The Reachability Problem for Petri Nets is Not Elementary

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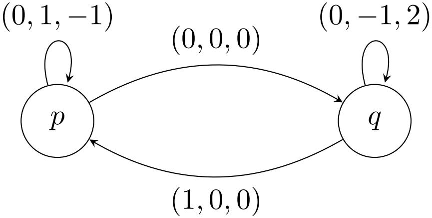
Introduction

VASS and Counter Programs

Vector addition systems with states (VASS)

(d, Q, T), where $T \subseteq Q \times \mathbb{Z}^d \times Q$

Example: $d = 3$, $Q = \{p, q\}$



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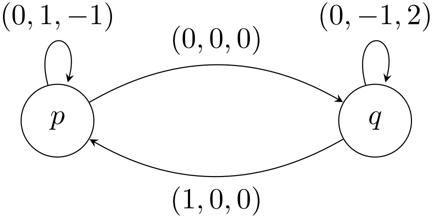

Configurations $q(\mathbf{v}) = (q, \mathbf{v}) \in Q \times \mathbb{N}^d$

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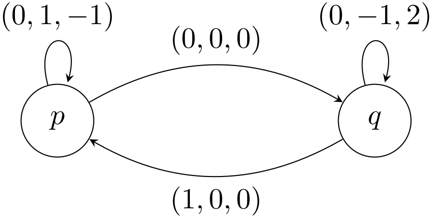

Configurations $q(\mathbf{v}) = (q, \mathbf{v}) \in Q \times \mathbb{N}^d$

Example run:

$p(0,0,1) \to p(0,1,0) \to q(0,1,0) \to q(0,0,2) \to p(1,0,2)$

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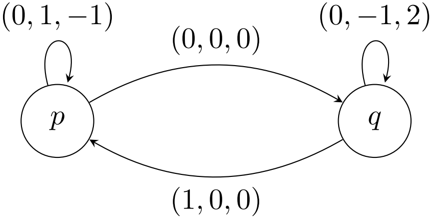

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$p(0,0,1) \to p(0,1,0) \to q(0,1,0) \to q(0,0,2) \to p(1,0,2)$

Notation: $p(0,0,1) \xrightarrow{*} p(1,0,2)$

Decision problems

Reachability problem:

GIVEN: VASS (d, Q, T) and configurations $p(\mathbf{u}), q(\mathbf{v})$

DECIDE: whether $p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})$?

Decision problems

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State reachability problem:

GIVEN: VASS (d, Q, T) a configuration $p(\mathbf{u})$ and a control-state q

DECIDE: whether exists $\mathbf{v}$ s.t. $p(\mathbf{u}) \xrightarrow{*} q(\mathbf{v})$?

Decision problems

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- State reachability can be reduced to reachability

Reachability state of art

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1976 — EXPSPACE-hard (Lipton)

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Reachability state of art

State reachability
Rackoff 1978
EXPSPACE-complete

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Counter programs

Counter programs

- Operations over a bounded counter $\bar{\mathsf{x}}$ ranges $\{0, \ldots, B\}$:
  $\bar{\mathsf{x}} \mathrel{+}= 1$
  $\bar{\mathsf{x}} \mathrel{-}= 1$
  zero? $\bar{\mathsf{x}}$
  max? $\bar{\mathsf{x}}$
- Operations over an unbounded counter x ranges $\{0, 1, \ldots\}$:
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$$\text{body} \quad := \quad \text{operation} \;\; \| \;\; \begin{array}{c}\text{body}\\ \vdots \\ \text{body}\end{array} \;\; \| \;\; \begin{array}{l}\textbf{loop}\\ \quad\text{body}\end{array}$$

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$$\text{counter program} \quad := \quad \begin{array}{l}\text{body}\\ \textbf{halt if } \mathsf{x}_1, \ldots, \mathsf{x}_n = 0\end{array}$$

Computed relations

Computed relations

A B-run is a run such that:

- bounded counters ranges in $\{0, \ldots, B\}$,
- unbounded counters ranges in $\mathbb{N}$.

A run is **complete** if it start with zero in every counter and it ends by executing the last **halt**.

The relation B-computed in some counters $\mathsf{x}_1, \ldots, \mathsf{x}_l$ is the set of tuples of values after a complete B-run in those counters.

An example

loop
  $\bar{\imath}$ += 1
  // assert $\bar{a} = 0$
  loop
    x += 1 $\bar{a}$ += 1
  max? $\bar{a}$
  loop
    $\bar{a}$ −= 1
  zero? $\bar{a}$
max? $\bar{\imath}$
halt

An example

loop

  $\bar{\imath}$ += 1

  // assert $\bar{a} = 0$

  loop

    x += 1 $\bar{a}$ += 1

  max? $\bar{a}$ $x+ = B$

  loop

    $\bar{a}$ −= 1

  zero? $\bar{a}$

max? $\bar{\imath}$

halt

An example

$$\begin{array}{l}
\textbf{loop}\\
\quad \bar{\imath} \mathrel{+}= 1\\
\quad \text{// assert } \bar{a} = 0\\
\left.\begin{array}{l} \quad\textbf{loop}\\ \qquad x \mathrel{+}= 1 \quad \bar{a} \mathrel{+}= 1\\ \quad \textbf{max? } \bar{a}\\ \quad\textbf{loop}\\ \qquad \bar{a} \mathrel{-}= 1\\ \quad\textbf{zero? } \bar{a}\end{array}\right\} x{+} = B\\
\textbf{max? } \bar{\imath}\\
\textbf{halt}
\end{array}$$

The relation B-computed in x is $x = B^2$.

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Reachability problem (for counter programs):

GIVEN: A counter program and a bound B.

DECIDE: Does it have a complete B-run ?

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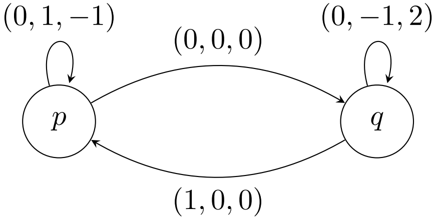

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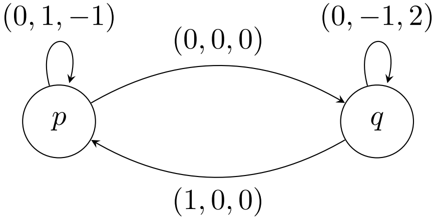

$p(0,0,1) \xrightarrow{*} p(1,0,2)$?

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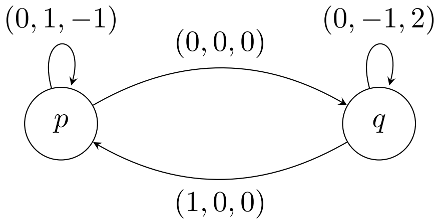

$p(0,0,1) \xrightarrow{*} p(1,0,2)$?

```
z += 1
loop
    loop
        y += 1    z -= 1
    loop
        y -= 1    z += 2
    x += 1
x -= 1    z -= 2
halt if x, y, z = 0.
```

Counter Programs = VASS

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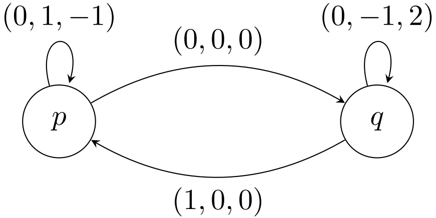

$p(0,0,1) \xrightarrow{*} p(1,0,2)$?

z += 1
loop
  loop
    y += 1   z −= 1
  loop
    y −= 1   z += 2
  x += 1
x −= 1   z −= 2
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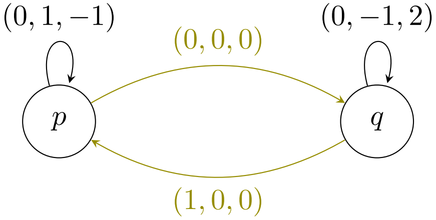

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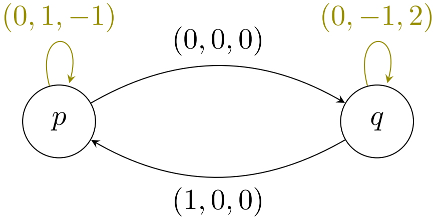

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  loop
    y += 1   z −= 1
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    y −= 1   z += 2
  x += 1
x −= 1   z −= 2
halt if x, y, z = 0.

Outline

- High level idea of the proof
- The factorial amplifier
- Composition operator

A TOWER-complete problem

The following problem is TOWER-complete (see Schmitz 2016)

GIVEN: A counter program without unbounded counters and n.

DECIDE: Does it have a complete $3\,\underbrace{!\cdots!}_{n\text{ times}}$-run ?

(B, R)-amplifier

A (B, R)-amplifier is a counter program that B-computes in $\mathsf{b}, \mathsf{c}, \mathsf{d}$ the relation $\mathsf{b} = R \ \wedge \ \mathsf{c} > 0 \ \wedge \ \mathsf{d} = \mathsf{c} \cdot R$.

(B, R)-amplifier

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Example: Counter program $\mathcal{A}_3$

```
b += 3    c += 1    d += 3
loop
    c += 1    d += 3
halt
```

$\mathcal{A}_3$ is a $(B, 3)$-amplifier for any $B \geq 0$.

Simulation with amplifiers

Simulation with amplifiers

We provide a composition operator $\mathcal{A} \triangleright \mathcal{P}$ such that if:

- $\mathcal{A}$ is a (B, R)-amplifier.
- $\mathcal{P}$ is a counter program.

Then:

$$\text{Relations } R\text{-computed by } \mathcal{P}$$
$$=$$
$$\text{Relations } B\text{-computed by } \mathcal{A} \triangleright \mathcal{P}$$

Factorial Amplifier

There exists a counter program $\mathcal{F}$ that is a $(B, B!)$-amplifier for every $B > 0$ called the **factorial amplifier**.

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$$\text{Relations } 3 \overbrace{! \cdots !}^{n \text{ times}}\text{-computed by } \mathcal{P}$$
$$=$$
$$\text{Relations } 0\text{-computed by } \mathcal{A}_3 \triangleright \underbrace{\mathcal{F} \triangleright \cdots \triangleright \mathcal{F}}_{n \text{ times}} \triangleright \mathcal{P}$$

The Factorial Amplifier

to B-compute the relation $\mathsf{b} = B! \ \wedge \ \mathsf{c} > 0 \ \wedge \ \mathsf{d} = \mathsf{c} \cdot B!$

Main idea

Implement with a counter program:

$$n \cdot \prod_{1 \leq i < B} \frac{i+1}{i} = n \cdot B$$

A weak multiplier by $\frac{3}{2}$

$//$ assert $\mathsf{x} = x \quad \mathsf{x}' = x'$

loop

$\quad \mathsf{x} \mathrel{-}= 2 \qquad \mathsf{x}' \mathrel{+}= 3$

loop

$\quad \mathsf{x}' \mathrel{-}= 1 \qquad \mathsf{x} \mathrel{+}= 1$

$//$ assert $\mathsf{x} + \mathsf{x}' \leq \frac{3}{2}(x + x')$

$//$ assert $\mathsf{x} + \mathsf{x}' = \frac{3}{2}(x + x') \Rightarrow x' = 0$

A weak multiplier by $\frac{3}{2}$

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// assert $\mathsf{x} + \mathsf{x}' \leq \frac{3}{2}(x + x')$

// assert $\mathsf{x} + \mathsf{x}' = \frac{3}{2}(x + x') \Rightarrow x' = 0$

x	x′	x + x′
15	0	15
13	3	16
11	6	17
9	9	18
7	12	19
5	15	20
3	18	21
1	21	$22 = \frac{3}{2}15 - \frac{1}{2}$
2	20	22
3	19	22
⋮	⋮	⋮
21	1	22
22	0	22

Implementing $n \cdot \prod_{1 \leq i < B} \frac{i+1}{i} = n \cdot B$

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$$
\begin{array}{l}
\bar{\imath} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1 \\
\textbf{loop} \\
\quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1 \\
\textbf{loop} \\
\quad \text{// assert } \mathsf{x} + \mathsf{x}' \le \mathsf{y} \cdot \bar{\imath} \\
\left.\begin{array}{l}
\quad \textbf{loop} \\
\quad\quad \mathsf{x} \mathrel{-}= \bar{\imath} \quad \mathsf{x}' \mathrel{+}= \bar{\imath} + 1 \\
\quad \textbf{loop} \\
\quad\quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1
\end{array}\right\} \text{weak multiplier by } \frac{\bar{\imath}+1}{\bar{\imath}} \\
\quad \text{// assert } \mathsf{x} + \mathsf{x}' \le \mathsf{y} \cdot (\bar{\imath} + 1) \\
\quad \bar{\imath} \mathrel{+}= 1 \\
\textbf{max? } \bar{\imath} \\
\textbf{loop} \\
\quad \mathsf{x} \mathrel{-}= \bar{\imath} \quad \mathsf{y} \mathrel{-}= 1 \\
\textbf{halt if } \mathsf{y} = 0
\end{array}
$$

How to simulate $\mathrm{x} \mathrel{-}= \bar{\mathrm{\imath}}$?

x is an unbounded counter
$\bar{\mathrm{\imath}}$ is a bounded counter and
$\bar{\mathrm{a}}$ is an auxiliary bounded counter assumed to be zero.

How to simulate $\mathsf{x} \mathrel{-}= \bar{\imath}$?

x is an unbounded counter
$\bar{\imath}$ is a bounded counter and
$\bar{\mathsf{a}}$ is an auxiliary bounded counter assumed to be zero.

$// \text{ assert } \mathsf{x} = x \quad \bar{\imath} = i \quad \bar{\mathsf{a}} = 0$
loop
$\quad \mathsf{x} \mathrel{-}= 1 \quad \bar{\imath} \mathrel{-}= 1 \quad \bar{\mathsf{a}} \mathrel{+}= 1$
zero? $\bar{\imath}$
$// \text{ assert } \mathsf{x} = x - i \quad \bar{\imath} = 0 \quad \bar{\mathsf{a}} = i$
loop
$\quad \bar{\imath} \mathrel{+}= 1 \quad \bar{\mathsf{a}} \mathrel{-}= 1$
zero? $\bar{\mathsf{a}}$
$// \text{ assert } \mathsf{x} = x - i \quad \bar{\imath} = i \quad \bar{\mathsf{a}} = 0$

How to simulate $\mathrm{x} \mathrel{+}= \bar{\imath} + 1$?

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$\bar{\imath}$ is a bounded counter and
$\bar{\mathrm{a}}$ is an auxiliary bounded counter assumed to be zero.

How to simulate $\mathsf{x} \mathrel{+}= \bar{\imath} + 1$?

x is an unbounded counter
$\bar{\imath}$ is a bounded counter and
$\bar{\mathsf{a}}$ is an auxiliary bounded counter assumed to be zero.

$\mathsf{x} \mathrel{+}= 1$
loop
    $\mathsf{x} \mathrel{+}= 1 \quad \bar{\imath} \mathrel{-}= 1 \quad \bar{\mathsf{a}} \mathrel{+}= 1$
zero? $\bar{\imath}$
loop
    $\bar{\imath} \mathrel{+}= 1 \quad \bar{\mathsf{a}} \mathrel{-}= 1$
zero? $\bar{\mathsf{a}}$

The factorial amplifier

The factorial amplifier

$\bar{\imath} \mathrel{+}= 1 \quad \mathsf{b} \mathrel{+}= 1 \quad \mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$

loop

$\quad \mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$

loop

Multiply	x	d	c	b
by	$\frac{\bar{\imath}+1}{\mathrm{i}}$	$\frac{\bar{\imath}+1}{\bar{\imath}}$	$\frac{1}{\bar{\imath}}$	$\bar{\imath}+1$

$\quad \bar{\imath} \mathrel{+}= 1$

max? $\bar{\imath}$

loop

$\quad \mathsf{x} \mathrel{-}= \bar{\imath} \quad \mathsf{y} \mathrel{-}= 1$

halt if $\mathsf{y} = 0$

Multiply	x	d	c	b
by	$\frac{\bar{\imath}+1}{\bar{\imath}}$	$\frac{\bar{\imath}+1}{\bar{\imath}}$	$\frac{1}{\bar{\imath}}$	$\bar{\imath}+1$

Multiply	x	d	c	b
by	$\frac{\bar{\imath}+1}{\bar{\imath}}$	$\frac{\bar{\imath}+1}{\bar{\imath}}$	$\frac{1}{\bar{\imath}}$	$\bar{\imath}+1$

// assert $x = d \leq y \cdot \bar{\imath} \quad \wedge \quad c \geq \frac{y}{(\bar{\imath}-1)!} \quad \wedge \quad b \leq \bar{\imath}!$

$$\left.\begin{array}{l}
\textbf{loop} \\
\quad c \mathrel{-}= \bar{\imath} \quad c' \mathrel{+}= 1 \\
\quad \textbf{loop at most } b \textbf{ times} \\
\qquad d \mathrel{-}= \bar{\imath} \quad x \mathrel{-}= \bar{\imath} \quad d' \mathrel{+}= \bar{\imath}+1
\end{array}\right\} \text{weak multiplier of } x, d \text{ by } \frac{\bar{\imath}+1}{\bar{\imath}}$$

$$\left.\begin{array}{l}
\textbf{loop} \\
\quad b \mathrel{-}= 1 \quad b' \mathrel{+}= \bar{\imath}+1 \\
\textbf{loop} \\
\quad b' \mathrel{-}= 1 \quad b \mathrel{+}= 1
\end{array}\right\} \text{weak multiplier of } b \text{ by } \bar{\imath}+1$$

loop

$c' \mathrel{-}= 1 \quad c \mathrel{+}= 1$

loop at most b **times**

$d' \mathrel{-}= 1 \quad d \mathrel{+}= 1 \quad x \mathrel{+}= 1$

Controled loops

loop at most b **times** $<$*body*$>$

b is an unbounded counter

b$'$ is an auxiliary unbounded counter

Controled loops

loop at most b **times** <*body*>

b is an unbounded counter

b′ is an auxiliary unbounded counter

loop
    $\mathsf{b} \mathrel{-}= 1 \quad \mathsf{b}' \mathrel{+}= 1$
loop
    $\mathsf{b}' \mathrel{-}= 1 \quad \mathsf{b} \mathrel{+}= 1$
    <*body*>

The composition operator

Composing with amplifiers

Composing with amplifiers

$$\text{Relations } R\text{-computed by } \mathcal{P}$$
$$=$$
$$\text{Relations } B\text{-computed by } \underbrace{\mathcal{A}}_{(B,R)\text{-amplifier}} \triangleright \mathcal{P}$$

Composing with amplifiers

$$\text{Relations } R\text{-computed by } \mathcal{P} = \text{Relations } B\text{-computed by } \underbrace{\mathcal{A}}_{(B,R)\text{-amplifier}} \triangleright \mathcal{P}$$

$$\boxed{\begin{array}{l}\text{body of } \mathcal{A} \\ \textbf{halt if } \mathsf{y}_1, \ldots = 0\end{array}} \triangleright \boxed{\begin{array}{l}\text{body of } \mathcal{P} \\ \textbf{halt if } \mathsf{z}_1, \ldots = 0\end{array}}$$

$$=$$

$$\boxed{\begin{array}{l}\text{body of } \mathcal{A} \\ \text{initialization body} \\ \text{modified body of } \mathcal{P} \\ \textbf{halt if } d, \mathsf{y}_1, \ldots, \mathsf{z}_1, \ldots = 0\end{array}}$$

Initialization body

$$\frac{\text{Invariants}}{b = R \wedge d = c \cdot R}$$

Encoding: We replace every bounded counter $\bar{\mathsf{x}}_i$ of $\mathcal{P}$ by two fresh unbounded counters x_i and x'_i

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To get $\mathsf{x}_i + \mathsf{x}_i' = R$, we start with:

$$\left.\begin{array}{l}
\textbf{loop} \\
\quad \mathsf{x}_1' \mathrel{+}= 1 \quad \cdots \quad \mathsf{x}_l' \mathrel{+}= 1 \\
\quad \mathsf{b} \mathrel{-}= 1 \quad \mathsf{d} \mathrel{-}= 1 \\
\mathsf{c} \mathrel{-}= 1
\end{array}\right\} \begin{array}{l}
\text{c decreased by } 1 \text{ and} \\
\text{d by at most } R \\
\text{So if not complete } \mathsf{d} > \mathsf{c} \cdot R
\end{array}$$

Invariants OK	Invariants NOK
$d = c \cdot R \wedge \mathsf{x}_i + \mathsf{x}_i' = R$	$d > c \cdot R \wedge \mathsf{x}_i + \mathsf{x}_i' \leq R$

Modified body of $\mathcal{P}$

Replace $\bar{\mathsf{x}}_i \mathrel{+}= 1$ with $\mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}'_i \mathrel{-}= 1$

Replace $\bar{\mathsf{x}}_i \mathrel{-}= 1$ with $\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}'_i \mathrel{+}= 1$

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Replace **zero?** $\bar{\mathsf{x}}_i$ with

loop

  $\mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}'_i \mathrel{-}= 1$

  $\mathsf{d} \mathrel{-}= 1$

$\mathsf{c} \mathrel{-}= 1$

loop

  $\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}'_i \mathrel{+}= 1$

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loop

  $\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}'_i \mathrel{+}= 1$

  $\mathsf{d} \mathrel{-}= 1$

$\mathsf{c} \mathrel{-}= 1$

c decreased by 2 and
d by at most $2R$

So if not complete $\mathsf{d} > \mathsf{c} \cdot R$

Invariants OK	Invariants NOK
$d = c \cdot R \wedge \mathsf{x}_i + \mathsf{x}'_i = R$	$d > c \cdot R \wedge \mathsf{x}_i + \mathsf{x}'_i \leq R$

To sum up

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We obtain that way a composition operator $\triangleright$ such that:

$$\text{Relations } 3\overbrace{!\cdots!}^{n \text{ times}}\text{-computed by } \mathcal{P}$$
$$=$$
$$\text{Relations } 0\text{-computed by } \mathcal{A}_3 \triangleright \underbrace{\mathcal{F} \triangleright \cdots \triangleright \mathcal{F}}_{n \text{ times}} \triangleright \mathcal{P}$$

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So maybe it's good to study restrictions of generalizations of etc...