \varepsilon \prec 10 \prec 1 \prec 11}$$

the linear order on bit strings induced by $0 \prec \varepsilon \prec 1$

TREE CODING LEMMA

LEMMA

If an ordered tree $\mathcal{T}$ has height h and $\leqslant n$ leaves then there is an order-preserving labelling of $\mathcal{T}$ by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leqslant \lg n$ bits are used



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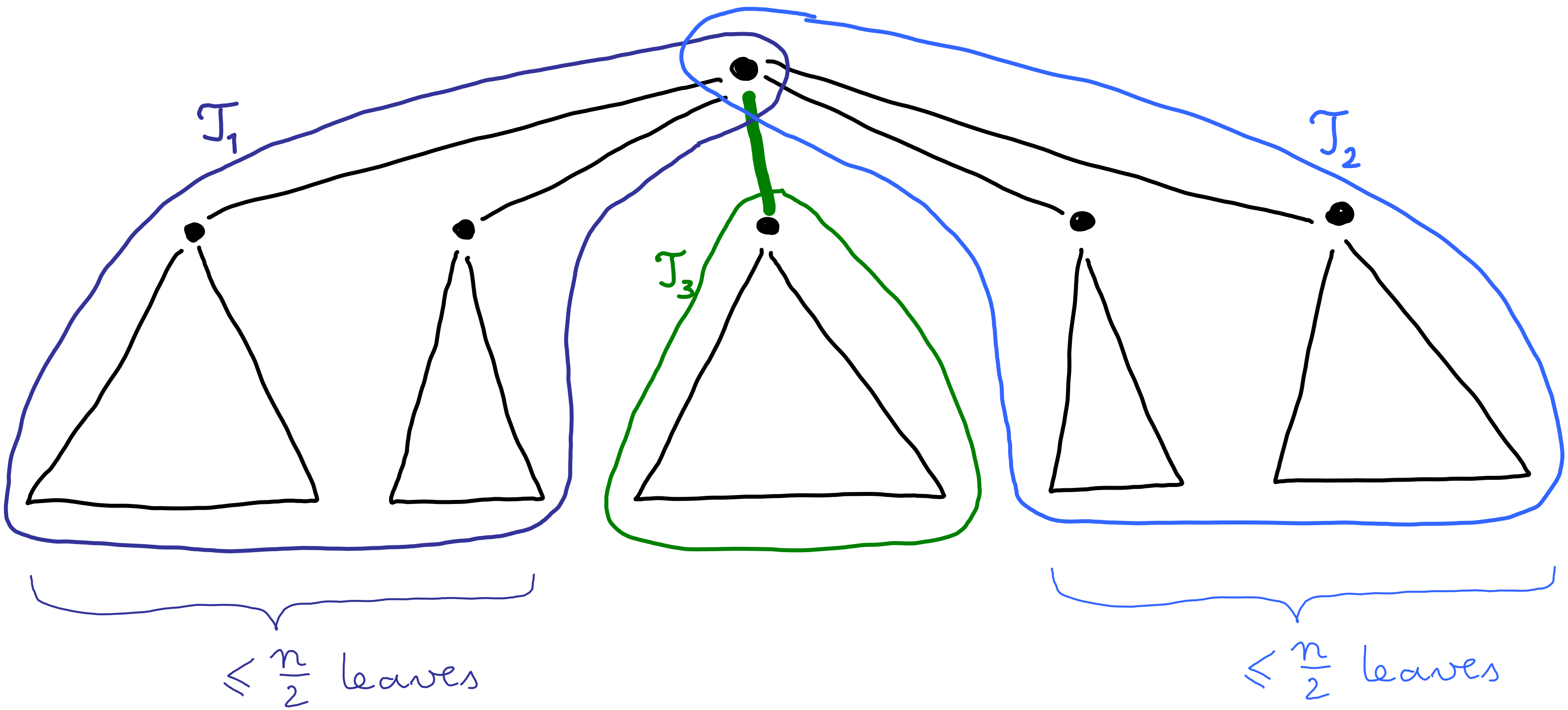

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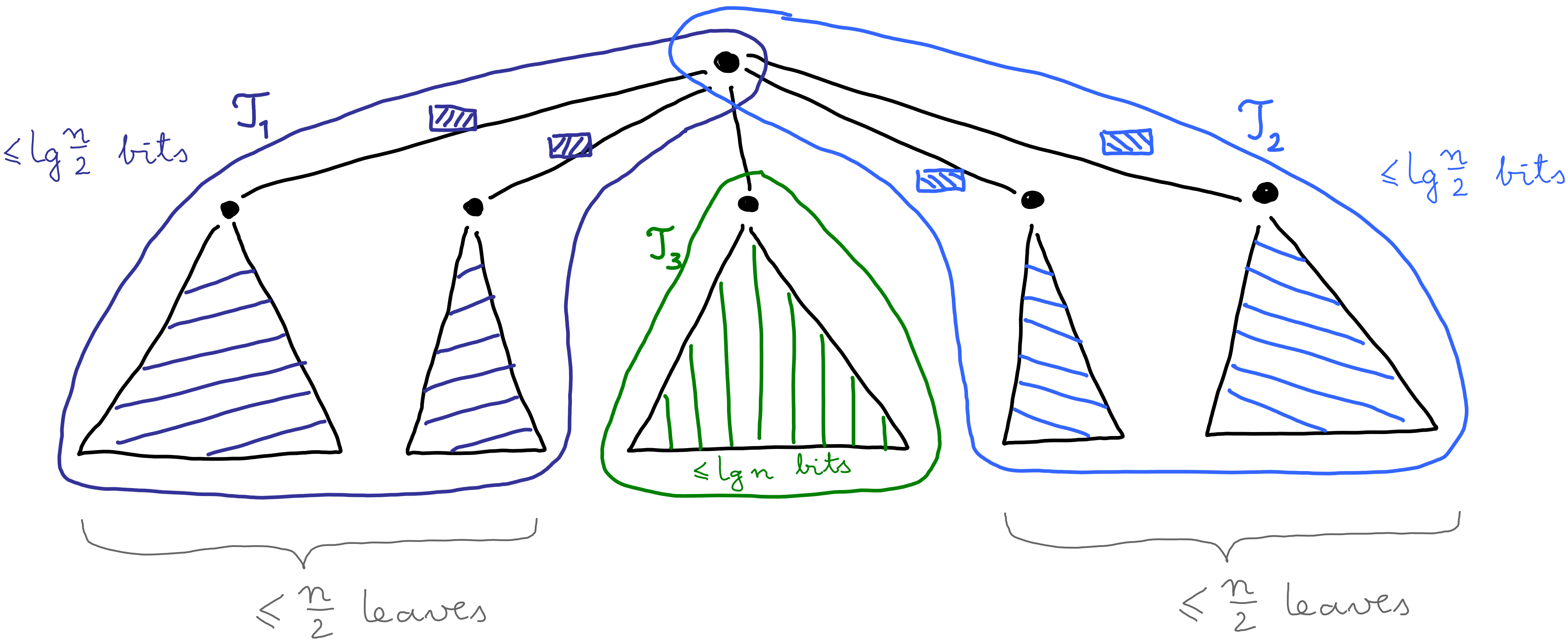

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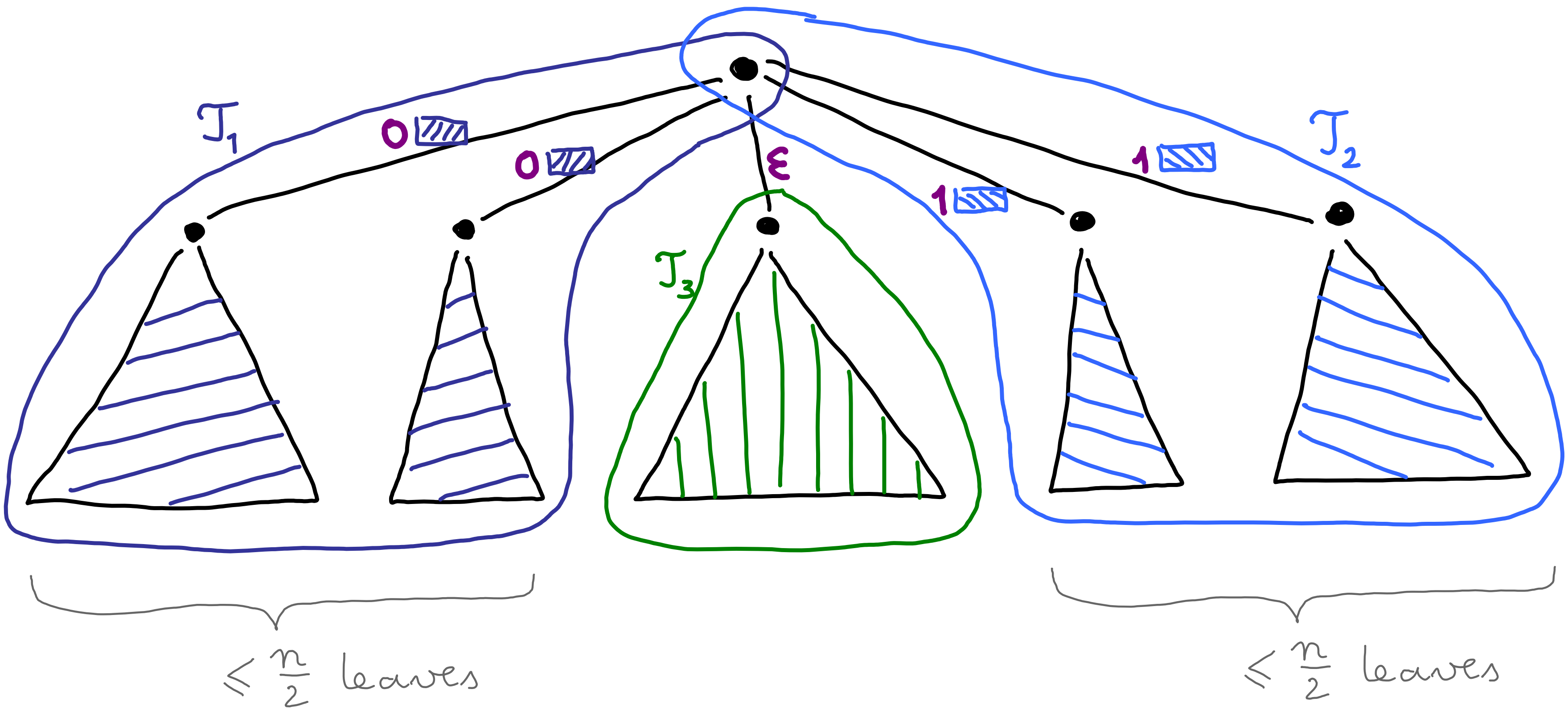
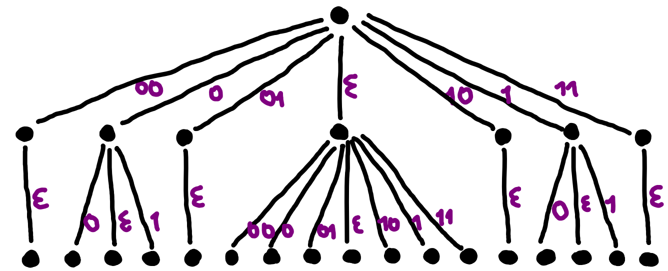


SUCCINCT MULTI-COUNTERS

$$S_{n,h} \stackrel{df}{=} \left\{ \langle s_h, s_{h-1}, \dots, s_1 \rangle : s_i \in \{0,1\}^* \text{ and } \sum_{i=1}^{h} |s_i| \leq \lceil \lg n \rceil \right\}$$

$S_{4,2}$



$\boxed{00 \prec 0 \prec 01 \prec \varepsilon \prec 10 \prec 1 \prec 11}$

<u>Fact</u> $|S_{n,h}| \leqslant 2^{\overbrace{\lceil \lg n \rceil \cdot (1 + \lceil \lg h \rceil)}^{\text{\# bits sufficient to represent a succinct multi-counter}}} = \boxed{n^{\lg h + O(1)}}$

For each bit,

the bit

the co-ordinate the bit belongs to

THE SIZE OF $S_{n,h}$

- $|S_{n,h}| \leq 2^{\lceil \lg n \rceil} \cdot \binom{\lceil \lg n \rceil + h}{h}$

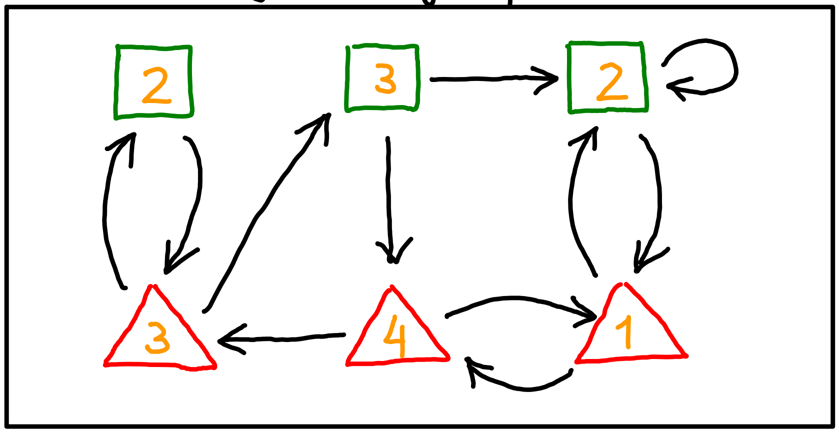
- $$|S_{n,h}| = \begin{cases} O(n \cdot \lg^h n) & \text{if } h = O(1) \\ O(n^{1+o(1)}) & \text{if } h = o(\log n) \\ \widetilde{\Theta}\left(n^{\lg(\delta+1) + \underbrace{\lg(e_\delta)+1}_{\leq 3.72}}\right) & \text{if } h = \lceil \delta \cdot \lg n \rceil \\ O\left(h \cdot n^{\lg\left(\frac{h}{\lg n}\right)+0.45}\right) & \text{if } h = \omega(\log n) \end{cases}$$

where $e_\delta = \left(1 + \frac{1}{\delta}\right)^\delta$

PARITY GAMES

$n = |V|$

A game graph



$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

$$\pi : V \to \{1, 2, 3, 4, 5, \dots, d\}$$

PARITY GAMES

$n = |V|$

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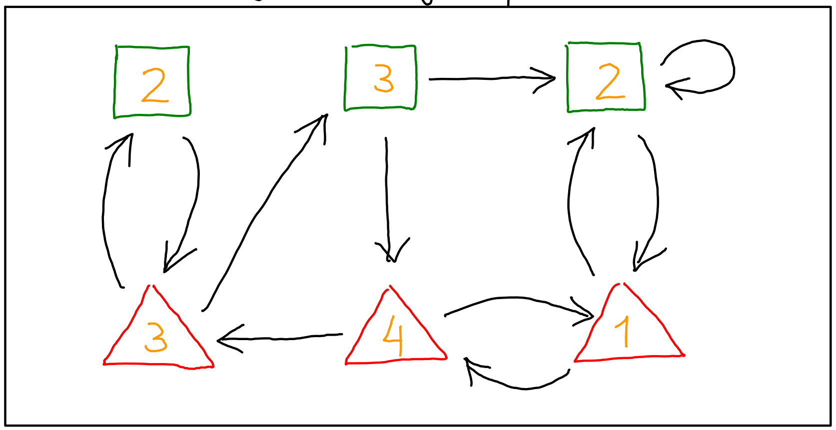

$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

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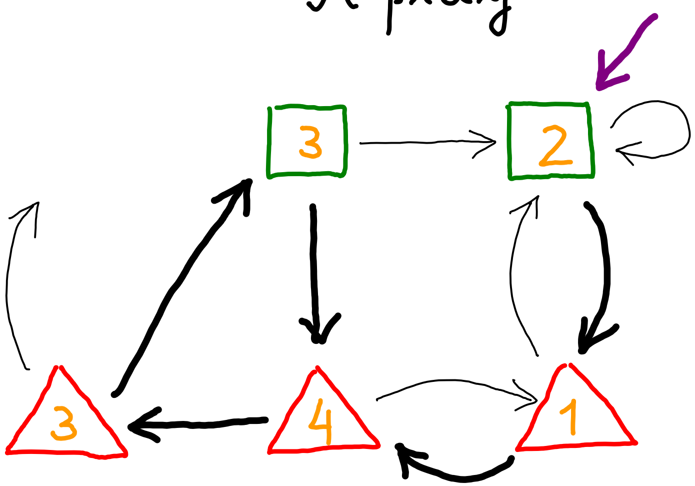
A play



Even wins $\langle p_1, p_2, p_3, \dots \rangle$

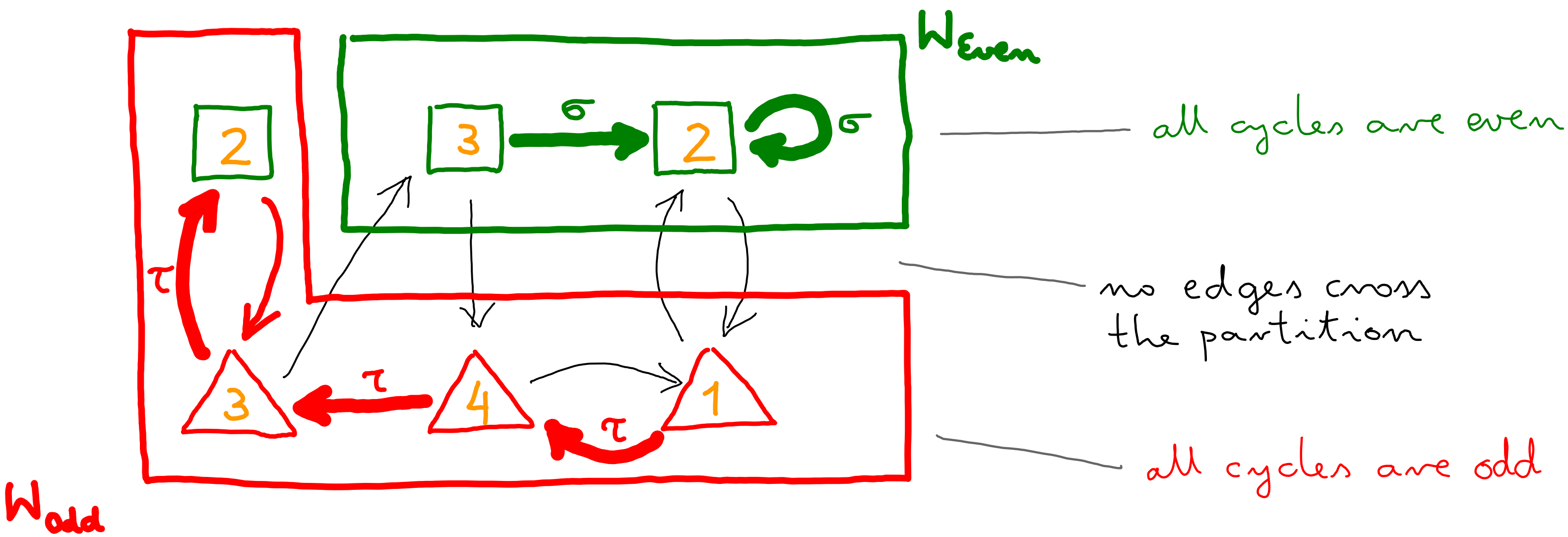
iff

$$\left[\limsup_{i \to \infty} p_i\right] \text{ is even}$$

SHORT WITNESSES FROM POSITIONAL DETERMINACY

THEOREM [Emerson, Jutla 1991; Mostowski 1991; re-proved since 1960's]

Parity games are positionally determined



COROLLARY [Emerson, Jutla, Sistla 1993]

Deciding the winner in parity games is $\boxed{\text{in NP} \cap \text{co-NP}}$

ALGORITHMS FOR SOLVING PARITY GAMES

- $\boxed{n^{d+O(1)}}$ [McNaughton 1993 ; Zielonka 1998]
- $\boxed{n^{\frac{d}{2}+O(1)}}$ [Browne, Clarke, Jha, Long, Marrero 1994; Seidl 1996; J. 2000]
- $\boxed{n^{d+O(1)}}$ strategy iteration [Vöge, J. 2000]

  $\boxed{2^{\Omega(n)}}$ [Friedmann 2009]

  - policy iteration for MDPs [Fearnley 2010]
  - randomized simplex [Hansen, Friedmann, Zwick 2011]
- $\boxed{n^{O(\sqrt{n})}}$ [Björklund, Sandberg, Vorobyov 2003; J., Paterson, Zwick 2006]
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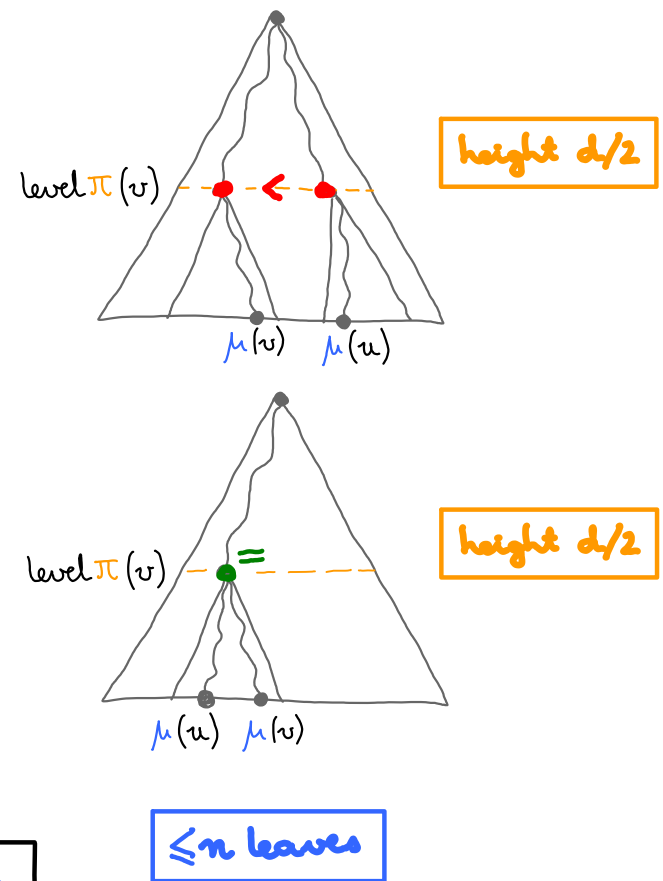
- $\boxed{n^{\lg d+O(1)}}$ **[Calude, Jain, Khoussainov, Li, Stephan 2017; J., Lazić 2017; Lehtinen 2018]**

TREE WITNESSES

DEFINITION $\mu : V \to \mathbb{T}$ is a *tree witness* for graph G if for every edge $(v,u) \in E$,

- $\boxed{\mu(v)|_{\pi(v)} < \mu(u)|_{\pi(v)}}$ if $\pi(v)$ is odd

- $\boxed{\mu(v)|_{\pi(v)} \leqslant \mu(u)|_{\pi(v)}}$ if $\pi(v)$ is even



a.k.a. *signature assignment*, *progress measure*

TREE WITNESSES

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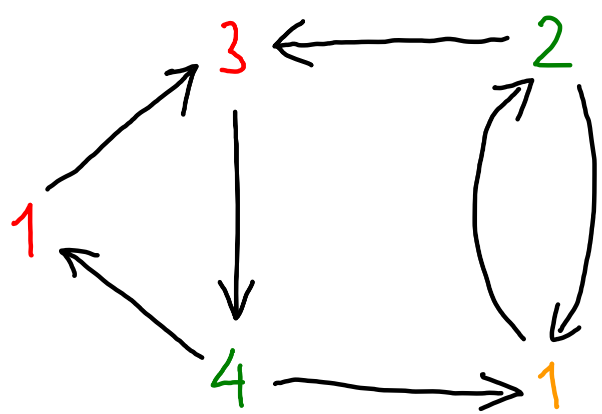

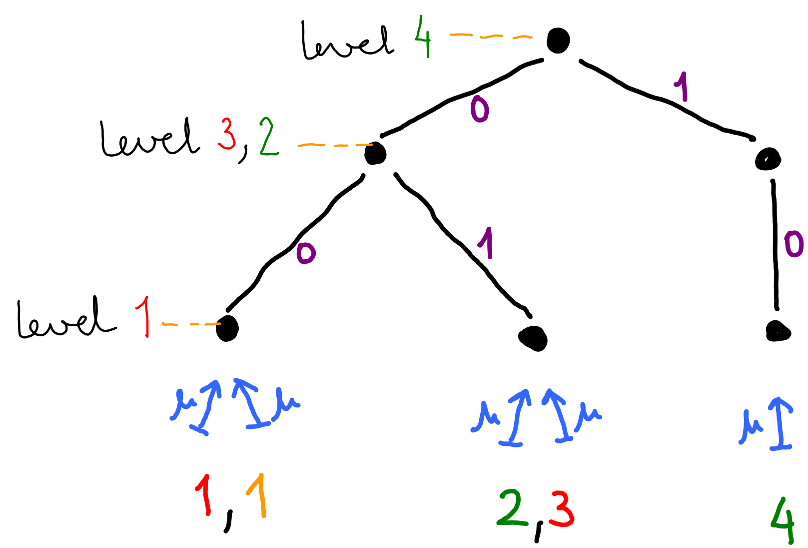

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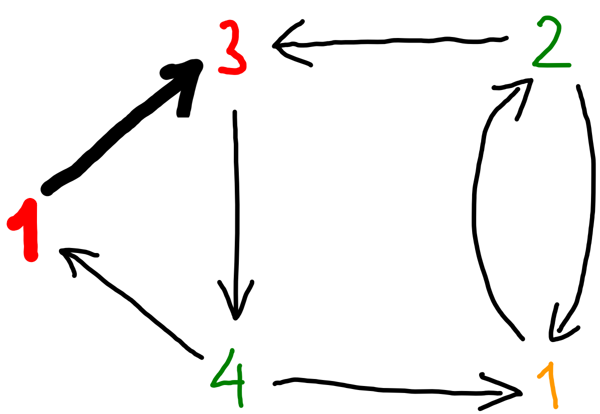

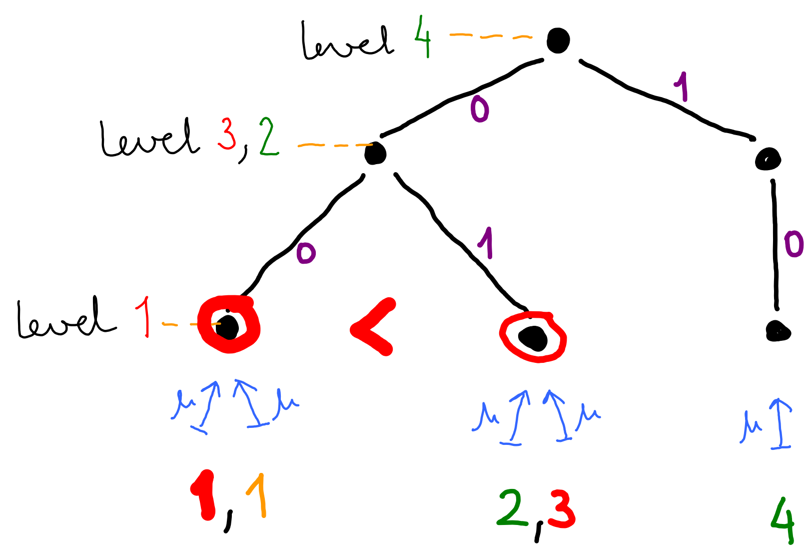

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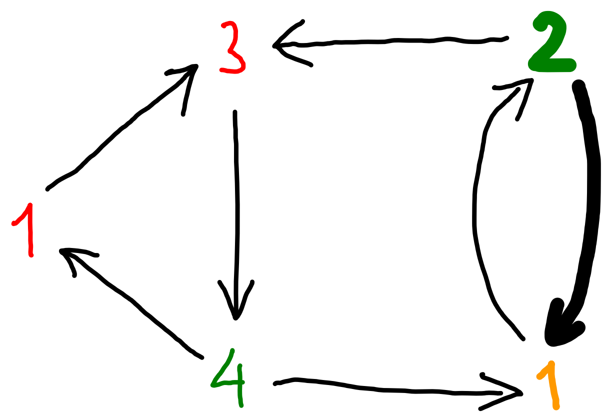

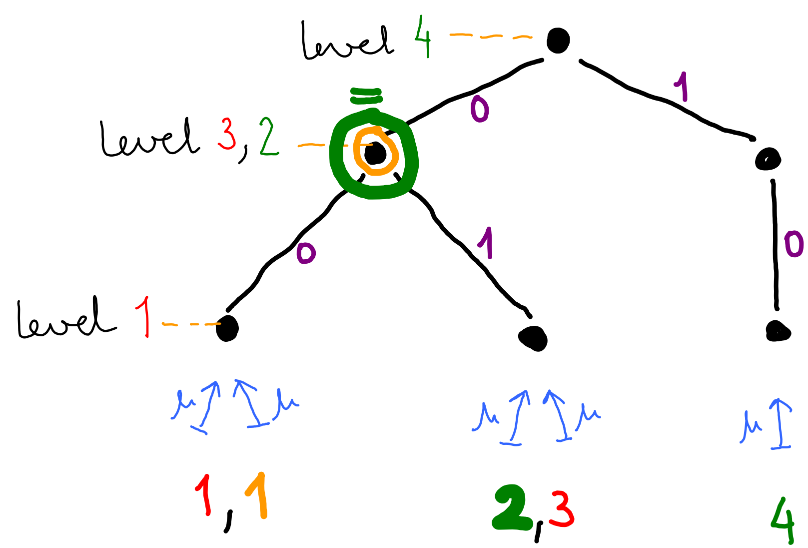

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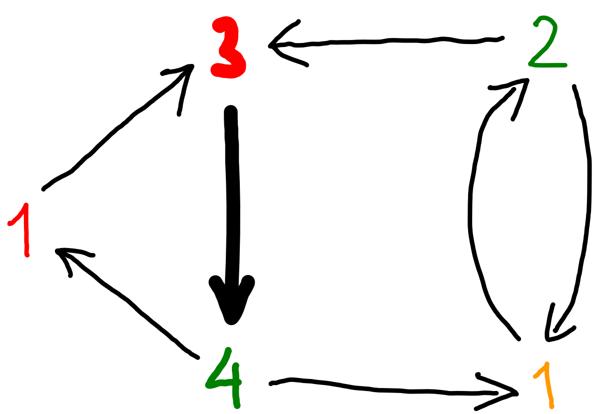
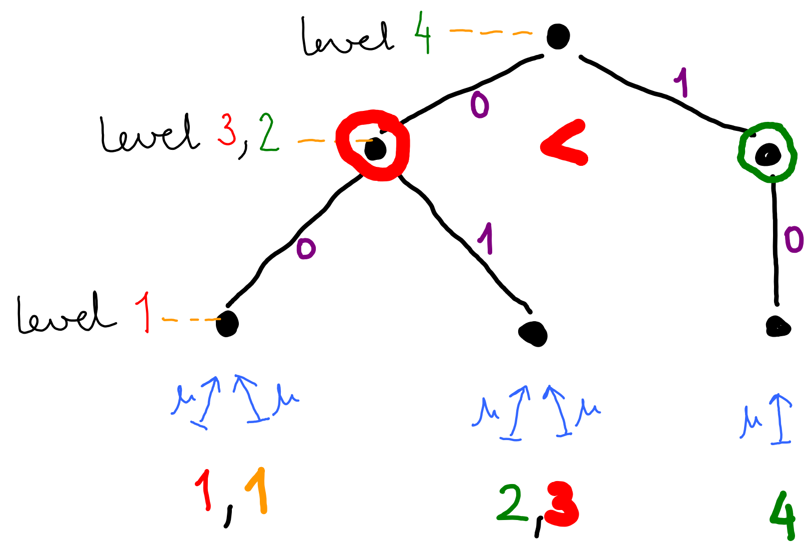
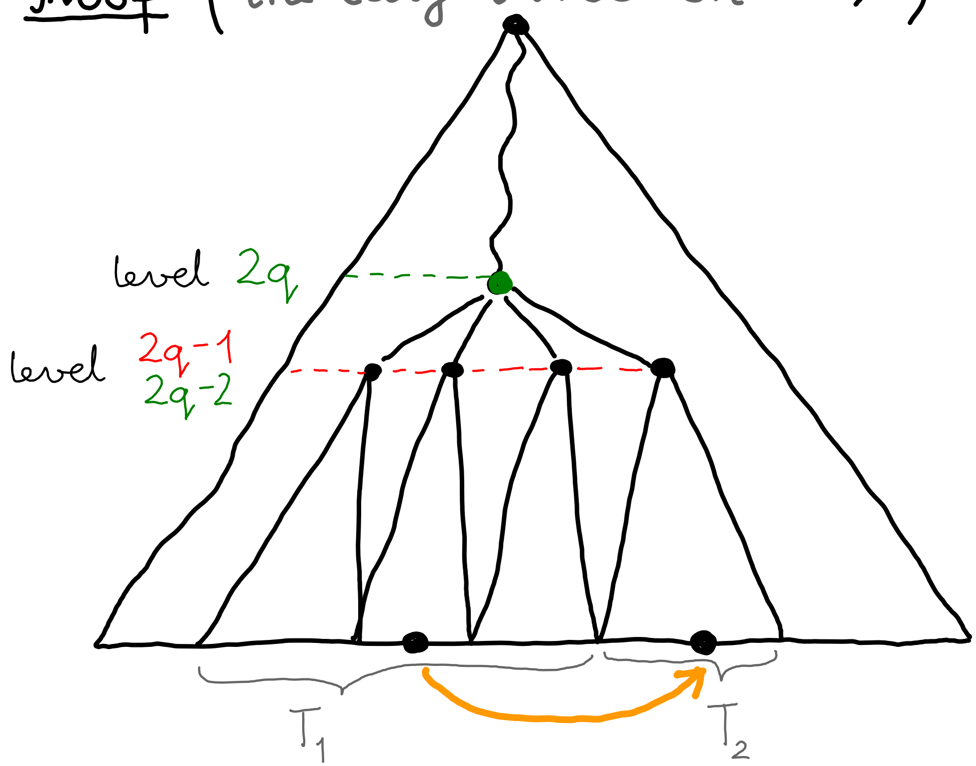




Tree witnesses

Theorem [...; Emerson, Jutla 1991; ...]
G has a tree witness iff every cycle in G is even

Proof (the easy direction "$\Rightarrow$")



- all odd priorities on the cycle are $< 2q$
- an even priority $2q$ or higher occurs on the cycle because there is an edge on the cycle from T_2 to T_1

SOLVING PARITY GAMES USING SAFETY AUTOMATA

[Bernet, Janin, Walukiewicz 2003]

parity game

deterministic **safety** automaton

the **"chained product"** game is played on G

$$\underbrace{G \triangleright A}_{\text{safety game}}$$

- A reads **priority sequences**
- A declares Odd the winner if **REJECT** state is reached

SOLVING PARITY GAMES USING SAFETY AUTOMATA

[Bernet, Janin, Walukiewicz 2003]

parity game — deterministic **safety** automaton

the **"chained product"** game is played on G

$$G \triangleright A$$

- A reads priority sequences
- A declares Odd the winner if **REJECT** state is reached

QUESTION What properties of A make G and $G \triangleright A$ have the same winners?

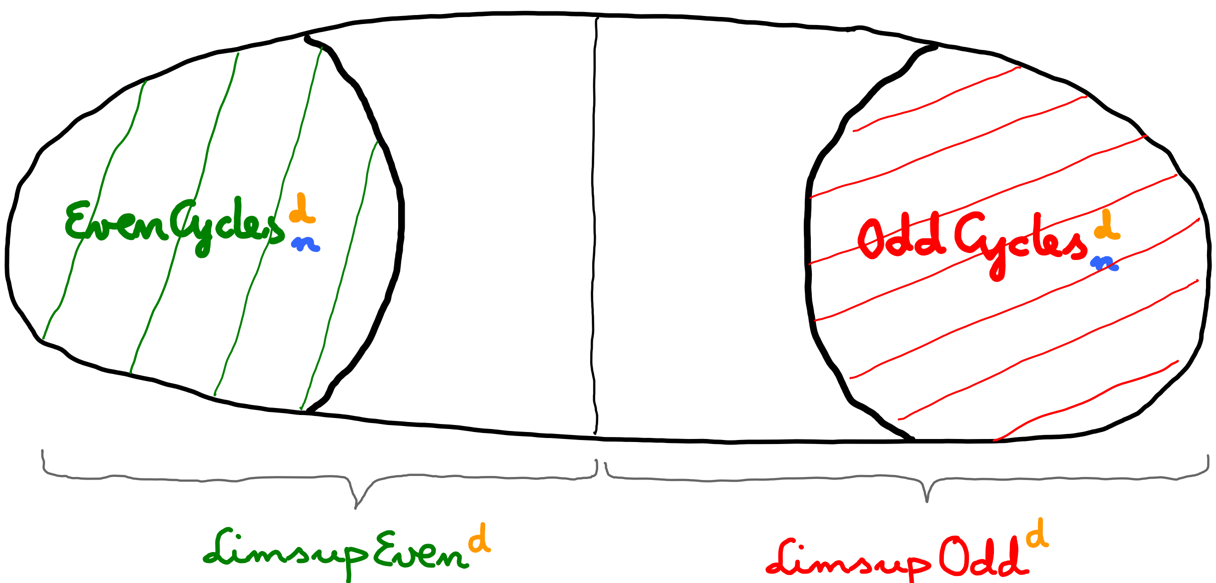
- If Even plays a positional winning strategy in G then A **accepts**
- If Odd plays a positional winning strategy in G then A **rejects**

LANGUAGES OF PRIORITY SEQUENCES

- limsupEven^d $\subseteq \{1,2,\ldots,d\}^\omega$: won by Even

LANGUAGES OF PRIORITY SEQUENCES

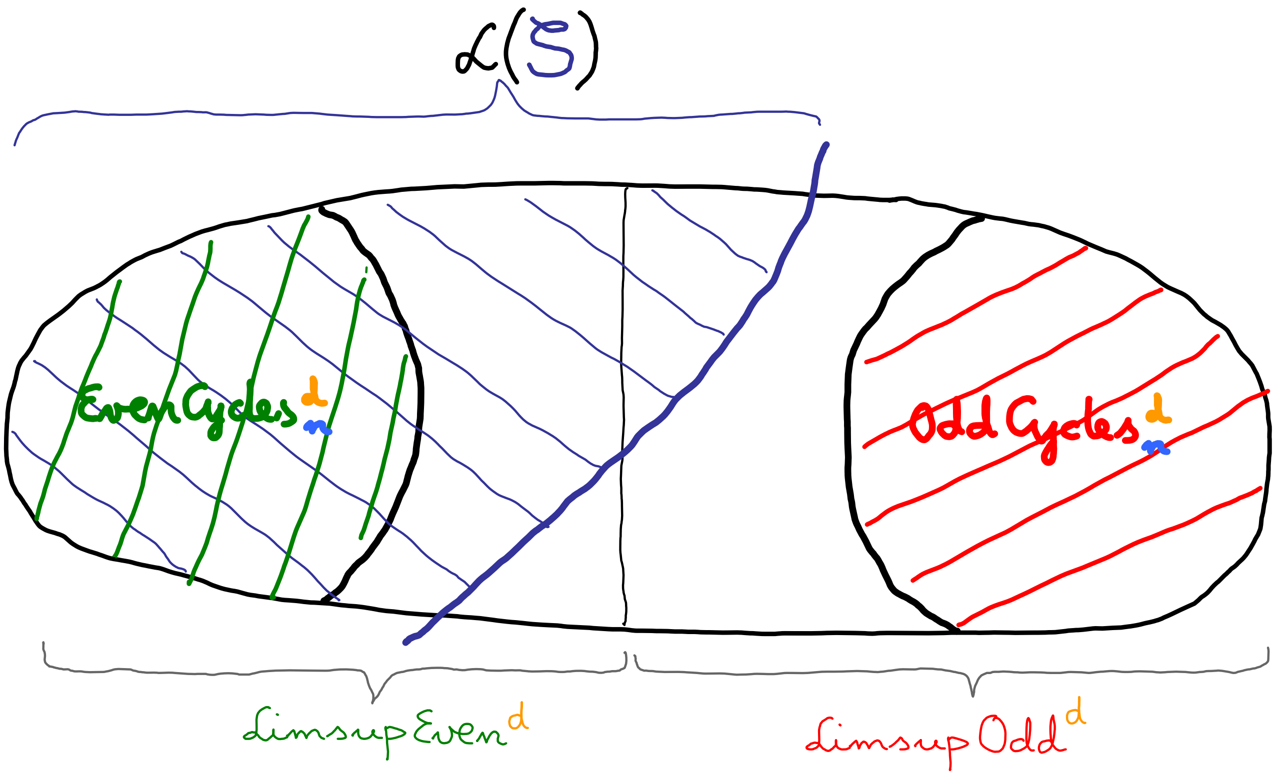
- $\text{LimsupEven}^d \subseteq \{1,2,\dots,d\}^\omega$: won by Even
- $\text{EvenCycles}^d_n \subseteq \{1,2,\dots,d\}^\omega$: arising from a game graph (with $\leq n$ vertices and $\leq d$ priorities) in which all cycles are even



SEPARATING (SAFETY) AUTOMATA

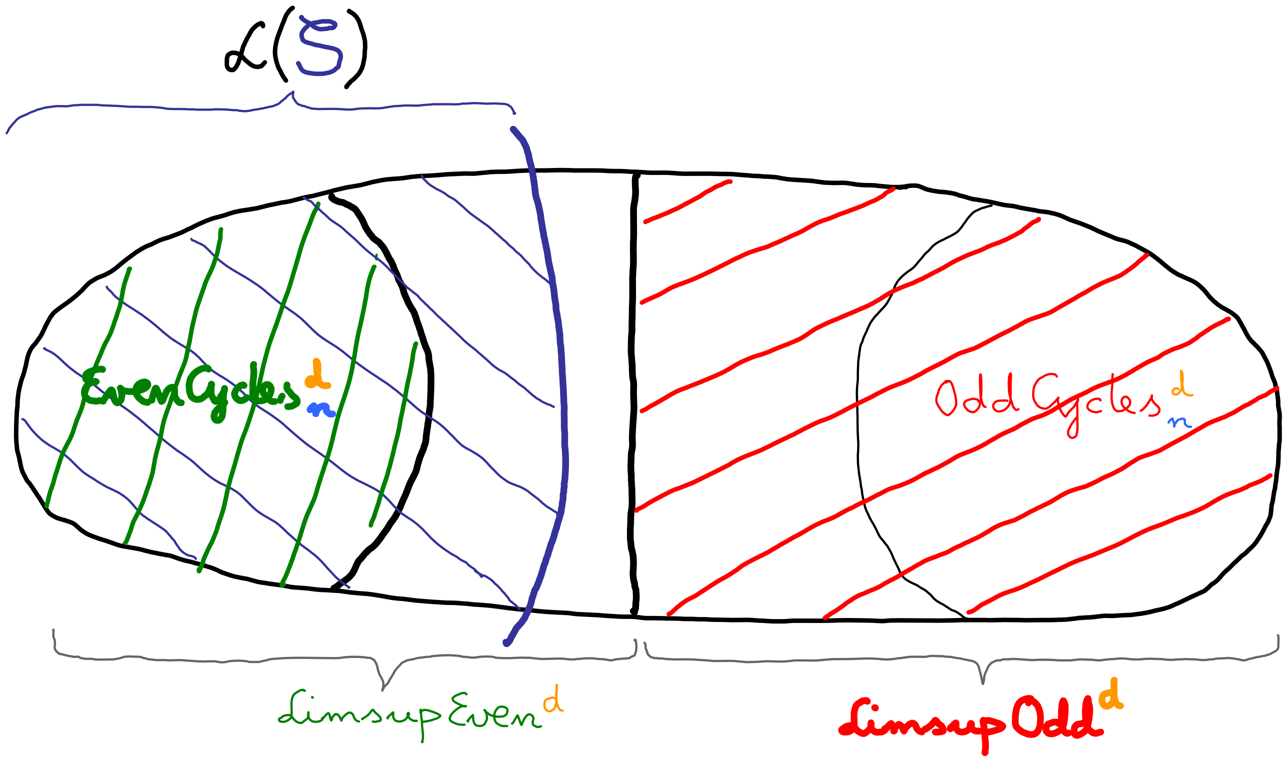
[Bojańczyk, Czerwiński 2018]

DEFINITION A finite (safety) automaton $\mathcal{S}$ is an (n,d)-separator if $\mathcal{L}(\mathcal{S})$ separates $\mathrm{EvenCycles}_n^d$ from $\mathrm{OddCycles}_n^d$



SEPARATING (SAFETY) AUTOMATA

DEFINITION A finite (safety) automaton $\mathcal{S}$ is a strong (n,d)-separator if $\mathcal{L}(\mathcal{S})$ separates $\mathrm{EvenCycles}^d_n$ from $\mathrm{LimsupOdd}^d$



THE SEPARATION APPROACH

The chained product $G \triangleright S$
is a safety game.

parity game

safety automaton

FACT If S is an (n,d)-separator
and the parity game G has $\leq n$ vertices and $\leq d$ priorities
then games G and $G \times S$ have the same winners

parity game

safety game

MULTI-COUNTER SEPARATOR OF SIZE $n^{d/2}$

[Bernet, Janin, Walukiewicz 2003]

- **States**: multi-counters $(c_{d-1}, c_{d-3}, \dots, c_3, c_1)$ s.t. $0 \leq c_p \leq n$
- **Initial state**: $(0, \dots, 0)$
- **Transitions**:

$$(c_{d-1}, \dots, c_{p+2}, c_p, c_{p-2}, \dots, c_1) \xrightarrow{p} \text{reject} \quad \text{if } c_p = n$$

$$(c_{d-1}, \dots, c_{p+2}, c_p, c_{p-2}, \dots, c_1) \xrightarrow{p} (c_{d-1}, \dots, c_{p+2}, c_p + 1, 0, \dots, 0)$$

$$(c_{d-1}, \dots, c_{p+2}, c_p, c_{p-2}, \dots, c_1) \xrightarrow{p-1} (c_{d-1}, \dots, c_{p+2}, c_p, 0, \dots, 0)$$

A QUASI-POLYNOMIAL SEPARATOR

THEOREM [Calude et al. 2017; Bojańczyk, Czerwiński 2019]
There is a strong (n,d)-separator of size $n^{\lg d + O(1)}$

- **States**: "play statistics" $(p_{\lg n}, p_{\lg n - 1}, \ldots, p_1, p_0) \in \{0,1,2,\ldots,d\}^{\lg n + 1}$

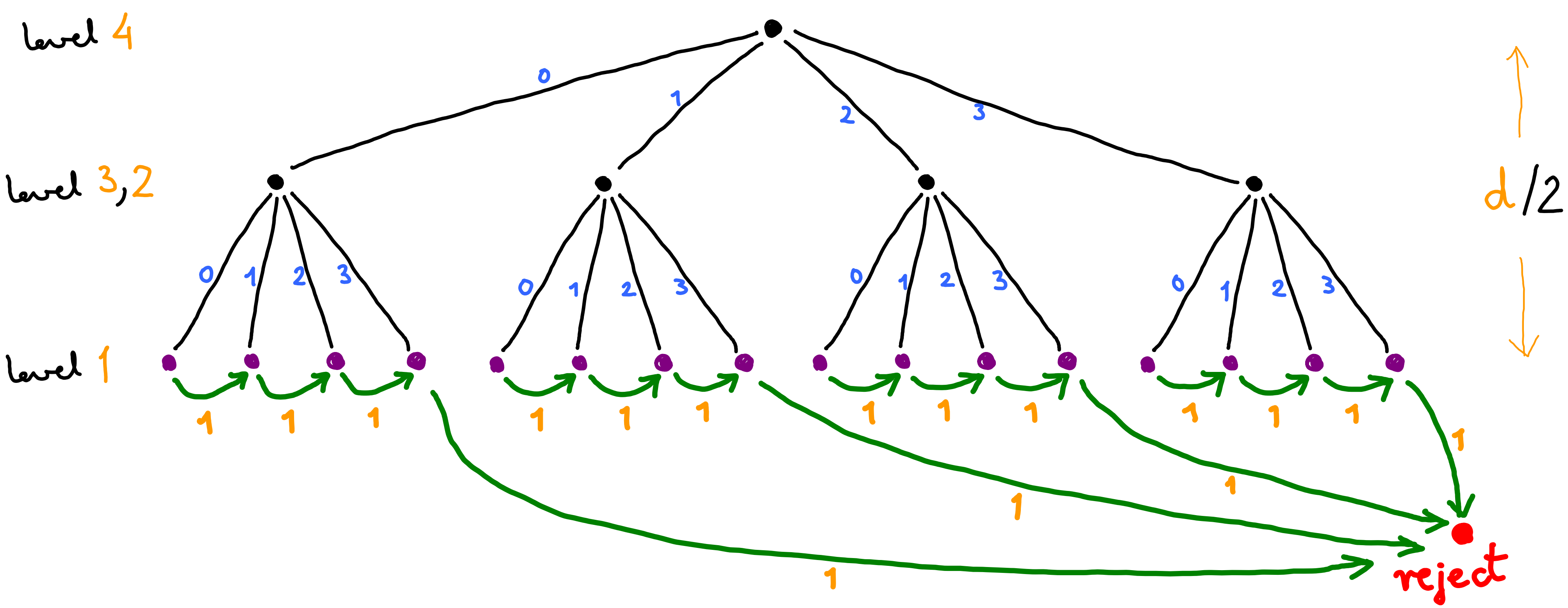
- **Transitions**: see [Calude et al. 2017] or [Bojańczyk, Czerwiński 2018]

$\lg d \cdot \lg n$-space **TM**

automaton of size $n^{\lg d + O(1)}$

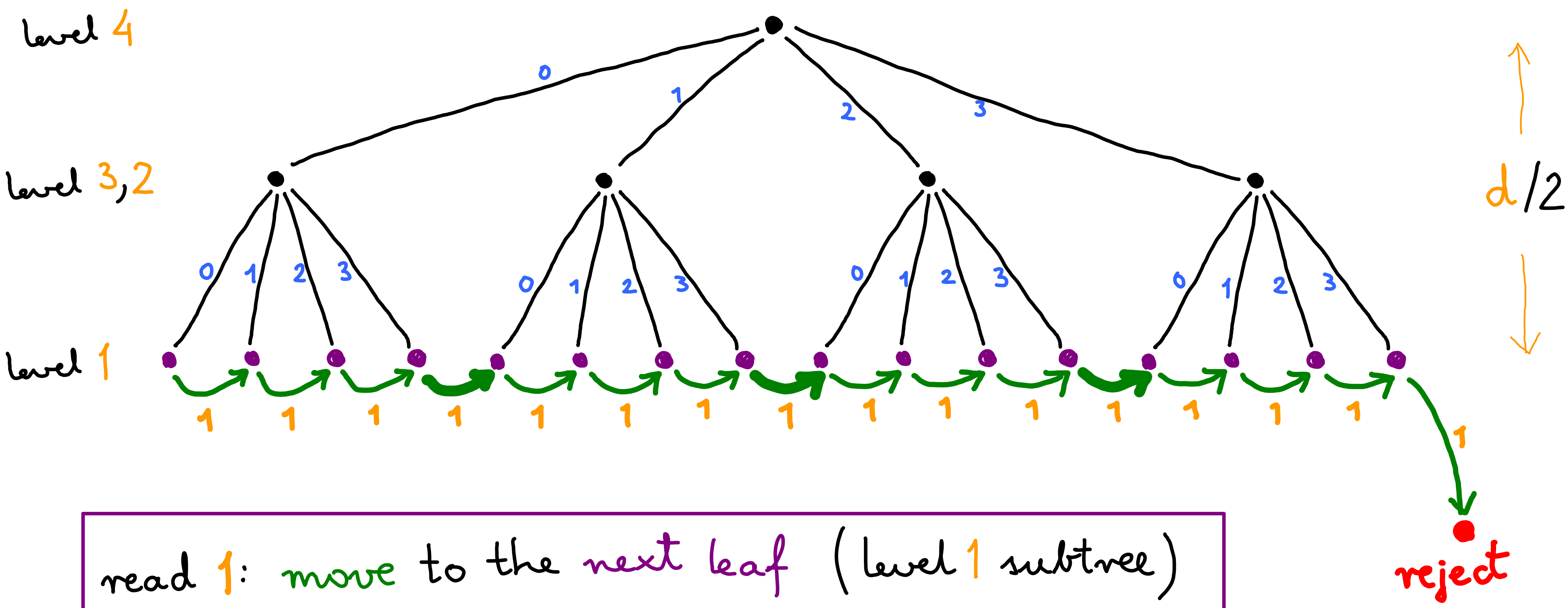
MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



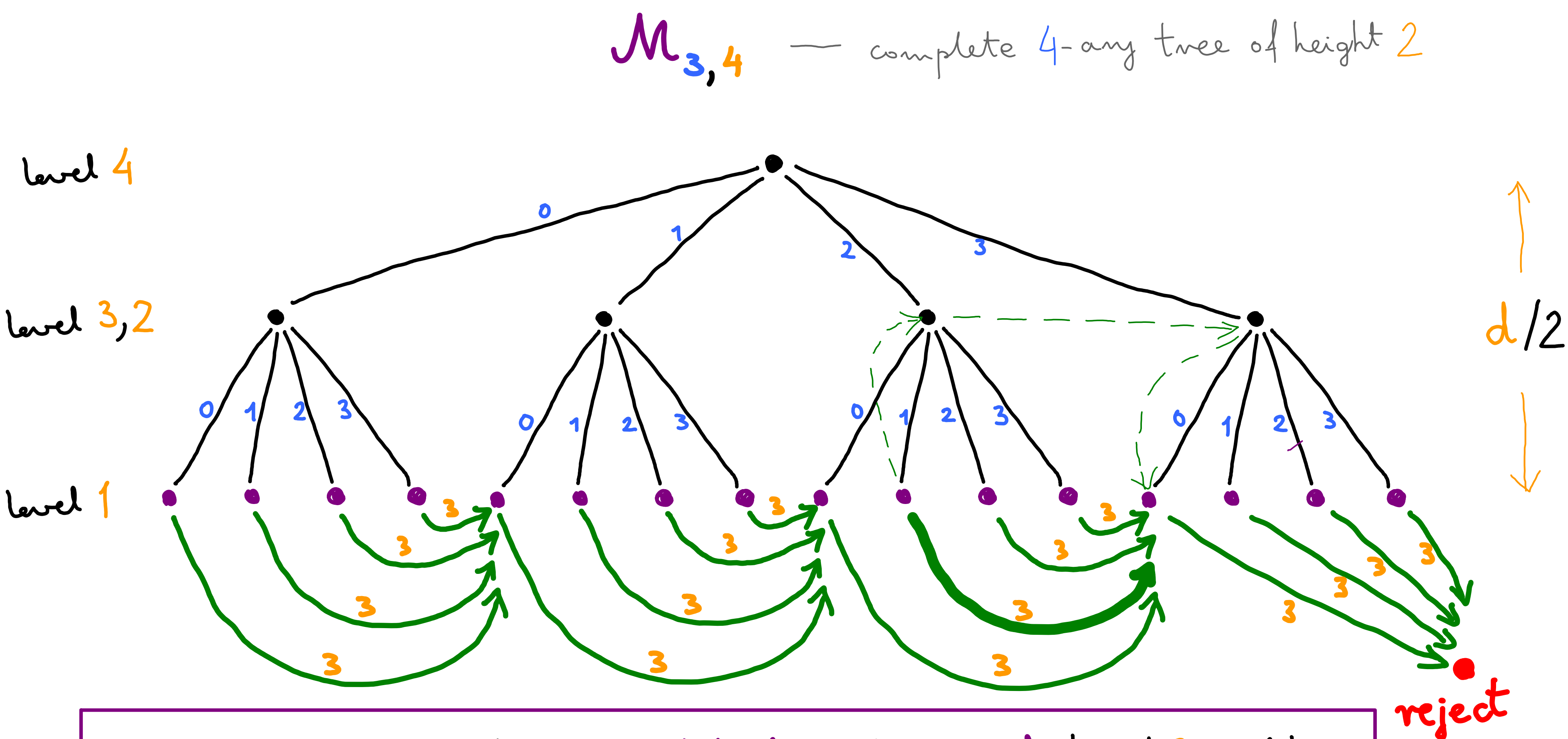
MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



read 1: move to the next leaf (level 1 subtree)

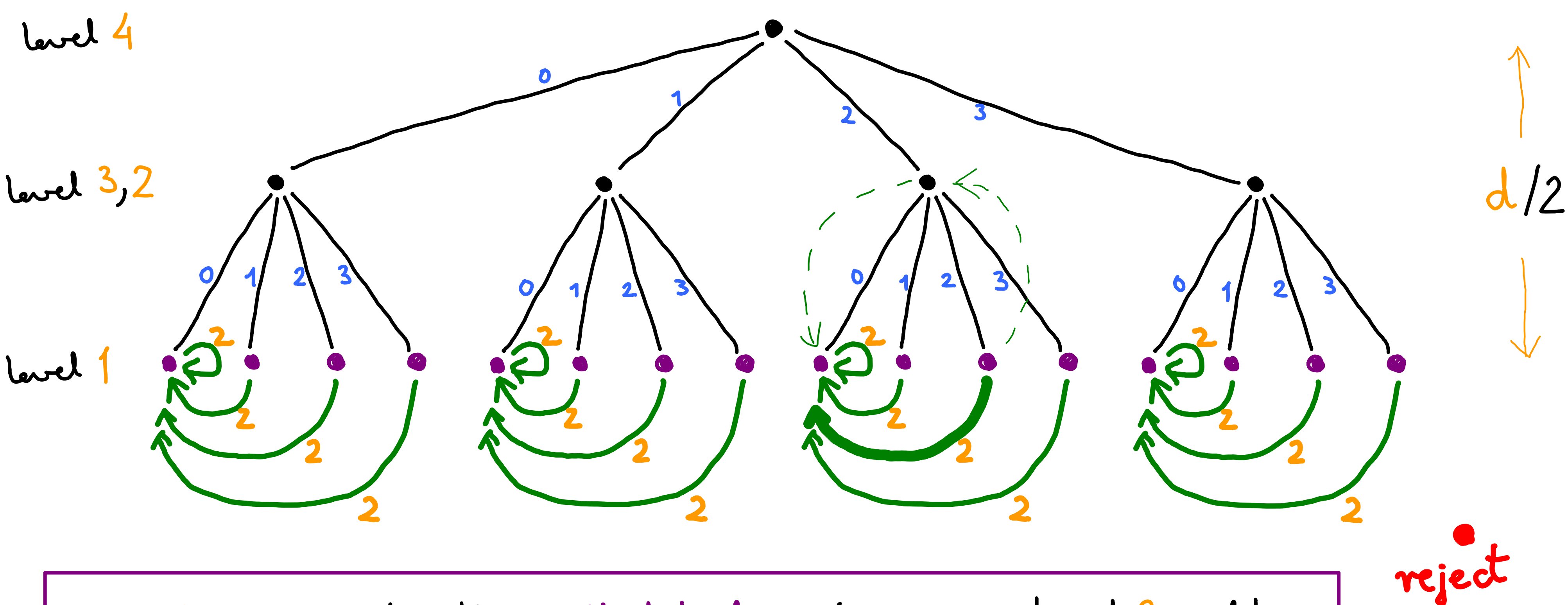
MULTI-COUNTERS AS AN ORDERED TREE



read 3: move to the smallest leaf in the next level 3 subtree

MULTI-COUNTERS AS AN ORDERED TREE

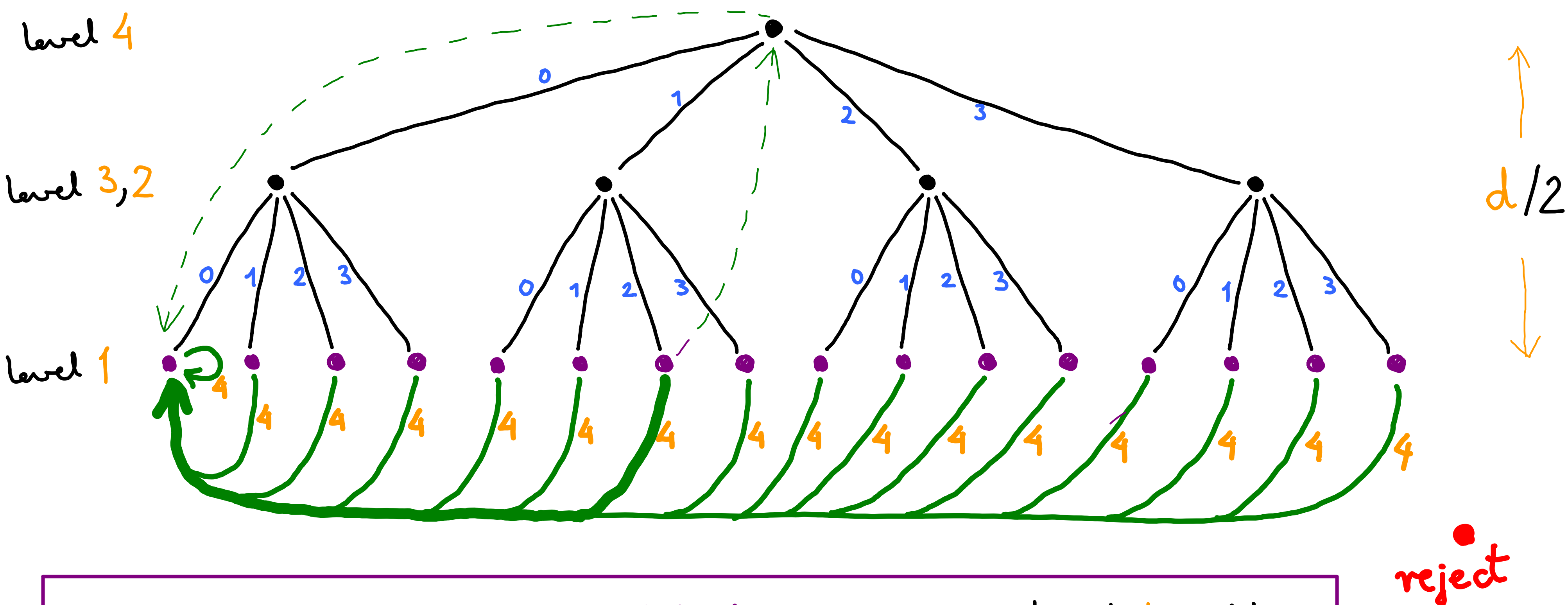
$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



read 2: move to the smallest leaf in the same level 2 subtree

MULTI-COUNTERS AS AN ORDERED TREE

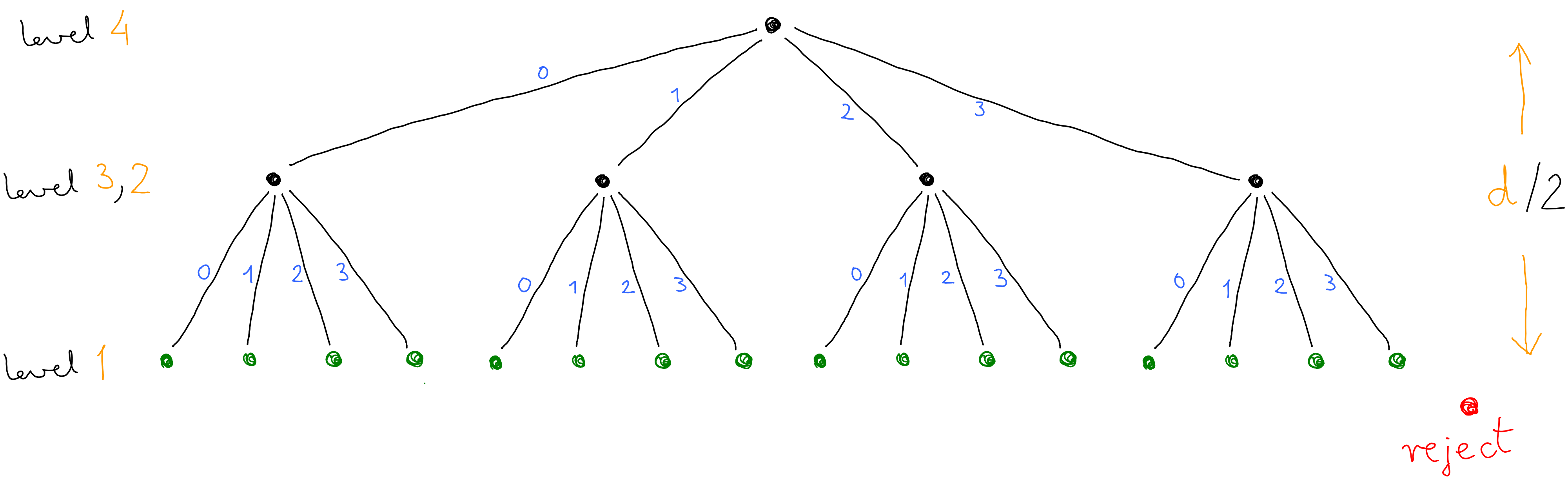
$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



read 4: move to the smallest leaf in the same level 4 subtree

MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



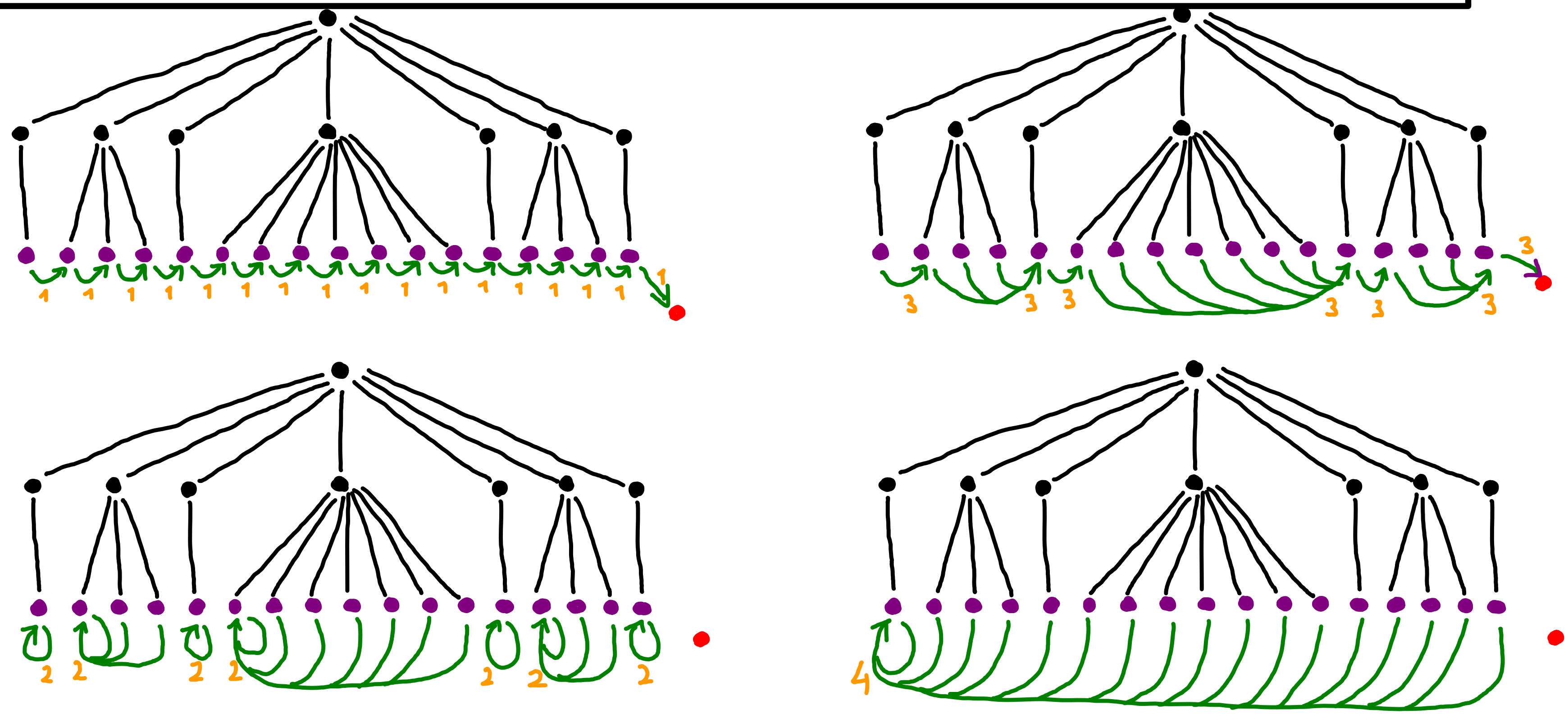
Question: Do we really need the complete $(n+1)$-ary tree of height $\frac{d}{2}$?

SEPARATING AUTOMATA FROM UNIVERSAL TREES

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

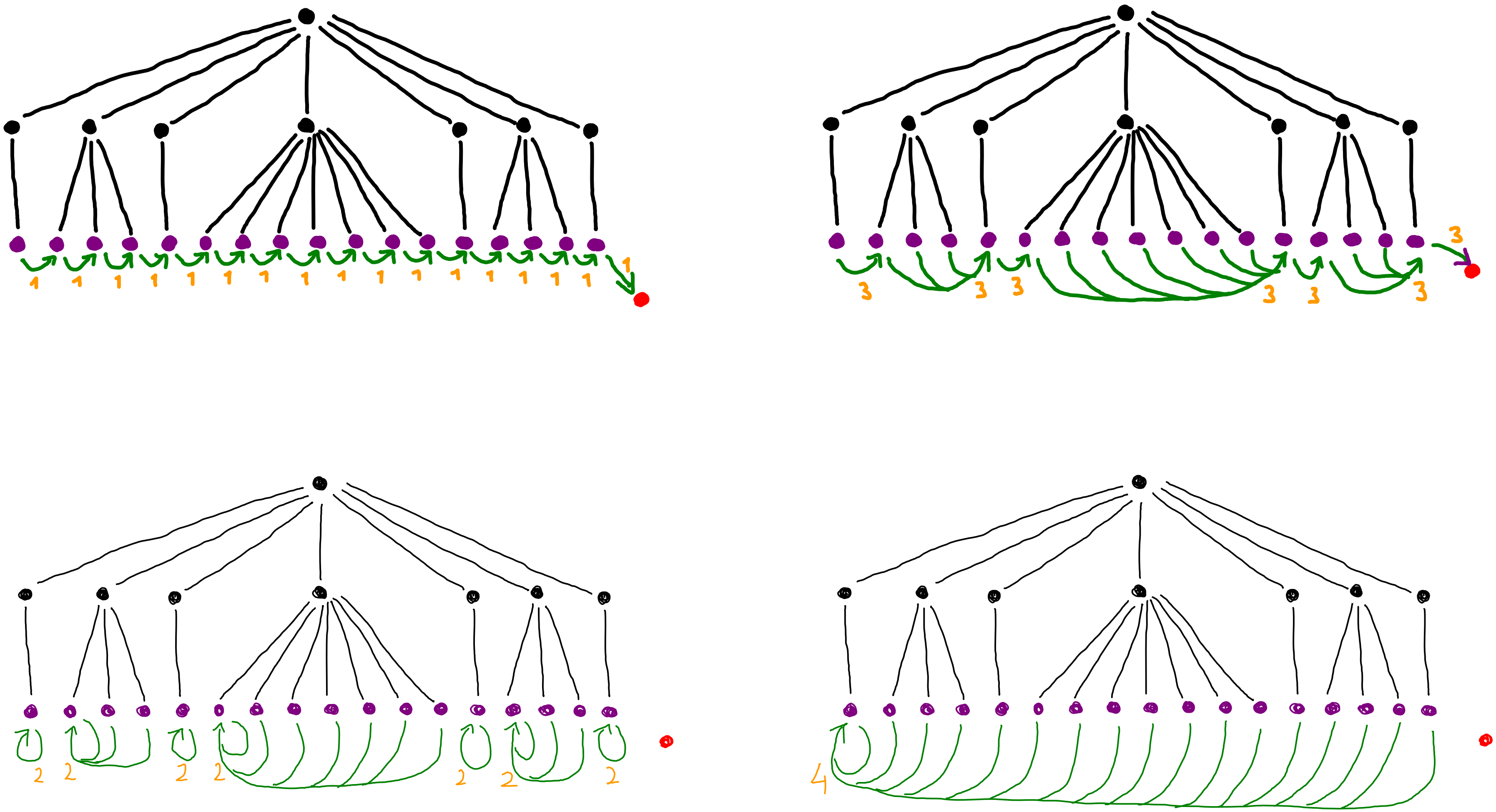
Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

$\mathcal{A}_{\mathcal{U}}$:



$\mathcal{A}_u$ REJECTS ALL WORDS IN Odd_d

$\mathcal{A}_u$:



$\mathcal{A}_{\mathcal{U}}$ ACCEPTS ALL WORDS IN $\mathrm{Even}_{n,d}$

- If all cycles in G are even then there is a tree witness

$$\mu : V \longrightarrow T \hookrightarrow \mathcal{U}$$

- Let $\mathcal{A}_{\mathcal{U}}$ run on G:

Invariant: $\boxed{\text{when } \mathcal{A}_{\mathcal{U}} \text{ visits vertex } v, \text{ its state is } \leq \mu(v)}$

UNIVERSAL TREES $\equiv$ SEPARATING AUTOMATA

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

States in every strong (n, d)-separator include all the leaves in an $(n, \frac{d}{2})$-universal tree

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

THEOREM [J., Lazić 2017]

There is an (n, h)-universal tree

of size $n \cdot \binom{\lg n + h}{h} = n^{\lg\left(\frac{h}{\lg n}\right) + O(1)}$

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

Every (n, h)-universal tree

is of size at least $\binom{\lg n + h - 1}{h - 1} \geqslant n^{\lg\left(\frac{h}{\lg n}\right) - 1}$

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

$L(n,h) \stackrel{\text{df}}{=}$ smallest # leaves in an (n,h)-universal tree

$$\boxed{\begin{aligned} L(n,1) &= n \\ L(1,h) &= 1 \end{aligned}}$$

Upper bound recurrence
$$\boxed{L(n,h) \leq L(n,h-1) + 2 \cdot L\left(\frac{n}{2},h\right)}$$

Lower bound recurrence
$$\boxed{L(n,h) \geq L(n,h-1) + L\left(\frac{n}{2},h\right)}$$

COROLLARY
$$\boxed{L(n,h) \geq \binom{\lg n + h - 1}{h-1}}$$

UNIVERSAL TREES AND SEPARATING AUTOMATA ARE QUASI-POLYNOMIAL

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

The sizes of smallest universal trees and of smallest separating automata are quasi-polynomial

UNIVERSAL TREES ≡ SEPARATING AUTOMATA

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

States in every strong (n, d)-separator include all the leaves in an $(n, \frac{d}{2})$-universal tree

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

THEOREM [J., Lazić 2017]

There is an $(n, \frac{d}{2})$-universal tree of size $n \cdot \binom{\lg n + \frac{d}{2}}{\lg n} = n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Every $(n, \frac{d}{2})$-universal tree is of size at least $\binom{\lg n + \frac{d}{2} - 2}{\lg n - 1} \geqslant n^{\lg\left(\frac{d}{\lg n}\right) - 2}$

ALTERNATING WEAK AUTOMATA FROM UNIVERSAL TREES

THEOREM

There is a translation
from alternating parity automata on words
(with n states and d priorities)
to alternating weak automata
whose state space blow-up is:

- $\Theta(n^d)$ [Kupferman, Vardi 2001]
- $n^{\Theta(\lg n \cdot \lg(d/\lg n))}$ [Boker, Lehtinen 2018]
- $\boxed{n^{\Theta(\lg(d/\lg n))}}$ [Daviaud, J., Lehtinen 2019]

LAZY TREE WITNESSES

THEOREM [Klarlund 1991]

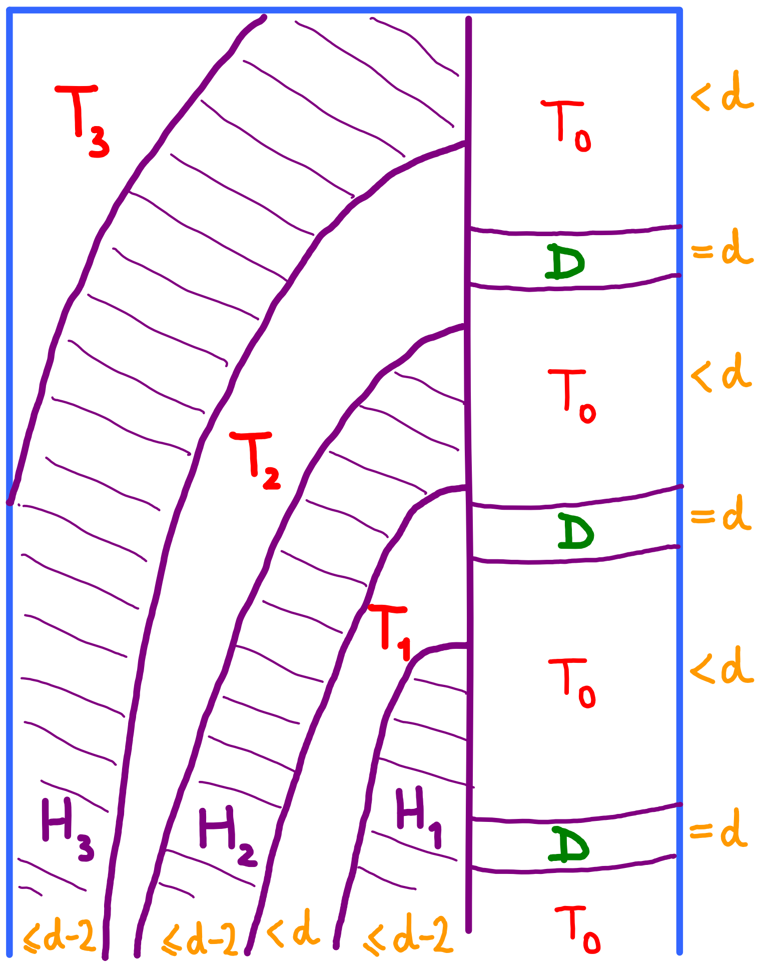
with n states and d priorities

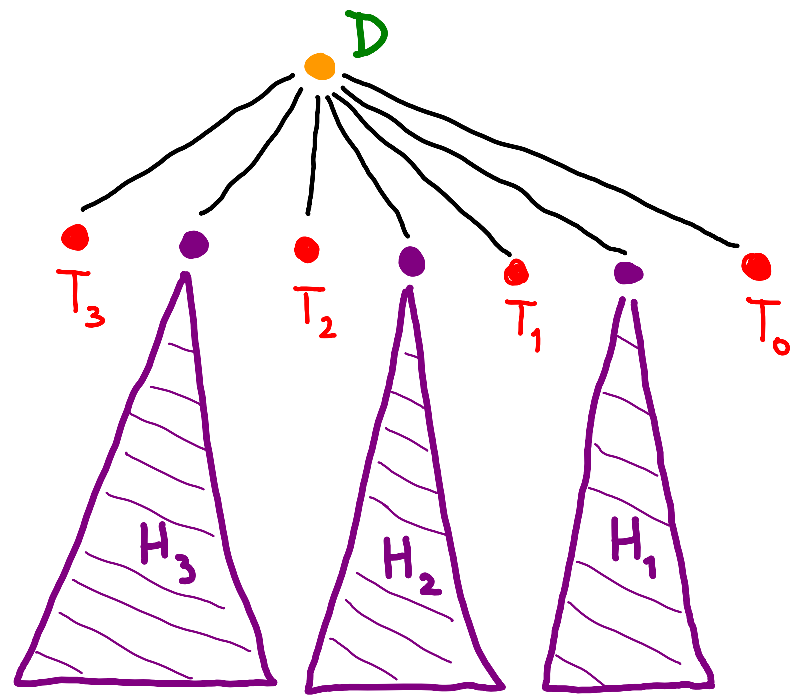
If a run dag of a parity automaton is accepting

then there is a lazy tree witness

of height $d/2$ and with $O(nd)$ nodes

HIERARCHICAL DECOMPOSITION OF ACCEPTING RUN DAGS





lazy tree witness

ALTERNATING WEAK AUTOMATA FROM UNIVERSAL TREES

guesses and certifies
a lazy tree witness
on an accepting run dag of $\mathcal{A}$

$$\mathcal{A} \xmapsto[\text{quasi-poly}]{\text{[Daviaud, J., Lehtinen 2019]}} \mathcal{A} \otimes \mathcal{U} \xmapsto[\text{quadratic}]{\text{[Kupferman, Vardi 2001]}} \mathcal{W}$$

$\mathcal{A}$: alternating parity automaton
- n states
- d priorities

$\mathcal{A} \otimes \mathcal{U}$: alternating Büchi automaton

$\mathcal{W}$: alternating weak automaton

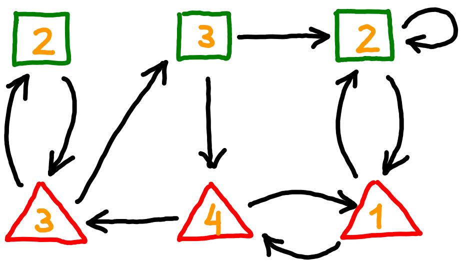
$\boxed{n^{\Theta(\lg(d/\lg n))} \text{ states}}$

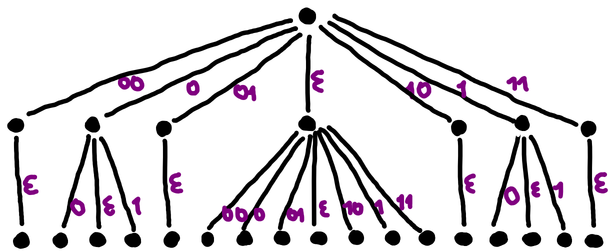
UNIVERSAL TREES

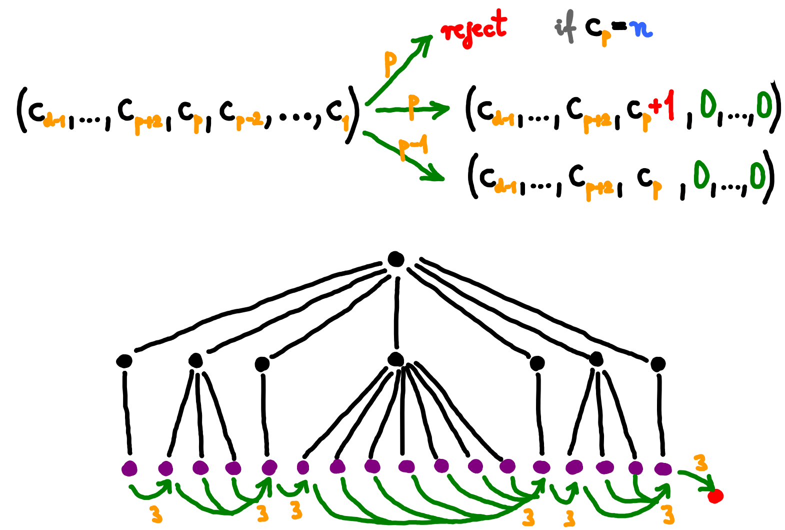
QUASI-POLYNOMIAL

PARITY GAMES

SEPARATING AUTOMATA







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