Shallow Packing Lemma and its Applications in Combinatorial Geometry

Arijit Ghosh[1]

[1]Indian Statistical Institute
Kolkata, India

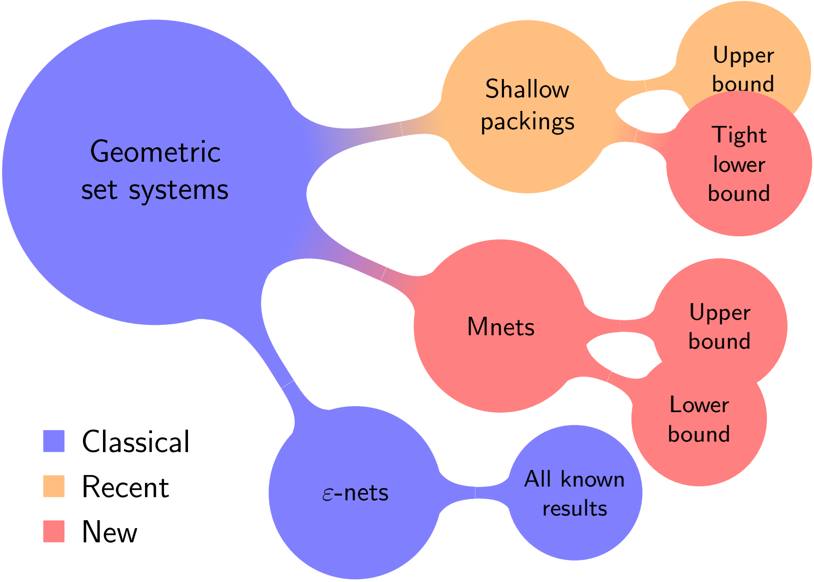
Coauthors

- Kunal Dutta (INRIA, DataShape)
- Esther Ezra (Georgia Tech, Math Dep.)
- Bruno Jartoux (Ben-Gurion Univ., CS Dep.)
- Nabil H. Mustafa (Université Paris-Est, LIGM)

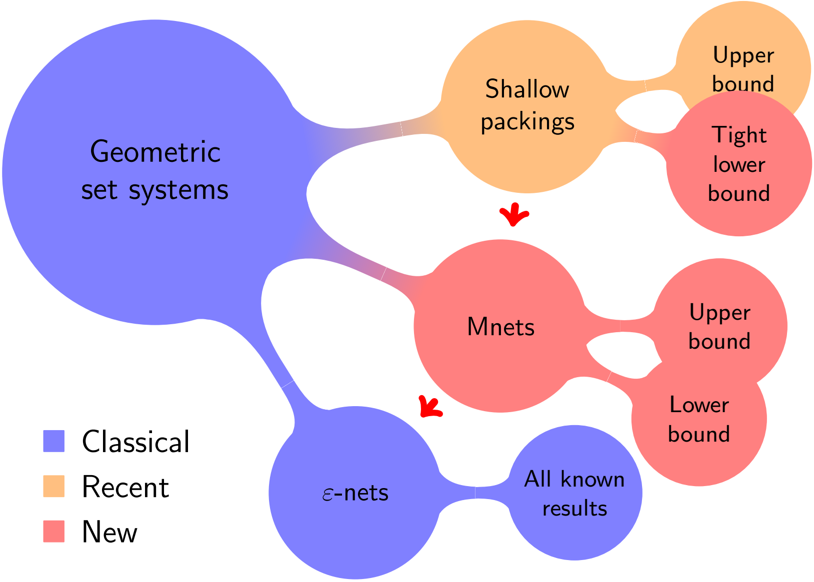
Talk will be based on the following papers

- **Shallow packings, semialgebraic set systems, Macbeath regions and polynomial partitioning**,
  with Bruno Jartoux, Kunal Dutta and Nabil Hassan Mustafa.
  Discrete & Computational Geometry, to appear.
- **A Simple Proof of Optimal Epsilon Nets**,
  with Kunal Dutta and Nabil Hassan Mustafa.
  Combinatorica, 38(5): 1269 – 1277, 2018.
- **Two proofs for Shallow Packings**,
  with Kunal Dutta and Esther Ezra.
  Discrete & Computational Geometry, 56(4): 910-939, 2016.

Situation map: three combinatorial structures

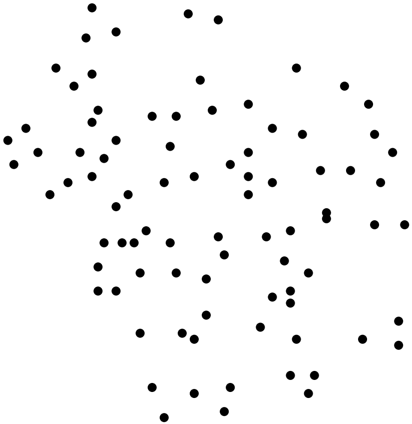


Situation map: three combinatorial structures



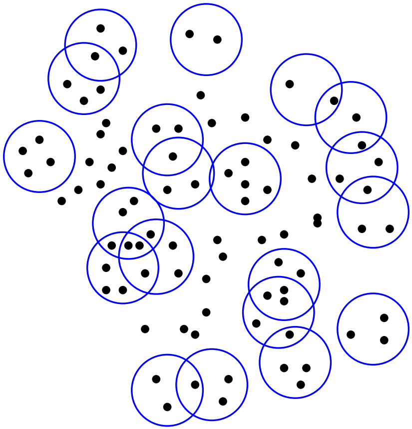
Geometric set systems

Point-disk incidences: an example of geometric *set system*



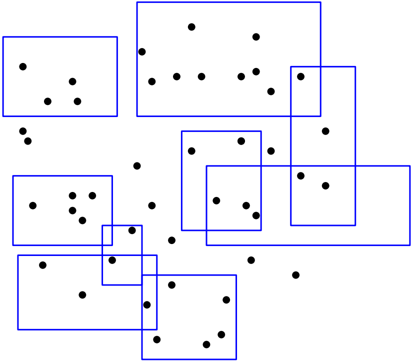
Geometric set systems

Point-disk incidences: an example of geometric *set system*



Geometric set systems

Typical applications: range searching, point set queries.



Macbeath regions

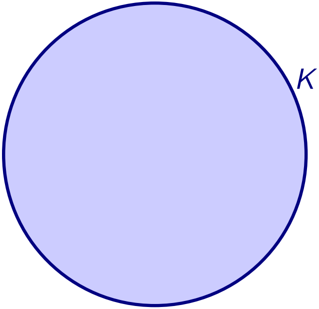
Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ contains one of them.

Macbeath regions

Macbeath decomposition (Macbeath 1952)

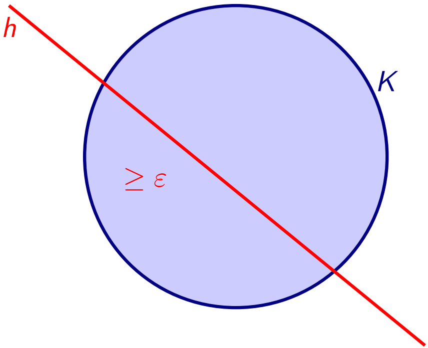
For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ contains one of them.



Macbeath regions

Macbeath decomposition (Macbeath 1952)

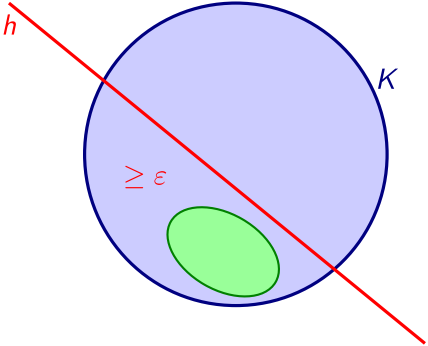
For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ contains one of them.



Macbeath regions

Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ contains one of them.



Mnets, or combinatorial Macbeath regions

Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ includes one of them.

Mnets, or combinatorial Macbeath regions

Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ includes one of them.

Mnets – for halfplanes

For a set K of n points and $\varepsilon > 0$, an **Mnet** is a collection of subsets of $\Theta(\varepsilon n)$ points such that any halfplane h with $|h \cap K| \geq \varepsilon n$ includes one of them.

Mnets, or combinatorial Macbeath regions

Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ includes one of them.

Mnets – for disks

For a set K of n points and $\varepsilon > 0$, an **Mnet** is a collection of subsets of $\Theta(\varepsilon n)$ points such that any disk h with $|h \cap K| \geq \varepsilon n$ includes one of them.

Mnets, or combinatorial Macbeath regions

Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ includes one of them.

Mnets – for [shapes]

For a set K of n points and $\varepsilon > 0$, an **Mnet** is a collection of subsets of $\Theta(\varepsilon n)$ points such that any [shape] h with $|h \cap K| \geq \varepsilon n$ includes one of them.

Mnets, or combinatorial Macbeath regions

Macbeath decomposition (Macbeath 1952)

For any convex body K with unit volume and $\varepsilon > 0$, there is a *small* collection of convex subsets of K with volume $\Theta(\varepsilon)$ such that any halfplane h with $vol(h \cap K) \geq \varepsilon$ includes one of them.

Mnets – for [shapes]

For a set K of n points and $\varepsilon > 0$, an **Mnet** is a collection of subsets of $\Theta(\varepsilon n)$ points such that any [shape] h with $|h \cap K| \geq \varepsilon n$ includes one of them.

Goal: discrete analogue of Macbeath's tool.

Bounds on Mnets

Question

What is the minimum size of an Mnet?

Bounds on Mnets

Question

What is the minimum size of an Mnet?

Theorem (D.–G.–J.–M. '17)

Semialgebraic *set systems with* **VC-dim**. $d < \infty$ *and* **shallow cell complexity** φ *have an* ε*-Mnet of size*

$$O\left(\frac{d}{\varepsilon} \cdot \varphi\left(\frac{d}{\varepsilon}, d\right)\right).$$

Bounds on Mnets

Question

What is the minimum size of an Mnet?

Theorem (D.–G.–J.–M. '17)

Semialgebraic *set systems with* **VC-dim**. $d < \infty$ *and* **shallow cell complexity** φ *have an* ε*-Mnet of size*

$$O\left(\frac{d}{\varepsilon} \cdot \varphi\left(\frac{d}{\varepsilon}, d\right)\right).$$

- ✓ Disks
- ✓ Rectangles
- ✓ Lines
- ✓ 'Fat' objects
- × General convex sets

Bounds on Mnets

Question

What is the minimum size of an Mnet?

Theorem (D.–G.–J.–M. '17)

Semialgebraic *set systems with* ***VC-dim.*** *$d < \infty$ and* ***shallow cell complexity*** *φ have an ε-Mnet of size*

$$O\left(\frac{d}{\varepsilon} \cdot \varphi\left(\frac{d}{\varepsilon}, d\right)\right).$$

- ✓ Disks
- ✓ Rectangles
- ✓ Lines
- ✓ 'Fat' objects
- × General convex sets

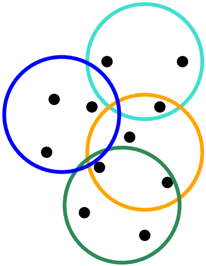
Theorem (D.–G.–J.–M. '17)

This is tight for hyperplanes.

Abstract set systems

X := arbitrary n-point set
Σ := collection of subsets of X, i.e., $\Sigma \subseteq 2^X$
The pair (X, Σ) is called a *set system*

Set systems (X, Σ) are also referred to as *hypergraphs*, *range spaces*



Abstract set systems

Projection:

For $Y \subseteq X$,

$$\Sigma_Y := \{S \cap Y : \; S \in \Sigma\}$$

and

$$\Sigma_Y^k := \{S \cap Y : \; S \in \Sigma \text{ and } |S \cap Y| \leq k\}$$

VC dimension and shallow cell complexity

Primal Shatter function Given (X, Σ), primal shatter function is defined as

$$\pi_{\Sigma}(m) := \max_{Y \subseteq X, |Y|=m} |\Sigma_Y|$$

VC dimension and shallow cell complexity

Primal Shatter function Given (X, Σ), primal shatter function is defined as

$$\pi_\Sigma(m) := \max_{Y \subseteq X, |Y|=m} |\Sigma_Y|$$

VC dimension: $d_0 := \max\{m \mid \pi_\Sigma(m) = 2^m\}$

VC dimension and shallow cell complexity

Primal Shatter function Given (X, Σ), primal shatter function is defined as

$$\pi_\Sigma(m) := \max_{Y \subseteq X, |Y|=m} |\Sigma_Y|$$

VC dimension: $d_0 := \max \{ m \mid \pi_\Sigma(m) = 2^m \}$

Sauer-Shelah Lemma: VC dim. d_0 implies for all $m \leq n$, $\pi_\Sigma(m) \leq O(m^{d_0})$.

VC dimension and shallow cell complexity

Primal Shatter function Given (X, Σ), primal shatter function is defined as

$$\pi_\Sigma(m) := \max_{Y \subseteq X, |Y| = m} |\Sigma_Y|$$

VC dimension: $d_0 := \max\{m \mid \pi_\Sigma(m) = 2^m\}$

Sauer-Shelah Lemma: VC dim. d_0 implies for all $m \leq n$, $\pi_\Sigma(m) \leq O(m^{d_0})$.

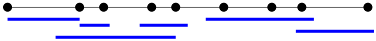
Shallow cell complexity $\varphi(\cdot, \cdot)$ If $\forall Y \subseteq X$,

$$\left|\Sigma_Y^k\right| \leq |Y| \times \varphi(|Y|, k).$$

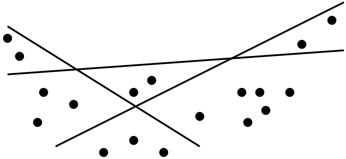
Shallow cell complexity of some geometric set systems

1. Points and half-spaces or orthants in $\mathbb{R}^d$ — $O(|Y|^{\lfloor d/2 \rfloor - 1} k^{\lceil d/2 \rceil})$
2. Points and balls in $\mathbb{R}^d$ — $O(|Y|^{\lfloor (d+1)/2 \rfloor - 1} k^{\lceil (d+1)/2 \rceil})$
3. $(d-1)$-variate polynomial function of constant degree and points in $\mathbb{R}^d$ — $|Y|^{d-2+\varepsilon} k^{1-\varepsilon}$

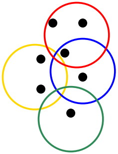
Some geometric set systems



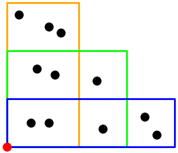
$\dim = 2$



$\dim = d + 1$



$\dim = d + 2$



$\dim = d + 1$

Epsilon-nets

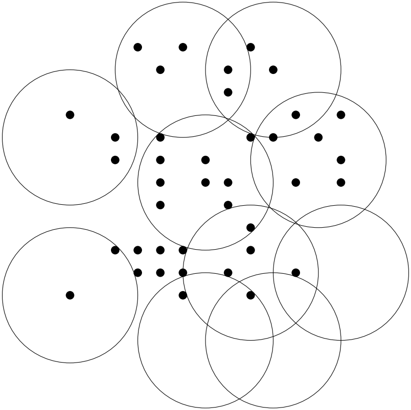
Epsilon-nets: For a set system (X, Σ), $Y \subseteq X$ is an ε-net if

$$\forall S \in \Sigma \ \text{ with } \ |S| \geq \varepsilon n, \quad Y \cap S \neq \emptyset$$

Epsilon-nets

Epsilon-nets: For a set system (X, Σ), $Y \subseteq X$ is an ε-net if

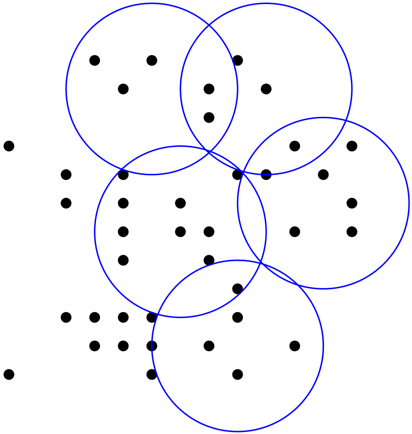
$$\forall S \in \Sigma \text{ with } |S| \geq \varepsilon n, \quad Y \cap S \neq \emptyset$$



Epsilon-nets

Epsilon-nets: For a set system (X, Σ), $Y \subseteq X$ is an ε-net if

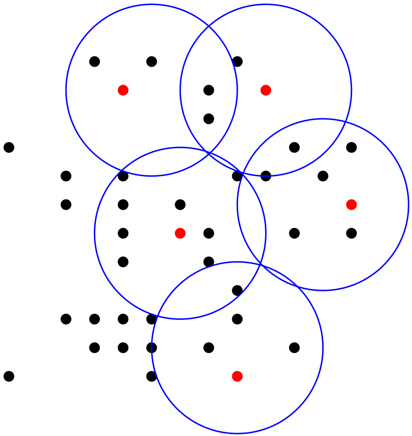
$$\forall S \in \Sigma \text{ with } |S| \geq \varepsilon n, \quad Y \cap S \neq \emptyset$$



Epsilon-nets

Epsilon-nets: For a set system (X, Σ), $Y \subseteq X$ is an ε-net if

$$\forall S \in \Sigma \text{ with } |S| \geq \varepsilon n, \quad Y \cap S \neq \emptyset$$



Epsilon-nets

Epsilon-nets: For a set system (X, Σ), $Y \subseteq X$ is an ε-net if

$$\forall S \in \Sigma \ \text{ with } \ |S| \geq \varepsilon n, \quad Y \cap S \neq \emptyset$$

Theorem (Haussler-Welzl'87)

For a set system with VC-dimen d there exists an ε-net of size $O\left(\frac{d}{\varepsilon} \log \frac{d}{\varepsilon}\right)$

Epsilon-nets

Epsilon-nets: For a set system (X, Σ), $Y \subseteq X$ is an ε-net if

$$\forall S \in \Sigma \ \text{ with } \ |S| \geq \varepsilon n, \quad Y \cap S \neq \emptyset$$

Theorem (Haussler-Welzl'87)

For a set system with VC-dimen d there exists an ε-net of size $O\left(\frac{d}{\varepsilon}\log\frac{d}{\varepsilon}\right)$

$O\left(\frac{1}{\varepsilon}\right)$-size ε-nets are known for special set systems
Half-spaces in $\mathbb{R}^2$ and $\mathbb{R}^3$, pseudo-disks, homothetic copies of convex objects, α-fat wedges etc ...
(Matousek-Seidel-Welzl, Buzaglo-Pinchasi-Rote, Pyra-Ray, . . .)

Optimal size Epsilon-nets

Theorem (Varadarajan'10, Aronov et al.'10, Chan et al.'12)

Let (X, Σ) be a set system with constant VC-dimen and shallow cell complexity $\varphi(\cdot)$. Then there exists an ε-net of (X, Σ) of size

$$O\left(\frac{1}{\varepsilon}\log\varphi\left(\frac{1}{\varepsilon}\right)\right).$$

Optimal size Epsilon-nets

Theorem (Varadarajan'10, Aronov et al.'10, Chan et al.'12)

Let (X, Σ) be a set system with constant VC-dimen and shallow cell complexity $\varphi(\cdot)$. Then there exists an ε-net of (X, Σ) of size

$$O\left(\frac{1}{\varepsilon}\log\varphi\left(\frac{1}{\varepsilon}\right)\right).$$

Remark: This result gives optimal size nets for all known geometric set systems.

Optimal size Epsilon-nets

Theorem (Varadarajan'10, Aronov et al.'10, Chan et al.'12)

Let (X, Σ) be a set system with constant VC-dimen and shallow cell complexity $\varphi(\cdot)$. Then there exists an ε-net of (X, Σ) of size

$$O\left(\frac{1}{\varepsilon}\log\varphi\left(\frac{1}{\varepsilon}\right)\right).$$

Remark: This result gives optimal size nets for all known geometric set systems. For example the above result implies $\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}$ size nets for points and rectangles in plane.

δ-packing number

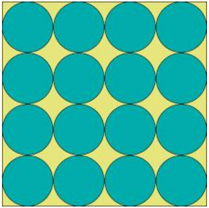
Parameter: Let $\delta > 0$ be a integer parameter

δ-separated: A set system (X, Σ) is δ-separated if for all S_1, S_2 in Σ, if the size of the symmetric difference (Hamming distance) $S_1 \Delta S_2$ is greater than δ, i.e. $|S_1 \Delta S_2| > \delta$.

δ-packing number: The cardinality of the largest δ-separated subcollection of Σ is called the δ-packing number of Σ.

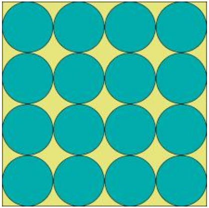
Connection to Euclidean packing

This is analogous to packing maximum number of Euclidean balls of radius $\delta/2$ in a box with edge length n.



Connection to Euclidean packing

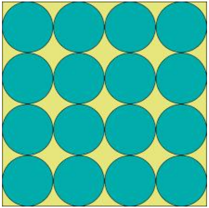
This is analogous to packing maximum number of Euclidean balls of radius $\delta/2$ in a box with edge length n.



In $\mathbb{R}^d$, this packing number is $O\left(\left(\frac{n}{\delta}\right)^d\right)$

Connection to Euclidean packing

This is analogous to packing maximum number of Euclidean balls of radius $\delta/2$ in a box with edge length n.



In $\mathbb{R}^d$, this packing number is $O\left(\left(\frac{n}{\delta}\right)^d\right)$

We have the same bound for the case of set systems with VC dimension d. (due to Haussler, Chazelle and Wernisch)

Shallow packing result

Theorem (Dutta-Ezra-G.'15 and Mustafa'16)

Let (X, Σ) be a set system with VC-dim d and shallow cell complexity $\varphi(\cdot)$ on a n-point set X. Let $\delta \geq 1$ and $k \leq n$ be two integer parameters such that:

1. $\forall S \in \sum$, $|S| \leq k$, *and*
2. $\sum$ *is* δ*-packed.*

Then

$$|\Sigma| \leq \frac{dn}{\delta}\varphi\left(\frac{d\,n}{\delta}, \frac{d\,k}{\delta}\right)$$

We can show that the above bound is tight.

Applications: Optimal nets

Theorem (Dutta-G.-Mustafa'17)

Let (X, Σ) be a set system with VC-dim d and shallow cell complexity $\varphi(\cdot)$ on a n-point set X. Then there exists an ε-net of size

$$\frac{1}{\varepsilon} \log \varphi\left(\frac{d}{\varepsilon}, d\right) + \frac{d}{\varepsilon}$$

Applications: Optimal nets

Theorem (Dutta-G.-Mustafa'17)

Let (X, Σ) be a set system with VC-dim d and shallow cell complexity $\varphi(\cdot)$ on a n-point set X. Then there exists an ε-net of size

$$\frac{1}{\varepsilon} \log \varphi\left(\frac{d}{\varepsilon}, d\right) + \frac{d}{\varepsilon}$$

Remark: The proof just uses the Shallow Packing Lemma and the Alteration Technique from *The Probabilistic Method*.

Application: Mnets bound

Theorem (D.–G.–J.–M. '17)

Semialgebraic *set systems with* **VC-dim**. $d < \infty$ *and* **shallow cell complexity** φ *have an* ε*-Mnet of size*

$$O\left(\frac{d}{\varepsilon} \cdot \varphi\left(\frac{d}{\varepsilon}, d\right)\right).$$

Application: Mnets bound

Theorem (D.–G.–J.–M. '17)

Semialgebraic *set systems with* **VC-dim**. $d < \infty$ *and* **shallow cell complexity** φ *have an* ε*-Mnet of size*

$$O\left(\frac{d}{\varepsilon} \cdot \varphi\left(\frac{d}{\varepsilon}, d\right)\right).$$

Remark: The proof just uses Guth-Katz's Polynomial Partitioning Theorem together with Shallow Packing Lemma.

From Mnets to ε-nets

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \text{ε-Mnet of size } M \\ \text{with sets of size} \geq \tau\varepsilon n \end{pmatrix} \implies \text{ε-net of size } \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

This gives ε-nets of size $\frac{d}{\varepsilon}\log\varphi\left(\frac{d}{\varepsilon}, d\right)$ for semialgebraic set systems.

From Mnets to ε-nets

Theorem (D.–G.–J.–M. '17)

$$\left(\begin{array}{c}\varepsilon\text{-Mnet of size } M \\ \text{with sets of size} \geq \tau\varepsilon n\end{array}\right) \implies \varepsilon\text{-net of size } \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

This gives ε*-nets of size* $\frac{d}{\varepsilon}\log\varphi\left(\frac{d}{\varepsilon}, d\right)$ *for semialgebraic set systems.*

- Yields best known bounds on ε-nets for geometric set systems with bounded VC-dim.

From Mnets to ε-nets

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \varepsilon\text{-Mnet of size } M \\ \text{with sets of size} \geq \tau\varepsilon n \end{pmatrix} \implies \varepsilon\text{-net of size } \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

This gives ε-nets of size $\frac{d}{\varepsilon} \log \varphi\left(\frac{d}{\varepsilon}, d\right)$ for semialgebraic set systems.

- Yields best known bounds on ε-nets for geometric set systems with bounded VC-dim.

Table: Upper bounds on Mnets and ε-nets

	Mnet	ε-net
Disks	ε^{-1}	ε^{-1}
Rectangles	$\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$	$\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}$
Halfspaces ($\mathbf{R}^d$)	$O\left(\varepsilon^{-\lfloor d/2 \rfloor}\right)$	$\frac{d}{\varepsilon} \log \frac{1}{\varepsilon}$

Probabilistic proof

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \varepsilon\text{-}\mathit{Mnet\ of\ size}\ M \\ \mathit{with\ sets\ of\ size} \geq \tau\varepsilon n \end{pmatrix} \implies \varepsilon\text{-}\mathit{net\ of\ size}\ \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

Proof.

1. $\mathcal{M}$ is such an Mnet. Let $p = \frac{1}{\tau\varepsilon n}\log(\varepsilon M)$.

$\square$

Probabilistic proof

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \varepsilon\text{-}Mnet\ of\ size\ M \\ with\ sets\ of\ size \geq \tau\varepsilon n \end{pmatrix} \implies \varepsilon\text{-}net\ of\ size\ \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

Proof.

1. $\mathcal{M}$ is such an Mnet. Let $p = \frac{1}{\tau\varepsilon n}\log(\varepsilon M)$.
2. Pick every point into a sample S with probability p.

$\square$

Probabilistic proof

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \varepsilon\text{-}Mnet\ of\ size\ M \\ with\ sets\ of\ size \geq \tau\varepsilon n \end{pmatrix} \implies \varepsilon\text{-}net\ of\ size\ \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

Proof.

1. $\mathcal{M}$ is such an Mnet. Let $p = \frac{1}{\tau\varepsilon n}\log(\varepsilon M)$.
2. Pick every point into a sample S with probability p.
3. $\forall m \in \mathcal{M}, \quad \Pr[S \cap m = \emptyset] = (1-p)^{|m|} \leq e^{-p|m|} \leq \frac{1}{\varepsilon M}$

$\square$

Probabilistic proof

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \varepsilon\text{-}\mathit{Mnet\ of\ size}\ M \\ \mathit{with\ sets\ of\ size} \geq \tau\varepsilon n \end{pmatrix} \implies \varepsilon\text{-}\mathit{net\ of\ size}\ \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

Proof.

1. $\mathcal{M}$ is such an Mnet. Let $p = \frac{1}{\tau\varepsilon n}\log(\varepsilon M)$.
2. Pick every point into a sample S with probability p.
3. $\forall m \in \mathcal{M}, \quad \Pr[S \cap m = \emptyset] = (1-p)^{|m|} \leq e^{-p|m|} \leq \frac{1}{\varepsilon M}$
4. In expectation, $|S| + |m \in \mathcal{M} : S \cap m = \emptyset| \leq np + \frac{1}{\varepsilon}$.

$\square$

Probabilistic proof

Theorem (D.–G.–J.–M. '17)

$$\begin{pmatrix} \varepsilon\text{-Mnet of size } M \\ \text{with sets of size} \geq \tau\varepsilon n \end{pmatrix} \implies \varepsilon\text{-net of size } \frac{\log(\varepsilon M)/\tau + 1}{\varepsilon}$$

Proof.

1. $\mathcal{M}$ is such an Mnet. Let $p = \frac{1}{\tau\varepsilon n}\log(\varepsilon M)$.
2. Pick every point into a sample S with probability p.
3. $\forall m \in \mathcal{M}, \quad \Pr[S \cap m = \emptyset] = (1-p)^{|m|} \leq e^{-p|m|} \leq \frac{1}{\varepsilon M}$
4. In expectation, $|S| + |m \in \mathcal{M} : S \cap m = \emptyset| \leq np + \frac{1}{\varepsilon}$.
5. so there is an ε-net of size $\leq np + \frac{1}{\varepsilon}$ (why?).

□

Proof of Shallow Packing

Theorem follows from the following result:

Theorem

Let $(X, \mathcal{R})$ be a δ-separated set system with VC dimension at most d. Then

$$|\mathcal{R}| \leq 2\mathbb{E}\left[|\mathcal{R}_{A'}|\right]$$

where A' is an uniformaly random subset of X of size $\frac{4dn}{\delta} - 1$.

Proof of Shallow Packing

Theorem follows from the following result:

Theorem

Let $(X, \mathcal{R})$ be a δ-separated set system with VC dimension at most d. Then

$$|\mathcal{R}| \leq 2\mathbb{E}\left[|\mathcal{R}_{A'}|\right]$$

where A' is an uniformaly random subset of X of size $\frac{4dn}{\delta} - 1$.

For the proof of the main result consider:

$$\mathcal{R}' := \left\{\sigma \in \mathcal{R} : |\sigma \cap A'| > 3 \times \frac{4dk}{\delta}\right\} \quad (1)$$

$$\mathcal{R}'' := \mathcal{R} \setminus \mathcal{R}' \quad (2)$$

Conclusion

- Ideally we want a combinatorial proof of the Mnets bound for set systems.
- Improve the current lower bound.
- Find more applications/connections of Mnets in combinatorial geometry.

Thank you.