Concatenation hierarchies and separation

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Based on joint work with Thomas Place

Regular Languages, Concatenation, Separation

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Regular expressions for describing languages

Regular language = set of words built from:

$\emptyset$, $\{\varepsilon\}$, $\{a\}$	$\emptyset$, ε, a
Union	$L_1 + L_2$
Concatenation	$L_1 L_2$
Iteration (star)	L^*

$$L_1 L_2 = \{u_1 \cdot u_2 \mid u_1 \in L_1 \text{ and } u_2 \in L_2\}$$
$$L^* = \{\varepsilon\} \cup L \cup L^2 \cup L^3 \cdots$$

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Example

$\left((aa)^* + b\right)^*$ = Words over $\{a, b\}$ with even blocks of a's

Regularity is robust (1): words containing an 'a'

$$(a + b)^* a (a + b)^*$$

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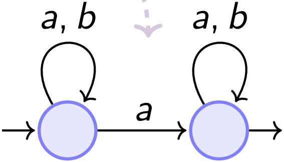

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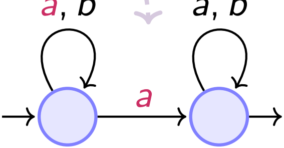

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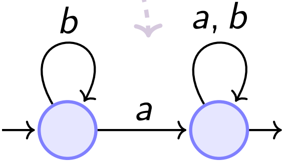


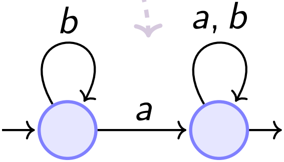
Regularity is robust (1): words containing an '*a*'

$$(a+b)^*a(a+b)^*$$

$$f : (a+b)^* \to (\{0,1\}, \times)$$

$$a \mapsto 0 \quad b \mapsto 1$$

$$f(bab) = f(b)f(a)f(b)$$



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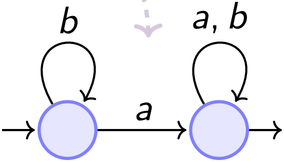
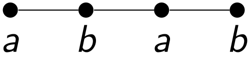


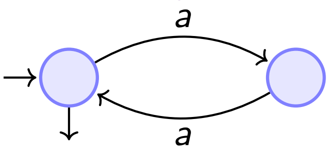
$$\exists x \; a(x)$$



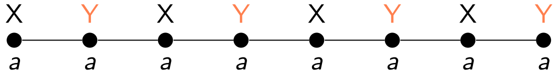
Regularity, robustness (2): words of even length

$$(aa)^*$$

$$f : a^* \to (\mathbb{Z}/2\mathbb{Z}, +)$$
$$a \mapsto 1$$



$$\forall X\, \forall Y$$
$$\big(\ \min \in \mathrm{X} \wedge \mathrm{Alternate}(\mathrm{X}, \mathrm{Y})\ \big) \Rightarrow \max \in Y$$



Robustness Theorem for Regular Word Languages

Kleene, Büchi, Elgot, Trakhtenbrot (60s)

For a language of finite words L, **TFAE**:

1. L is described by a regular expression ($\cup$, $\bullet$, $\star$).
2. L is recognized by an NFA.
3. L is recognized by a DFA.
4. L is described by an MSO sentence.
5. L is recognized by a morphism into a finite monoid.

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$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ **Built-in complement**

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5. L is recognized by a morphism into a finite monoid.
6. L is described by a generalized regular expression.

Built-in complement

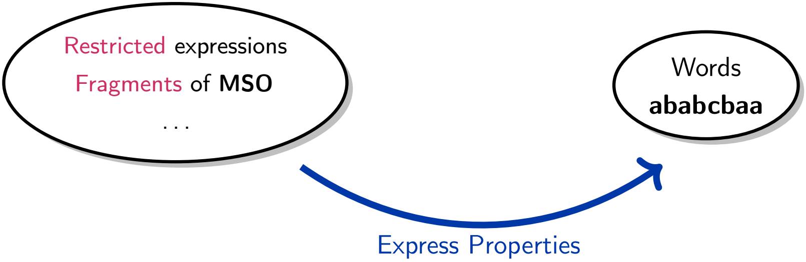
Generalized regular expression:

- Built from singletons, using $\cup$, $\bullet$, $\star$ and complement.

Goal: Understanding expressiveness of fragments

Descriptive Formalisms
Syntax

Structures
Semantics



We want to understand what a formalism can express

What does "understand" mean?

Meaningful fragments of regular languages

What languages can be expressed by a simple expression/formula?

What does simple mean? Several possible choices, e.g.:

- For (generalized) expressions: number of nested stars.
- For formulas: number of alternations between $\exists$ and $\forall$.

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What is the GSH of b^* over $A = \{a, b\}$?

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- Can we do better?

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Notation. GSH0 = Star-free = SF

Temporary conclusion

- Regular languages are easy, but complement is hard.
- Understanding a class = designing algorithms testing membership

Outline

1. The Membership Problem

2. Concatenation Hierarchies

3. Beyond Membership: Separation

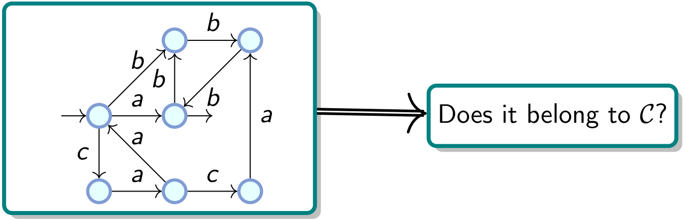
4. Generalizing Separation

The Membership Problem

Capturing expressiveness: seminal result

Membership problem for a class $\mathcal{C}$

- **INPUT** A (regular) language L.
- **QUESTION** Does L belong to $\mathcal{C}$?



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- Languages definable in **FO**.
- Languages of SH k.
- Languages of GSH $k \geq 1$.
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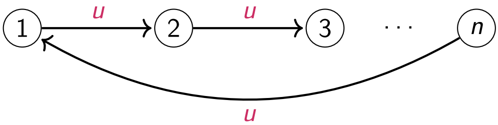

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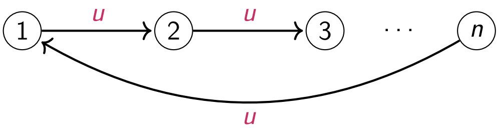

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Example

Minimal DFA of b^* has no counter	$\Rightarrow$	Star-free
Minimal DFA of $(a(bb)^*a)^*$ has a counter	$\Rightarrow$	Not star-free

Star-free expressions vs. first-order logic

First-order logic, with only the linear order '$<$'.

$$\begin{array}{ccccccccc} a & b & b & c & a & a & a & c & a \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

Word = sequence of labeled positions.

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Example: in the future of every 'a', there is a 'b'

$$\forall x \left(a(x) \Rightarrow \exists y \left((y > x) \wedge b(y) \right) \right)$$

Why is Schützenberger's theorem interesting?

1. Link with first-order logic **FO**.

Schützenberger '65, McNaughton, Papert '71

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SF	FO
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2. Provides an effective characterization of **SF** and **FO**.
3. Constructive proof $\Rightarrow$ normal forms for **SF**-expressions/**FO**.

Recap

- Understanding fragment $\mathcal{C}$ = solving $\mathcal{C}$-**membership**

- **Successful methodology for SF = FO**, reproduced
  - For other logical classes on words (eg, several restrictions of FO).
  - For other structures: infinite words, trees.

- Proof provides a **canonical representation** of languages in $\mathcal{C}$.

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- Understanding fragment $\mathcal{C}$ = solving $\mathcal{C}$-**membership**
- **Successful methodology for SF = FO**, reproduced
  - For other logical classes on words (eg, several restrictions of FO).
  - For other structures: infinite words, trees.
- Proof provides a **canonical representation** of languages in $\mathcal{C}$.
- Still, the methodology seems to **fail for some major classes**.

Concatenation Hierarchies

Concatenation hierarchies: Motivation

Definition of SF

- **SF** = smallest class such that:
  - $\emptyset \in \mathrm{SF}$ and $A^* \in \mathrm{SF}$.
  - SF is closed under Boolean operations over A^*.
  - SF is closed under marked concatenation $K, L \mapsto KaL$.

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Goal

Classify SF languages according to some complexity measure.

Complexity measures of SF/FO languages

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- For **SF**: number of alternations complement/concatenation.
- For **FO**: number of alternations between $\exists$ and $\forall$.

Two standard complexity hierarchies inside SF

Two classes built on top of $\mathcal{C}$

- Boolean closure $Bool(\mathcal{C})$.
- Polynomial closure $Pol(\mathcal{C})$ = closure under **marked concatenation** + **union**

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$$0 \xrightarrow{Pol} \frac{1}{2} \xrightarrow[Bool]{} 1 \xrightarrow{Pol} \frac{3}{2} \xrightarrow[Bool]{} 2 \xrightarrow{Pol} \frac{5}{2} \cdots\cdots$$

Brzozowski-Cohen and Straubing-Thérien hierarchies

Natural questions

- Are the hierarchies strict?
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What is known

1. Both are strict (Brzozowski-Knast 1978 + interleaving),

$$(a \cdots (a(ab)^* b)^* \cdots b)^*$$

2. Natural logical description wihin FO.
3. Membership for BC reduces to membership for ST.
4. Membership solved for only few levels.

Logical counterpart: quantifier alternation hierarchies

Intuition: marked concatenation corresponds to $\exists$.

- $\Sigma_i = \underbrace{\exists^*\forall^*\exists^*\forall^*\exists^*\cdots}_{\text{at most } i \text{ blocks } \exists^* \text{ or } \forall^*} \quad \varphi,$ (φ quantifier free).
- $\mathcal{B}\Sigma_i$ = Finite Boolean combinations of Σ_i.

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Quantifier Alternation Hierarchies

$$\Sigma_1 \subsetneq \mathcal{B}\Sigma_1 \subsetneq \Sigma_2 \subsetneq \mathcal{B}\Sigma_2 \subsetneq \Sigma_3 \subsetneq \mathcal{B}\Sigma_3 \subsetneq \Sigma_4 \cdots \subsetneq \cdots \mathrm{FO}$$

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$$\Sigma_1 \subsetneq \mathcal{B}\Sigma_1 \subsetneq \Sigma_2 \subsetneq \mathcal{B}\Sigma_2 \subsetneq \Sigma_3 \subsetneq \mathcal{B}\Sigma_3 \subsetneq \Sigma_4 \cdots\cdots \subsetneq \cdots\cdots \mathrm{FO}$$

Two versions

- Order signature: $<$ and $a()$.
- Enriched signature: $<$, $a()$, $+1$, $\min()$, $\max()$ and ε.

Logical counterpart: quantifier alternation hierarchies

Intuition: marked concatenation corresponds to $\exists$.

- $\Sigma_i = \underbrace{\exists^*\forall^*\exists^*\forall^*\exists^*\cdots}_{\text{at most } i \text{ blocks } \exists^* \text{ or } \forall^*} \quad \varphi, \qquad (\varphi \text{ quantifier free}).$
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$$\begin{array}{ccccccccccccccc} \Sigma_1 & \subsetneq & \mathcal{B}\Sigma_1 & \subsetneq & \Sigma_2 & \subsetneq & \mathcal{B}\Sigma_2 & \subsetneq & \Sigma_3 & \subsetneq & \mathcal{B}\Sigma_3 & \subsetneq & \Sigma_4 & \subsetneq & \mathrm{FO} \\ \frac{1}{2} & & 1 & & \frac{3}{2} & & 2 & & \frac{5}{2} & & 3 & & \frac{7}{2} & & \end{array}$$

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- Order signature: $<$ and $a()$.
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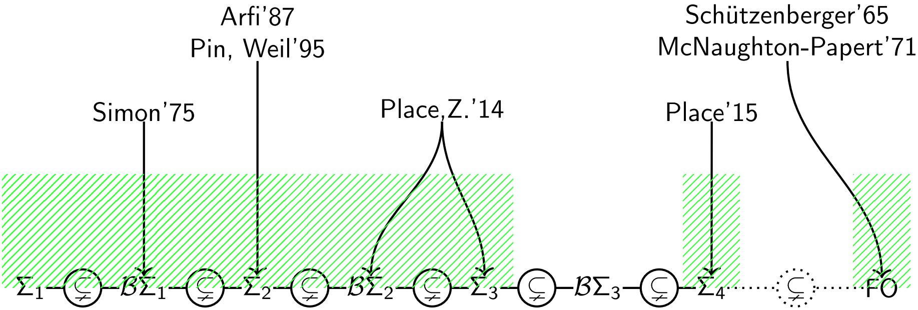
Logical Correspondence Theorem (Thomas '82, Perrin-Pin '86)

- Straubing-Thérien hierarchy = order quantifier alternation hierarchy.
- Brzozowski-Cohen hierarchy = enriched quantifier alternation hierarchy.

The membership problem for BC and ST hierarchies



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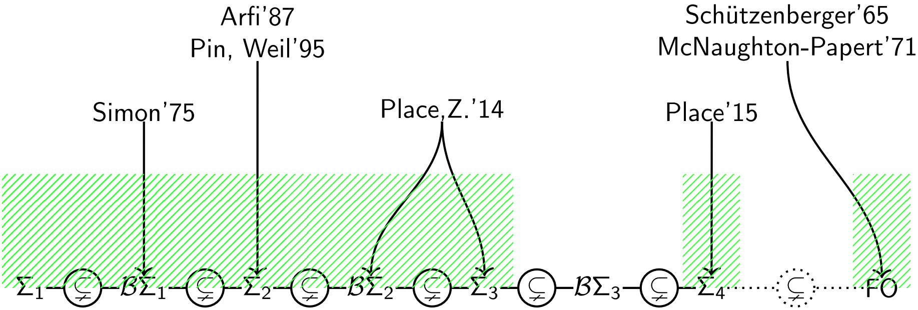

Enrichment Theorem for membership (Straubing, 1985 – Pin, Weil 1997)

Membership for a level in the enriched hierarchy (ie, BC)
reduces to
Membership for the same level in the order hierarchy (ie, ST).

Generalizations in two directions

1. Proofs are ad hoc for BC and ST: obtain generic theorems.
   For given $\mathcal{C}$, what about $Pol(\mathcal{C})$, $Bool(Pol(\mathcal{C}))$,...

2. Recent results via generalizations of membership:
   - separation,
   - covering.

Generic concatenation hierarchies

Generic pattern parametrized by the basis

- $\mathcal{C}[0]$ (basis) Boolean algebra in REG closed under left/right quotients.

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- $\mathcal{C}[n+1]$: close $\mathcal{C}[n+\frac{1}{2}]$ under Boolean operations.

$$0 \xrightarrow{Pol} \tfrac{1}{2} \xrightarrow[Bool]{} 1 \xrightarrow{Pol} \tfrac{3}{2} \xrightarrow[Bool]{} 2 \xrightarrow{Pol} \tfrac{5}{2} \cdots\cdots$$

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Examples

- Straubing-Thérien: $\mathcal{C}[0] = \{\emptyset, A^*\}$.
- Brzozowski-Cohen: $\mathcal{C}[0] = \{\emptyset, \{\varepsilon\}, A^*, A^+\}$.
- Pin-Margolis: $\mathcal{C}[0] =$ group languages.

Generic Hierarchies

Natural questions

- Are the hierarchies strict?
- Logical description of each level?
- What is known about membership?

Strictness of generic hierarchies

Strictness Theorem (Place, Z. '17)

Any hierarchy whose basis is finite is **strict**.

Generic logical correspondence

Logical Correspondence Theorem (Place, Z. '17)

For any basis $\mathcal{C}$, there is a natural set $\mathcal{S}$ of first order predicates, st.

Concatenation hierarchy of basis $\mathcal{C}$

=

Quantifier alternation hierarchy over signature $\mathcal{S}$

Generalizes the correspondences discovered for BC and ST hierarchies.

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For each $L \in \mathcal{C}$, add 4 predicates in addition to $<$ and $a(), b(), \dots$

- $w \models I_L(x, y)$ when $x < y$ and $w]x, y[\in L$ (*Infix*).

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- $w \models P_L(y)$ when $w[1, y[\in L$ (*P*refix).
- $w \models S_L(x)$ when $w]x, n] \in L$ (*S*uffix).

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- $w \models W_L$ when $w \in L$ (*Whole word*).

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For any finite basis $\mathcal{C}$, levels $\frac{1}{2}, 1, \frac{3}{2}, \quad \frac{5}{2}$ have decidable membership.

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Languages in ST $\left[\frac{3}{2}\right]$ (Pin and Straubing '85)

Languages of level ST $\left[\frac{3}{2}\right]$ are unions of languages of the form $B_0^* a_1 B_1^* \cdots a_n B_n^*$

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Recap

- Generic construction process for concatenation hierarchies.
- Generic logical correspondence.
- Generic strictness theorem.
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Nice idea, Henckell and Rhodes '88

Prove more on $\mathcal{C}$ to recover membership decidability for $\mathrm{Op}(\mathcal{C})$.

Nice statement, Almeida '96

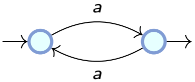
Almeida'96: a problem introduced by Henckell can be formulated as **separation**.

Beyond membership: Separation

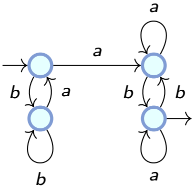
Decide the following problem:

Take **2** regular languages L_1, L_2

L_1



L_2



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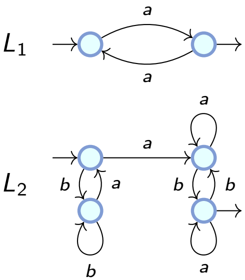

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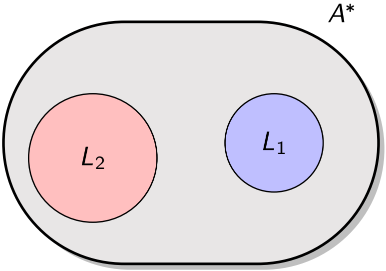

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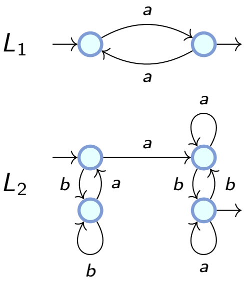

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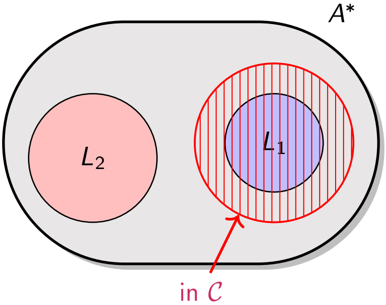

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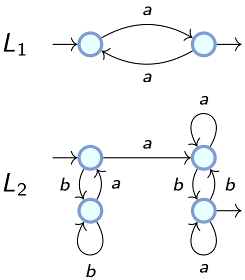

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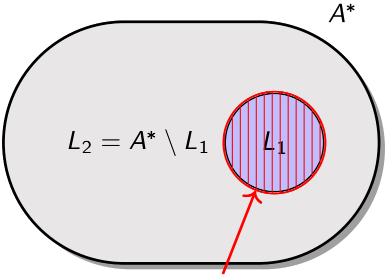
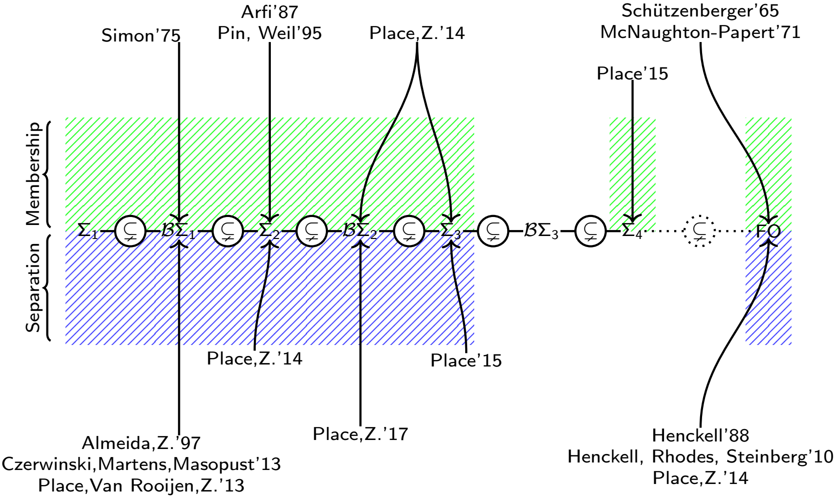


$\mathcal{C}$-separable from complement
$\Leftrightarrow$
in $\mathcal{C}$

Membership can be formally reduced to separation

Separation for classical hierarchies



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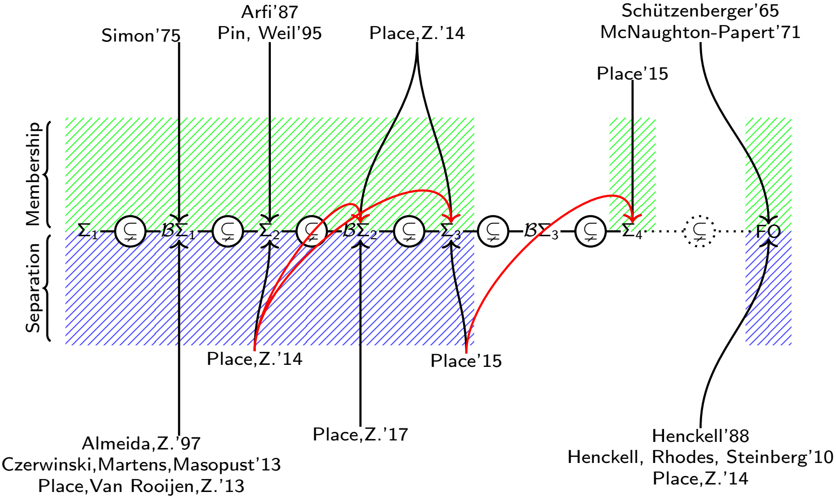

Some membership algorithms come from separation algorithms for simpler levels

Separation results for generic hierarchies

Generic Separation Theorem (Place, Z. '17, Place '15)

In any hierarchy of **finite basis**, separation is decidable for levels $\frac{1}{2}, 1, \frac{3}{2}$.

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Corollary (by Alphabet trick + Enrichment)

Levels $\frac{1}{2}, 1, \frac{3}{2}$, 2 and $\frac{5}{2}$ have decidable separation in ST and BC hierarchies.

The Jump Theorem

Jump Theorem for quotienting lattices (Place, Z. '15)

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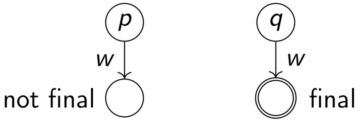

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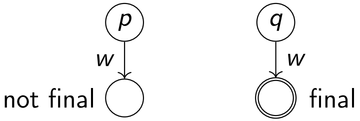

where $L_{p,q}$ is **not $\mathcal{C}\left[\mathbf{n} - \frac{1}{2}\right]$-separable** from $L_{p,p} \cap L_{q,q}$

$$L_{p,q} = \{w \mid p \xrightarrow{w} q\}$$

Recap

Current knowledge is captured by these 3 generic results:

1. Separation theorem
   $\mathcal{C}$ finite $\Rightarrow$ separation decidable for $Pol(\mathcal{C})$, $BPol(\mathcal{C})$, and $Pol(BPol(\mathcal{C}))$.
   In particular, unable to deal with 2 levels of complement.

2. Jump theorem
   $\mathcal{C}$-separation decidable $\Rightarrow$ $Pol(\mathcal{C})$-membership decidable.

3. Enrichment theorem.

Generalizing separation

Beyond Separation: Covering

- If $\mathcal{A}$ is the minimal DFA for L

$$L \in \mathcal{C} \qquad \text{iff} \qquad \forall p, q, \qquad L_{p,q} \in \mathcal{C}$$

Comes from reasonable closure properties of usual classes.

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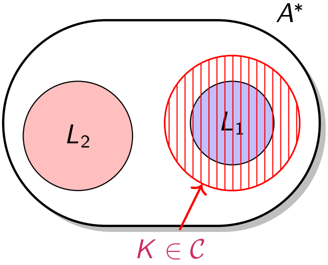
- We should actually consider a **set** of languages as **input**.
$\Longrightarrow$ natural to extend separation to **several input languages**.

The Covering Problem

- **Recall:** L_1, L_2 $\mathcal{C}$-separable if

$$\exists K \in \mathcal{C}, \qquad L_1 \subseteq K \text{ and } L_2 \cap K = \emptyset$$



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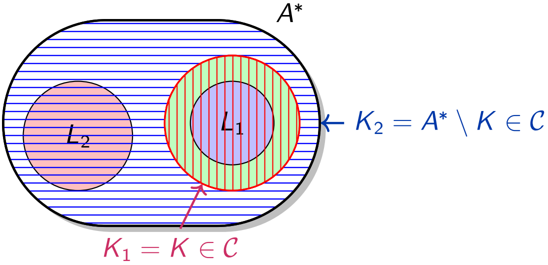

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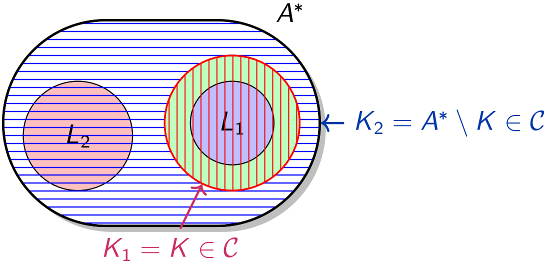

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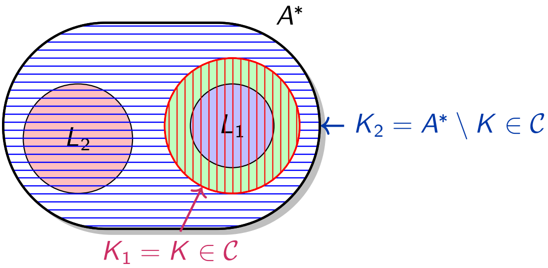
$\}$ $\{K_1, K_2\}$ **covers** $\{L_1, L_2\}$...



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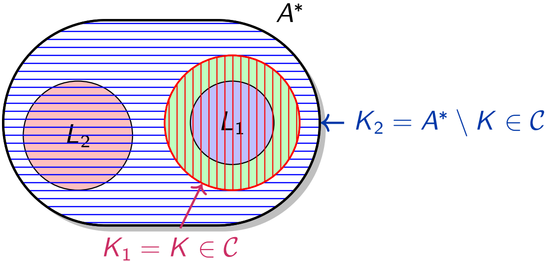

$\mathcal{C}$-Covers

- $\mathbf{L} = \{L_1, \ldots, L_n\}$ = set of languages.
- $\mathcal{C}$-**cover** of $\mathbf{L}$ = finite set of languages $\mathbf{K} = \{K_1, \ldots, K_m\}$ from $\mathcal{C}$ st.

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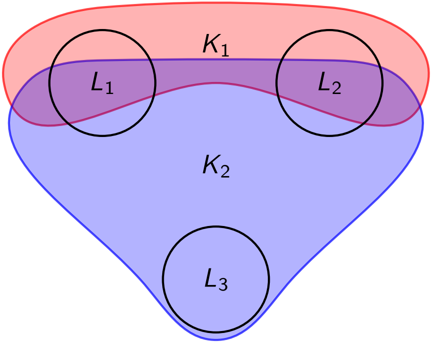
- **Goal**: Measure **how good** a cover is at "separating" input set $\mathbf{L}$.
  We define the **imprint** of $\mathbf{K}$ on $\mathbf{L}$ for this.

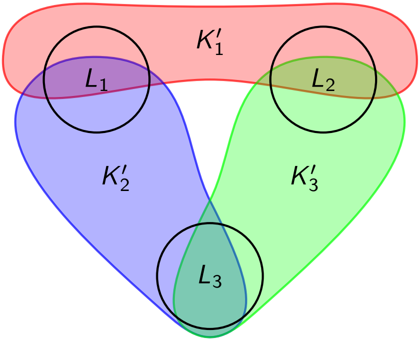
Imprint = Quality of a $\mathcal{C}$-Cover — Example 1



$\mathcal{C}$-Cover $\mathbf{K} = \{K_1, K_2\}$

Imprint $\mathcal{I}[\mathbf{L}](\mathbf{K}) = \Big\{\text{all subsets of } \{L_1, L_2, L_3\}\Big\}$

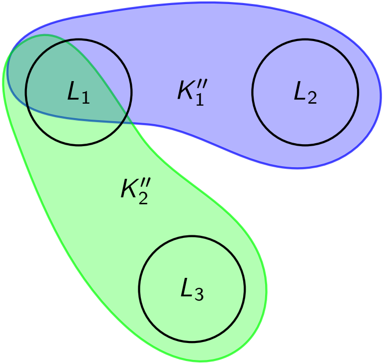
Imprint = Quality of a $\mathcal{C}$-Cover — Example 2 (better)



$\mathcal{C}$-Cover $\mathbf{K}' = \{K_1', K_2', K_3'\}$

Imprint $\mathcal{I}[\mathbf{L}](\mathbf{K}') = \Big\{$all subsets of $\{L_1, L_2, L_3\}$ but $\{L_1, L_2, L_3\}\Big\}$

Imprint = Quality of a $\mathcal{C}$-Cover — Example 3 (even better)



$\mathcal{C}$-Cover $\mathbf{K}'' = \{K_1'', K_2''\}$

Imprint $\mathcal{I}[\mathbf{L}](\mathbf{K}'') = \Big\{$all subsets of $\{L_1, L_2, L_3\}$ but $\{L_1, L_2, L_3\}$ and $\{L_2, L_3\}\Big\}$

Recap: Quality of a $\mathcal{C}$-Cover

- Goal: Measure **how good** a cover is at “separating” an input set.
- Captured by **imprint** of **K** on **L**.
- The smaller the imprint, the better.

Optimal $\mathcal{C}$-covers

- A $\mathcal{C}$-cover $\mathbf{K}$ is optimal if **it has minimal imprint**.

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- $\mathcal{C}$ = Boolean algebra generated by languages A^*aA^* for $a \in A$.
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- Trivial, but non-constructive proof.

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- $\mathcal{C}$-Optimal imprints capture more information than $\mathcal{C}$-separation.

The $\mathcal{C}$-Covering Problem

$\mathcal{C}$-**Optimal imprint:** $\mathcal{I}_{\mathcal{C}}[\mathbf{L}] \stackrel{\text{def}}{=} \mathcal{I}[\mathbf{L}](\mathbf{K})$ for any optimal $\mathcal{C}$-cover $\mathbf{K}$ of $\mathbf{L}$.

Theorem (Place Z.'16)

Let $\mathcal{C}$ be a Boolean algebra and $\mathbf{L}$ be a finite set of languages.
Given $L_1, L_2 \in \mathbf{L}$, **TFAE**:

1. L_1 and L_2 are $\mathcal{C}$-separable.
2. $\{L_1, L_2\} \notin \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$.

What did we gain?

Contrary to separation information alone, $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$,

- has nice, generic properties,
- can be computed for all classes where separation is known decidable.

Conclusion

- Overview of membership and separation for concatenation hierarchies.
- Membership is a natural problem, but too rigid as a setting.
- Separation more flexible, easier to cope with for low levels.
- Often requires solving even more general problems.

Thanks!