Beyond NP Revolution

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CAALM Workshop

Turing, 1950: "Opinions may vary as to the complexity which is suitable in the child machine. One might try to make it as simple as possible consistent with the general principles. Alternatively one might have a complete system of logical inference "built in". In the latter case the store would be largely occupied with definitions and propositions. The propositions would have various kinds of status, e.g., well-established facts, conjectures, mathematically proved theorems, statements given by an authority, expressions having the logical form of proposition but not a belief-value"

Aristotle's Syllogisms

- All men are mortal
- Socrates is a man

Socrates is a mortal

Boole's Symbolic Logic

Boole's insight: Aristotle's syllogisms are about *classes* of objects, which can be treated *algebraically*.

> *"If an adjective, as 'good', is employed as a term of description, let us represent by a letter, as y, all things to which the description 'good' is applicable, i.e., 'all good things', or the class of 'good things'. Let it further be agreed that by the combination xy shall be represented that class of things to which the name or description represented by x and y are simultaneously applicable. Thus, if x alone stands for 'white' things and y for 'sheep', let xy stand for 'white sheep'.*

Boolean Satisfiability (SAT); Given a Boolean expression, using "and" ($\wedge$) "or", ($\vee$) and "not" ($\neg$), *is there a satisfying solution* (an assignment of 0's and 1's to the variables that makes the expression equal 1)?

Example:

$$(\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4) \wedge (x_3 \vee x_1 \vee x_4)$$

Solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$

History:

- William Stanley Jevons, 1835-1882: "I have given much attention, therefore, to lessening both the manual and mental labour of the process, and I shall describe several devices which may be adopted for saving trouble and risk of mistake."
- Ernst Schröder, 1841-1902: "Getting a handle on the consequences of any premises, or at least the fastest method for obtaining these consequences, seems to me to be one of the noblest, if not the ultimate goal of mathematics and logic."

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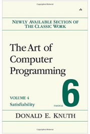
- Cook, 1971, Levin, 1973: Boolean Satisfiability is NP-complete.
- Clay Institute, 2000: $1M Award!

Algorithmic Boolean Reasoning: Early History

- Davis and Putnam, 1958: "Computational Methods in The Propositional calculus", unpublished report to the NSA
- Davis and Putnam, JACM 1960: "A Computing procedure for quantification theory"
- Davis, Logemman, and Loveland, CACM 1962: "A machine program for theorem proving"
- Marques-Silva and Sakallah 1996, Zhang et al. 2001, Een and Sorensson 2003, Simon and Audemard 2009, Liang et al 2016
  CDCL = conflict-driven clause learning
  - Smart but cheap branching heuristics
  - Quick detection of unit clauses
  - Conflict Driven Clause Learning
  - Restarts

(Laurent Simon's talk will give a behind the scenes peek into SAT revolution)

Modern SAT solvers are able to deal routinely with practical problems that involve many thousands of variables, although such problems were regarded as hopeless just a few years ago. (Donald Knuth, 2016)



The Tale of Triumph of SAT Solvers

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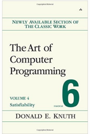

Industrial usage of SAT Solvers: Hardware Verification, Planning, Genome Rearrangement, Telecom Feature Subscription, Resource Constrained Scheduling, Noise Analysis, Games, $\cdots$

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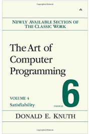

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Now that SAT is "easy", it is time to look beyond satisfiability

Constrained Counting and Sampling

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$

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- Constrained Counting: Determine $|\mathsf{Sol}(F)|$
- Constrained Sampling: Randomly sample from $\mathsf{Sol}(F)$ such that $\Pr[\text{y is sampled}] = \frac{1}{|\mathsf{Sol}(F)|}$

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  - $F := (X_1 \vee X_2)$
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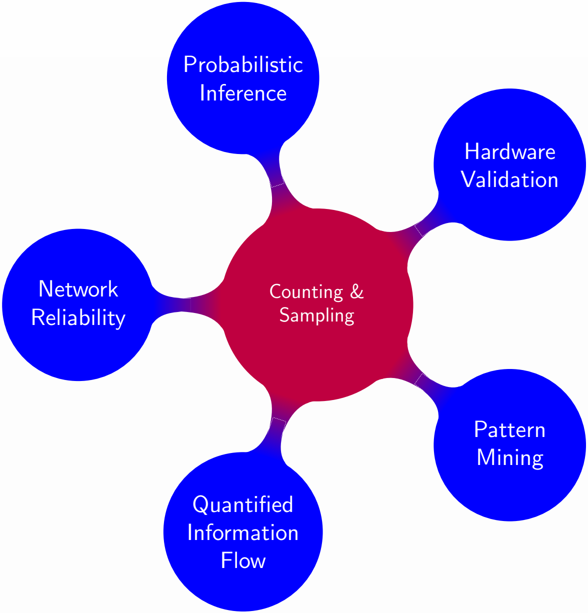
Applications across Computer Science



Today's Menu

Network Reliability

Probabilistic Inference

Part I Constrained Counting

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Part I Constrained Counting Hashing Framework

Can we reliably predict the effect of natural disasters on critical infrastructure such as power grids?

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Can we predict likelihood of a region facing blackout?

Reliability of Critical Infrastructure Networks

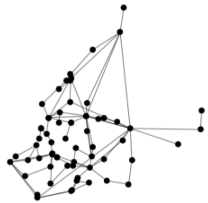

Figure: Plantersville, SC

- $G = (V, E)$; source node: s and terminal node t
- failure probability $g : E \rightarrow [0, 1]$
- Compute Pr[s and t are disconnected]?

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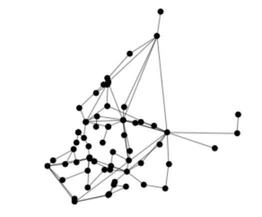

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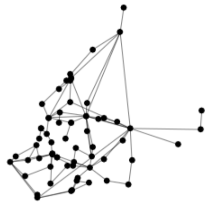

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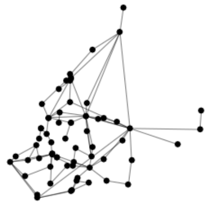

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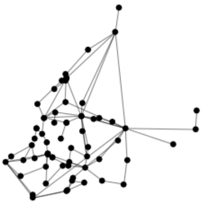

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(DMPV, AAAI 17, RESS 2018)

Probabilistic Models

Patient	Cough	Smoker	Asthma
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Bob	0	0	1
Randee	1	0	0
Tova	1	1	1
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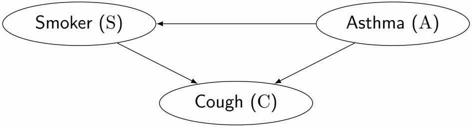

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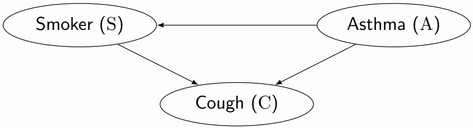

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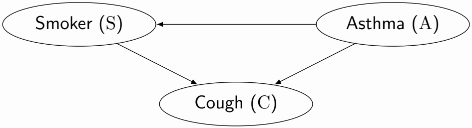

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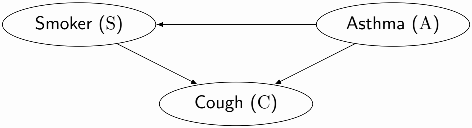

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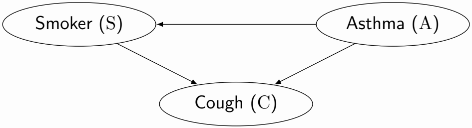

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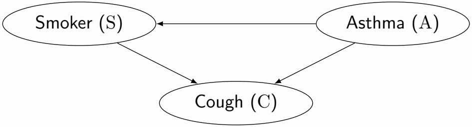

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Constrained Counting

(Roth, 1996)

Prior Work

Strong guarantees but poor scalability

- Exact counters (Birnbaum and Lozinskii 1999, Jr. and Schrag 1997, Sang et al. 2004, Thurley 2006)
- Hashing-based approach (Stockmeyer 1983, Jerrum Valiant and Vazirani 1986)

Weak guarantees but impressive scalability

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How to bridge this gap between theory and practice?

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- ApproxCount($F, W, \varepsilon, \delta$): Compute C such that

$$\Pr[\frac{W(F)}{1+\varepsilon} \leq C \leq W(F)(1+\varepsilon)] \geq 1-\delta$$

From Weighted to Unweighted Counting

Boolean Formula F and weight function $W : \{0,1\}^n \to \mathbb{Q}^{\geq 0}$

Boolean Formula F'

$$W(F) = c(W) \times |\mathsf{Sol}(F')|$$

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How do we estimate $|\mathsf{Sol}(F')|$?

Counting in Chennai

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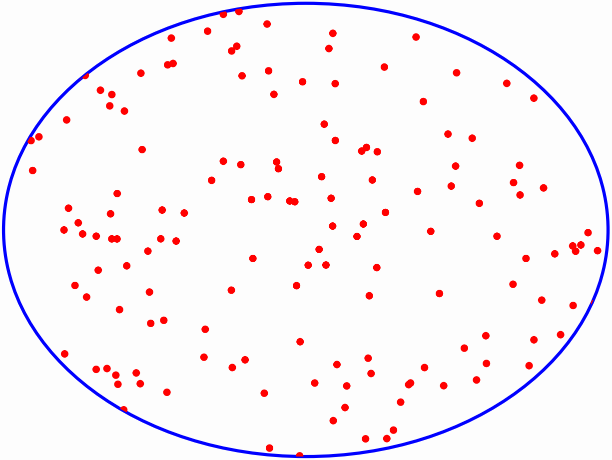
  - Potentially 2^n queries

Can we do with lesser # of SAT queries – $\mathcal{O}(n)$ or $\mathcal{O}(\log n)$?

As Simple as Counting Dots



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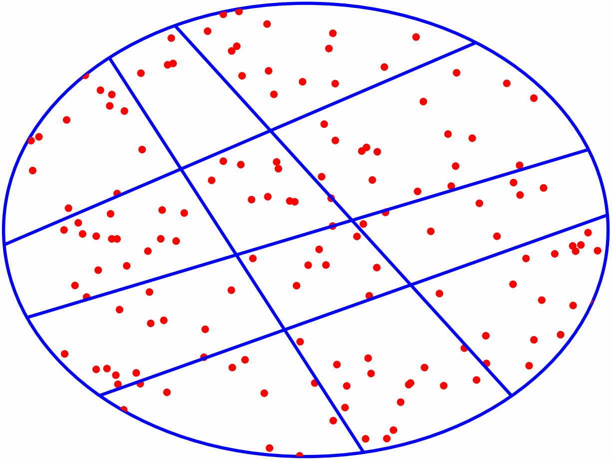
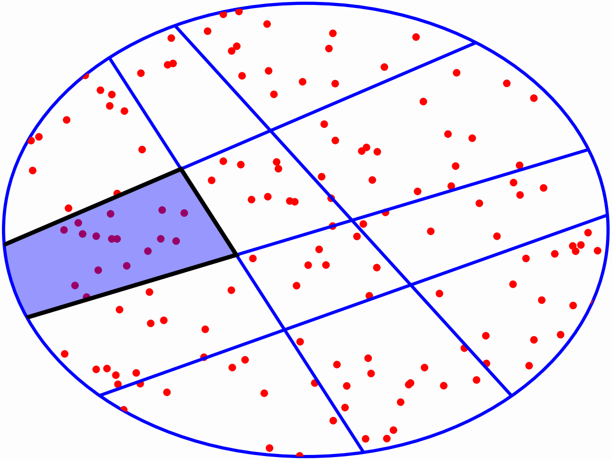


As Simple as Counting Dots

Pick a random cell



Estimate = Number of solutions in a cell $\times$ Number of cells

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- Choose h randomly from a large family H of hash functions

  Universal Hashing (Carter and Wegman 1977)

2-Universal Hashing

- Let H be family of 2-universal hash functions mapping $\{0,1\}^n$ to $\{0,1\}^m$

$$\forall y_1, y_2 \in \{0,1\}^n, \alpha_1, \alpha_2 \in \{0,1\}^m, h \xleftarrow{R} H$$
$$\Pr[h(y_1) = \alpha_1] = \Pr[h(y_2) = \alpha_2] = \left(\frac{1}{2^m}\right)$$

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- The power of 2-universality
  - Z be the number of solutions in a randomly chosen cell
  - $\mathsf{E}[Z] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z] \leq \mathsf{E}[Z]$

2-Universal Hash Functions

- Variables: $X_1, X_2, \cdots X_n$
- To construct $h : \{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every X_i with prob. $\frac{1}{2}$ and XOR them
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- Solutions in a cell: $F \wedge Q_1 \cdots \wedge Q_m$
- Performance of state of the art SAT solvers degrade with increase in the size of XORs (SAT Solvers != SAT oracles)

Improved Universal Hash Functions

- Not all variables are required to specify solution space of F
  - $F := X_3 \iff (X_1 \vee X_2)$
  - X_1 and X_2 uniquely determines rest of the variables (i.e., X_3)
- Formally: if I is independent support, then $\forall \sigma_1, \sigma_2 \in \mathsf{Sol}(F)$, if σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
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Algorithmic procedure to determine I?

- FP^{NP} procedure via reduction to Minimal Unsatisfiable Subset
- Two orders of magnitude runtime improvement
  (IMMV CP15, Best Student Paper) (IMMV Constraints16, Invited Paper)

Challenges

Challenge 1 How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

- Independent Support-based 2-Universal Hash Functions

Challenge 2 How many cells?

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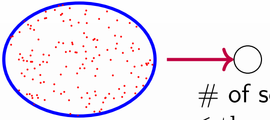
  - Check for every $m = 0, 1, \cdots n$ if the number of solutions $\leq \text{thresh}$

ApproxMC(F, ε, δ)



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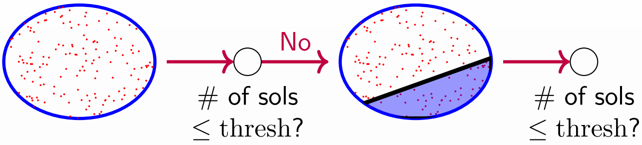

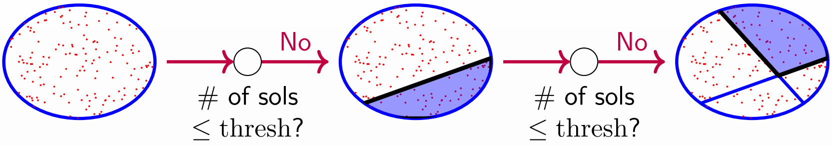
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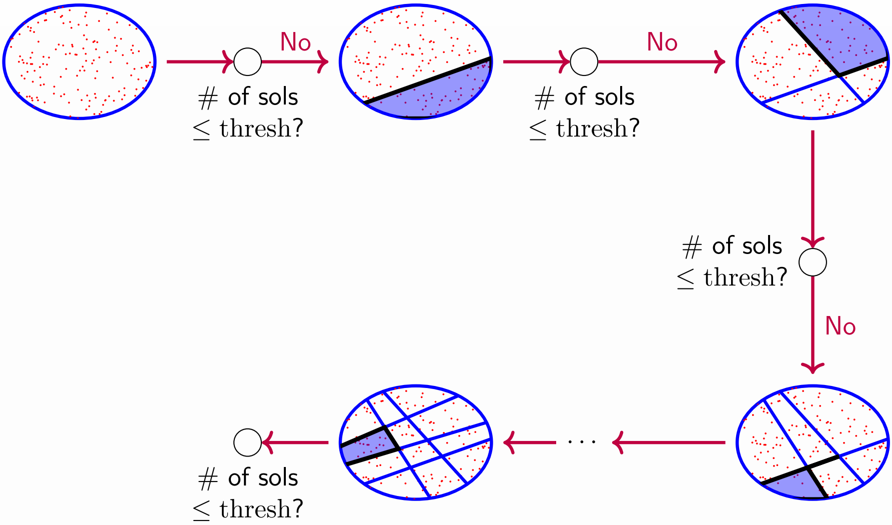

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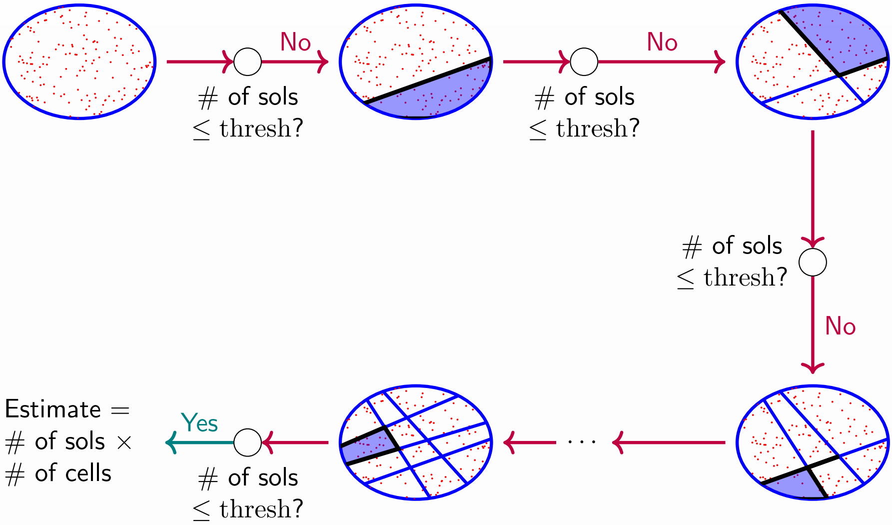

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- Stop at the first m where Query m returns YES and return estimate as $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_m) \times 2^m$
- Observation: $\#(F \wedge Q_1 \cdots \wedge Q_i \wedge Q_{i+1}) \leq \#(F \wedge Q_1 \cdots \wedge Q_i)$
  - If Query i returns YES, then Query $i+1$ must return YES

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  - Independence crucial to analysis (Stockmeyer 1983, $\cdots$)
  - Key Insight: The probability of making a bad choice of Q_i is very small for $i \ll m^*$

(CMV, IJCAI16)

Taming the Curse of Dependence

Let $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\text{thresh}}$ $(m^* = \log(\frac{|\mathsf{Sol}(F)|}{\text{thresh}}))$

Lemma (1)

ApproxMC (F, ε, δ) *terminates with* $m \in \{m^* - 1, m^*\}$ *with probability* $\geq$ *0.8*

Lemma (2)

For $m \in \{m^* - 1, m^*\}$*, estimate obtained from a randomly picked cell lies within a tolerance of* ε *of* $|\mathsf{Sol}(F)|$ *with probability* $\geq$ *0.8*

ApproxMC(F, ε, δ)

Theorem (Correctness)

$$\Pr\left[\frac{|\mathsf{Sol}(F)|}{1+\varepsilon} \leq ApproxMC(F, \varepsilon, \delta) \leq |\mathsf{Sol}(F)|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem (Complexity)

ApproxMC(F, ε, δ) makes $\mathcal{O}(\frac{\log n \log(\frac{1}{\delta})}{\varepsilon^2})$ calls to SAT oracle.

- *Prior work required $\mathcal{O}(\frac{\boldsymbol{n} \log \boldsymbol{n} \log(\frac{1}{\delta})}{\varepsilon})$ calls to SAT oracle (Stockmeyer 1983)*

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Theorem (FPRAS for DNF; (MSV, FSTTCS 17; CP 18, Invited Paper))

If F is a DNF formula, then ApproxMC is FPRAS – fundamentally different from the only other known FPRAS for DNF (Karp, Luby 1983)

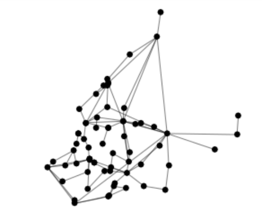
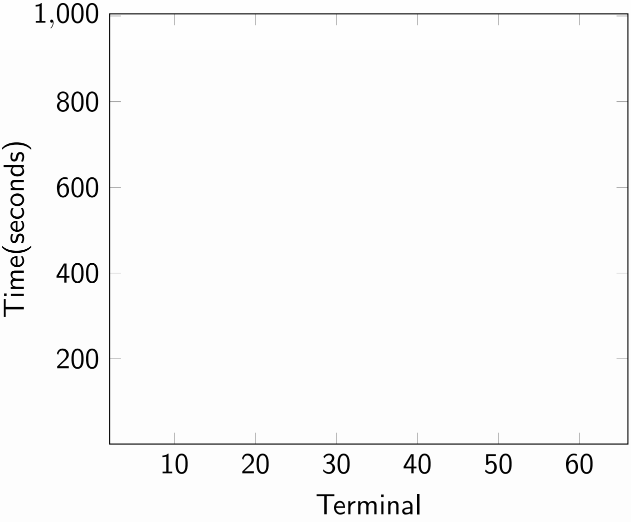


Figure: Plantersville, SC

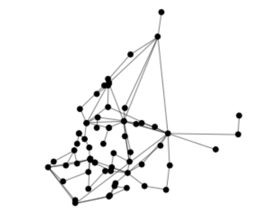
- $G = (V, E)$; source node: s
- Compute Pr[t is disconnected]?

Timeout = 1000 seconds



(DMPV, AAAI17)

Reliability of Critical Infrastructure Networks



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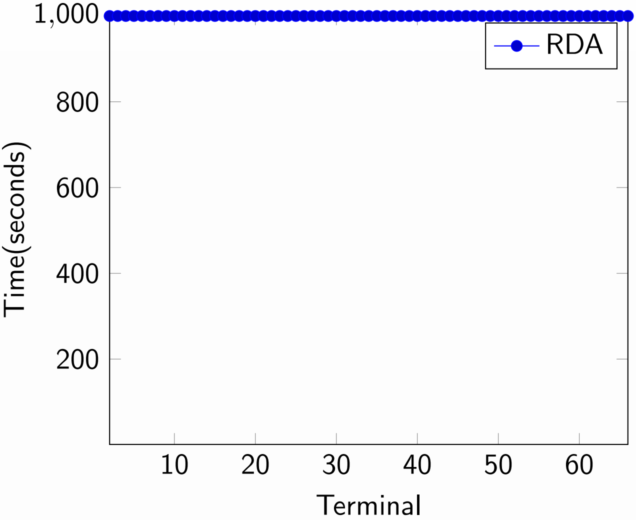

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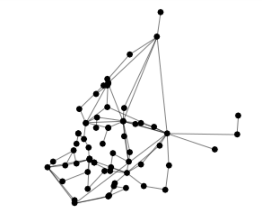

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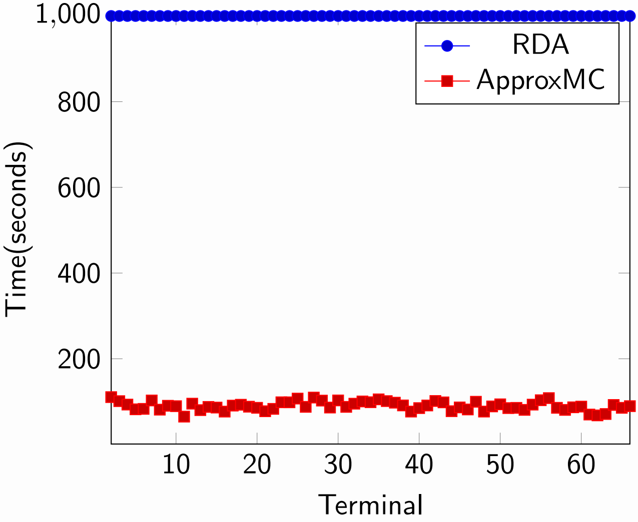

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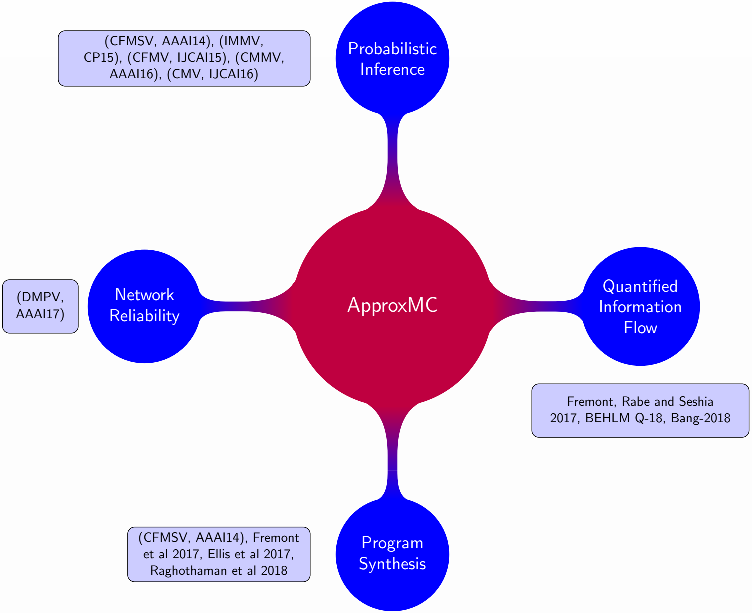
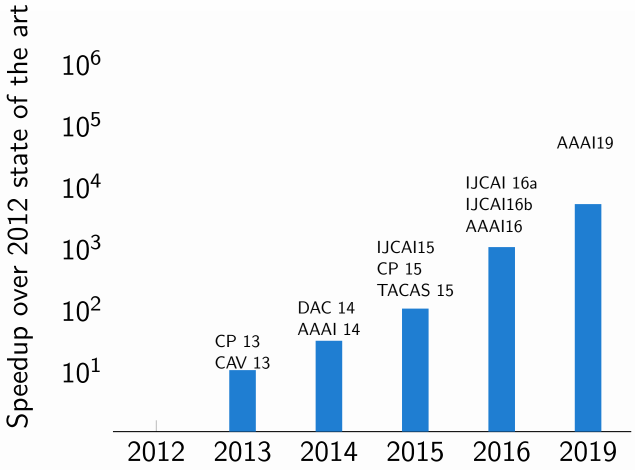
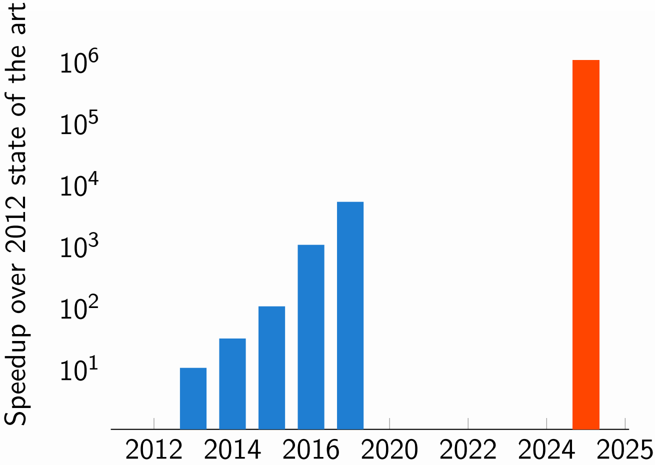
Beyond Network Reliability




Speedup over 2012 state of the art
10^6
10^5
10^4
10^3
10^2
10^1
CP 13
CAV 13
DAC 14
AAAI 14
IJCAI15
CP 15
TACAS 15
IJCAI 16a
IJCAI16b
AAAI16
AAAI19
2012
2013
2014
2015
2016
2019

Mission 2025: Constrained Counting Revolution



Requires combinations of ideas from theory, statistics and systems

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