Complexity of Parikh's Theorem

Anthony Widjaja Lin

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$$T = \begin{array}{ccccc} & & a & & \\ & \swarrow & & \searrow & \\ & b & & & b \\ \swarrow & & \searrow & & \\ a & & & a & \end{array}$$

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Parikh (1961) suggested to remove ordering

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Q: What's the expressive power of standard automata models "modulo Parikh mapping $\mathcal{P}$" (i.e. treated as sets of vectors)?

Parikh's Theorem

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Semilinear sets = Presburger-definable subsets of $\mathbb{N}^k$.

Parikh's Theorem Almost Everywhere

- Verification of concurrent systems
  - Bounded context-switch analysis [Esparza-Ganty'11, Hague-Lin'12]
  - Asynchronous programs [Ganty-Majumdar'10]
  - Message-passing programs [Abdulla-Atig-Cederberg'13]
- Verification of (restrictions of) counter machines:
  - Reversal-bounded verification [Ibarra'79]
  - Flat counter machines [Fribourg-Olsen'97, Comon-Jurski'98]
  - Flattable counter machines [Bardin-Finkel-Leroux-Schnoebelen'05, Leroux-Sutre'05]
- Path logics over graph databases [Barcelo-Libkin-Lin-Woods'12]
- Cryptographic Analysis of C programs [Verma *et al.*'06]

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Computational complexity: Can we compute semilinear sets for Parikh images of these models efficiently?

Parikh's Theorem (more precisely)

Parikh Images

- Parikh Image $\mathcal{P}(L)$ of a language L over $\Sigma = \{a_1, \ldots, a_k\}$ is a subset of $\mathbb{N}^k$

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  - $\mathcal{P}(L) = \{(0,0), (1,1), (2,2), (3,3), \ldots\}$

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  - $\mathcal{P}(L) = \{(0,0), (1,1), (2,2), (3,3), \ldots\}$
- **Note**: $L' = (ab)^*$ and $\mathcal{P}(L) = \mathcal{P}(L')$.

Semilinear Sets

- A linear set (over $\mathbb{N}^k$) is a set of the form

$$L(\mathbf{v}_0; \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}) := \left\{ \mathbf{v}_0 + \sum_{i=1}^{m} a_i \mathbf{v}_i : a_1, \ldots, a_m \in \mathbb{N} \right\}$$

for some offset $\mathbf{v}_0 \in \mathbb{N}^k$ and periods $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{N}^k$ and $m \in \mathbb{N}$.

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- **Example**: $\{(i, i) : i \in \mathbb{N}\} = L((0, 0); \{(1, 1)\})$
- A semilinear set over $\mathbb{N}^k$ is a finite union of linear sets.

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Theorem (Parikh): Parikh images of context-free and regular languages are effectively semilinear.

- Descriptional complexity: size of smallest description
- Computational complexity: how efficient to compute

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Parikh's original proof gives exponential complexity bound with linearly many periods for each linear set!

CFG Case

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$$A \rightarrow aBb \tag{2}$$
$$B \rightarrow aAb \tag{3}$$
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- $\{1, 3, 4\}$ — 3 cannot be fired without 2 (C2)

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Note: This has been successfully used in many applications in infinite-state verification.

Lower bound for semilinear sets for CFGs

Proposition: There is an infinite sequence $\{G_n\}_{n\in\mathbb{N}}$ of CFGs over $\Sigma = \{a\}$ s.t. $\mathcal{P}(L(G_n))$ must have at least $2^{\Omega(|G_n|)}$ linear sets.

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\begin{aligned}
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A_i &\to \varepsilon \quad \text{for each } 0 \le i < n \\
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n.b. this kind of encoding for CFG is from (Stockmeyer-Meyer'73)

NFA Case

Exponential lower bound for DFAs

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Note: Σ_n grows with n

What about a fixed alphabet size?

Unary alphabet case: Chrobak-Martinez Theorem

Let us assume that the alphabet is unary, i.e., $\Sigma = \{a\}$.

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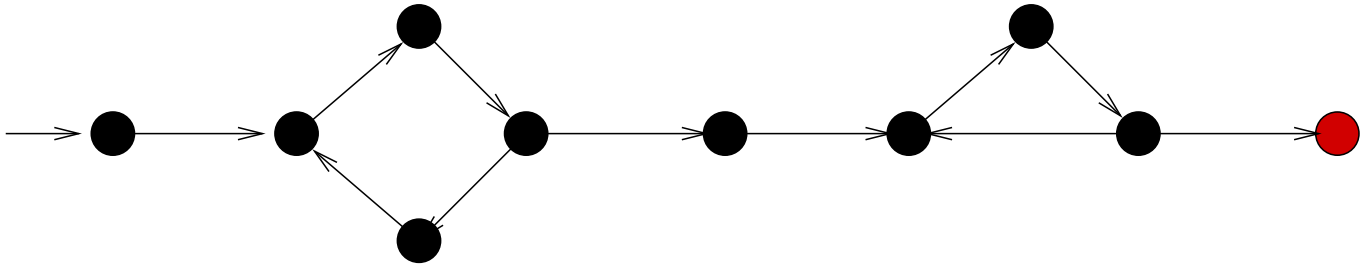
Note:

- quadratically many union of arithmetic progressions with periods of linear size suffice.

Chrobak-Martinez Theorem in Action



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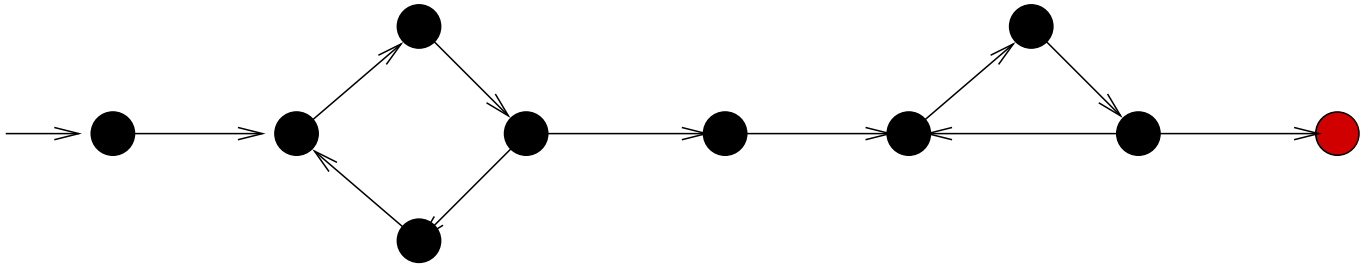

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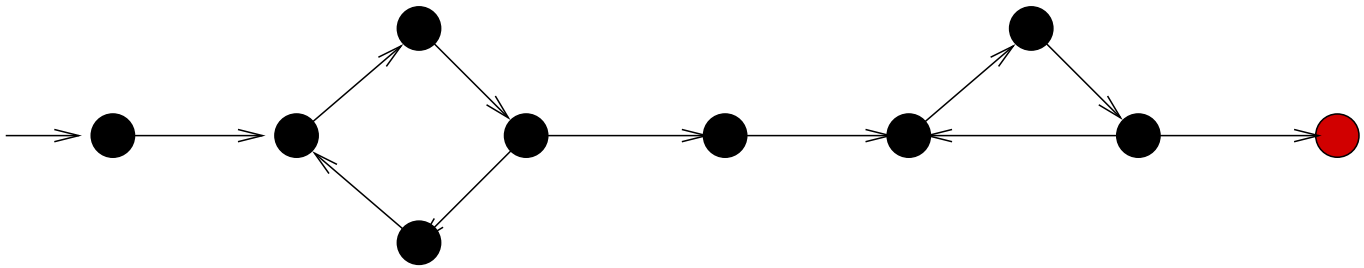

Parikh image of $L(\mathcal{A})$ is $8 + 4\mathbb{N} + 3\mathbb{N}$ which is equal to

$$(8 + 4\mathbb{N}) \cup (11 + 4\mathbb{N}) \cup (14 + 4\mathbb{N}) \cup (17 + 4\mathbb{N})$$

Generalised Chrobak-Martinez Theorem

Let $\Sigma := \{a_1, \ldots, a_k\}$ for fixed $k \in \mathbb{Z}_{>0}$.

Generalised Chrobak-Martinez Theorem

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Theorem (Kopczynski & Lin'10): Descriptional and computational complexity of Parikh Images of NFAs are polynomial.

- union of polynomially many linear sets with at most k polynomially-bounded periods
- Complexities are exponential in k
- Generalizes Chrobak-Martinez Theorem (case $k = 1$).

Outline of proof

- Normal-Form Theorem for Semilinear sets.
- Normal-Form Theorem for Parikh images of NFA

Normal Form Theorem for Semilinear Sets

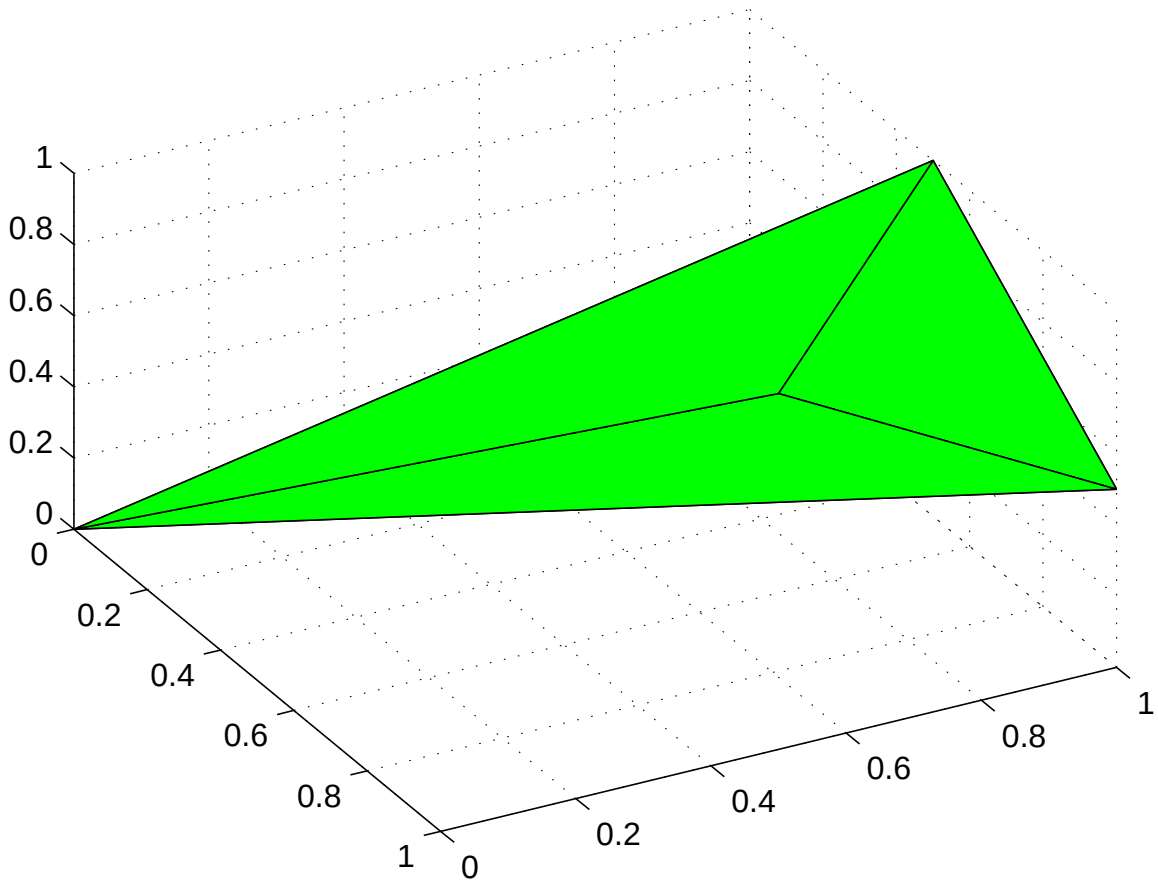
Digression to Convex Geometry: Real Cones

Given $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^k$, define the real cone over V:

$$\textbf{cone}(V) := \{\Sigma_{i=1}^{n} a_i \mathbf{v}_i : a_i \in \mathbb{R}_{\geq 0}\}.$$

Note: akin to definition of vector subspace of $\mathbb{R}^k$ "spanned" by some set V (but ...)

Examples



Real cone of $(1, 1, 0.5), (1, 0.5, 1), (0.5, 1, 1)$

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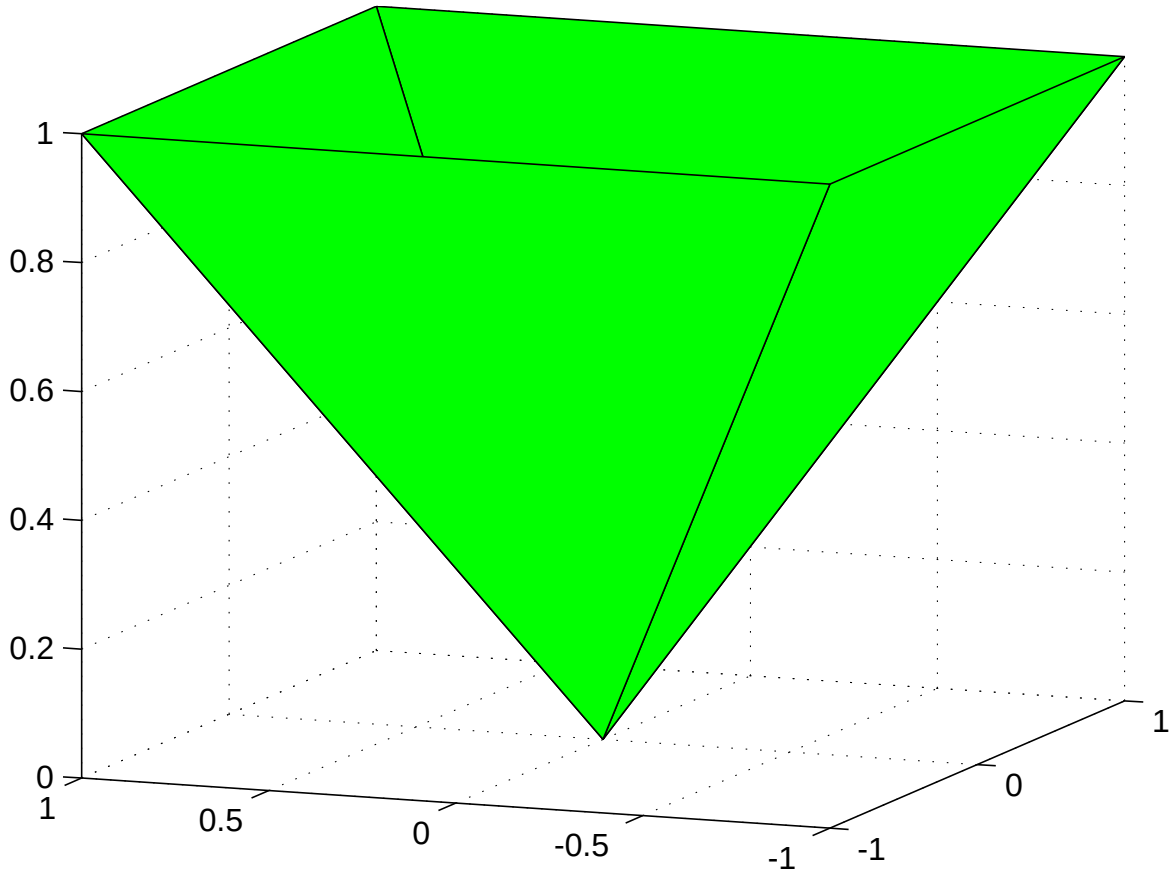

Real cone over $\{1, -1\} \times \{1, -1\} \times \{1\}$

Caratheodory's theorem

Theorem: Given a finite $V \subseteq \mathbb{R}^k$ of rank $d \leq k$, we have

$$\textbf{cone}(V) = \bigcup_{V' \subseteq V, |V'| = \|V'\| = d} \textbf{cone}(V').$$

- *Cones over $\mathbb{R}^k$ can be decomposed into smaller subcones with $\leq k$ vertices*

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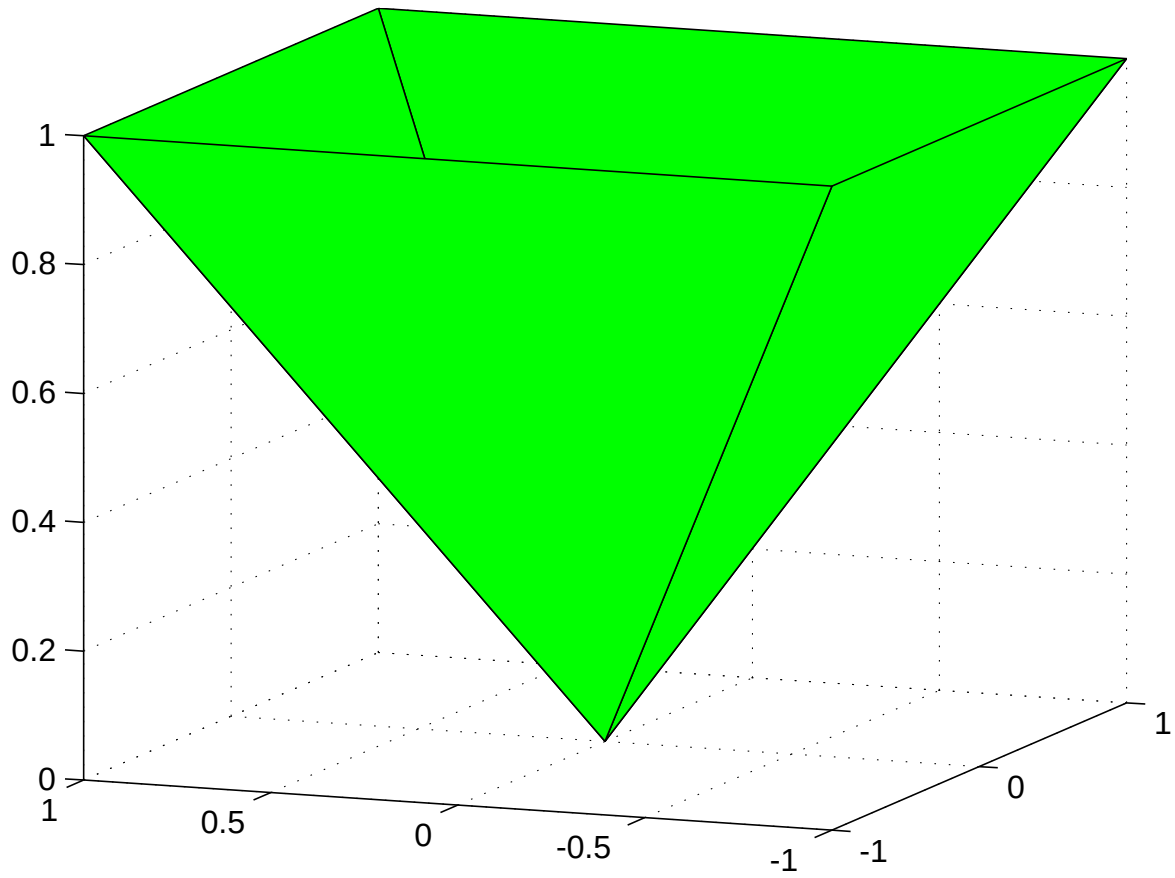
- ***Note**: if k is fixed, there are only polynomially many such subcones.*

Caratheodory's theorem (example)



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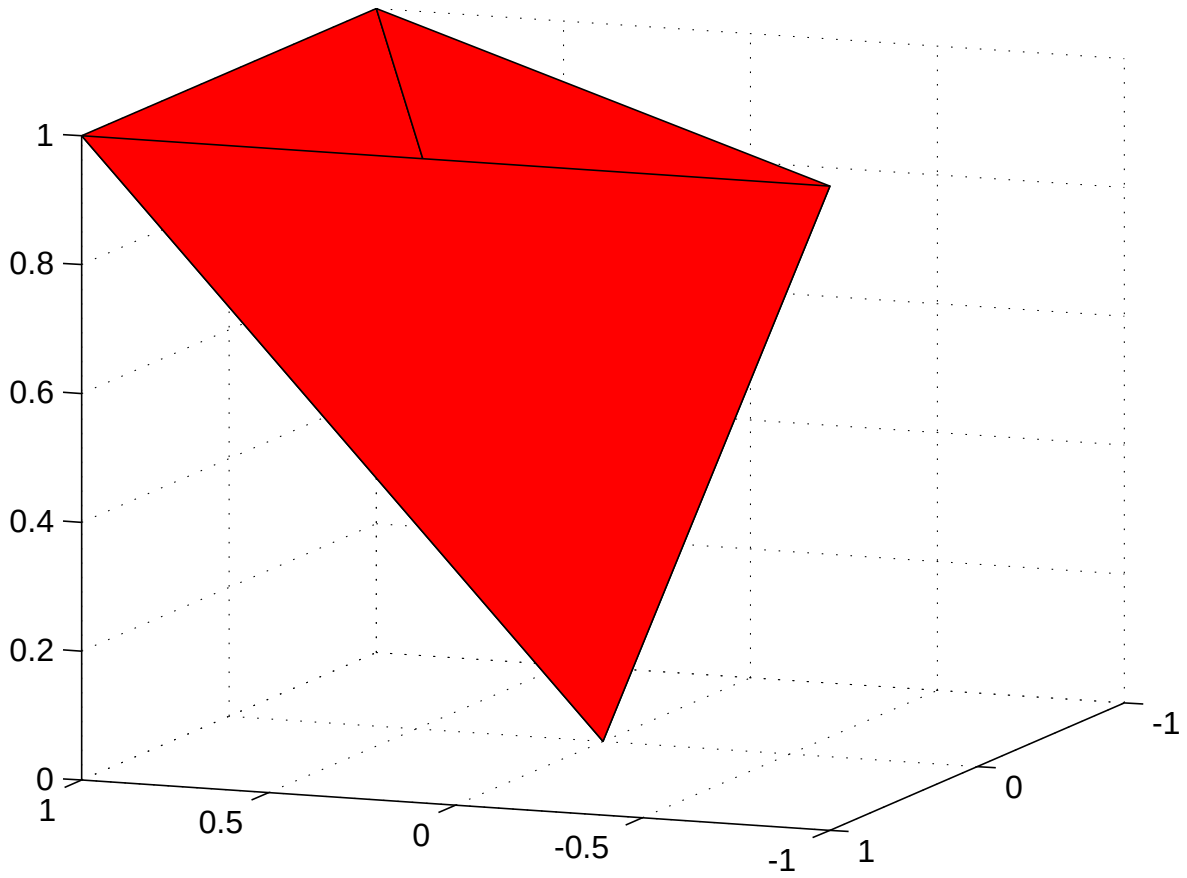

Real cone over $(1, 1, 1), (1, -1, 1), (-1, 1, 1)$

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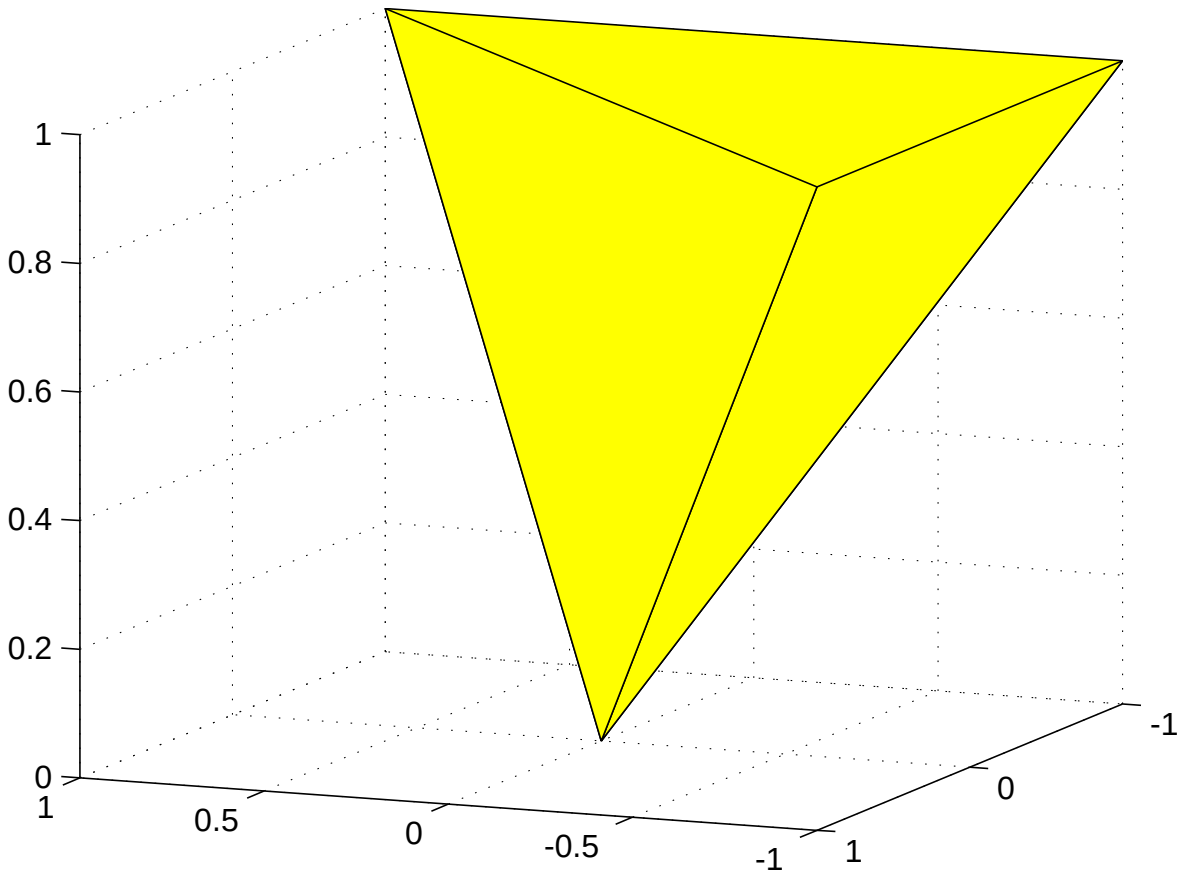

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Caratheodory-like theorem for linear sets

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- Fortunately, it does if you allow nonzero offsets :)

Caratheodory-like theorem for linear sets

Theorem: Fix $k \in \mathbb{N}$. Given a finite $V \subseteq \mathbb{Z}^k$ of rank $d \leq k$, we can compute in pseudopolynomial time linear sets $L(\mathbf{w}_1; S_1), \ldots, L(\mathbf{w}_s; S_s)$ s.t. each $S_i \subseteq V$ and $|S_i| = d$

$$L(\mathbf{0}; V) = \bigcup_{i=1}^{s} L(\mathbf{w}_i; S_i).$$

Each number in $\mathbf{w}_i$ is polynomially large (in unary)

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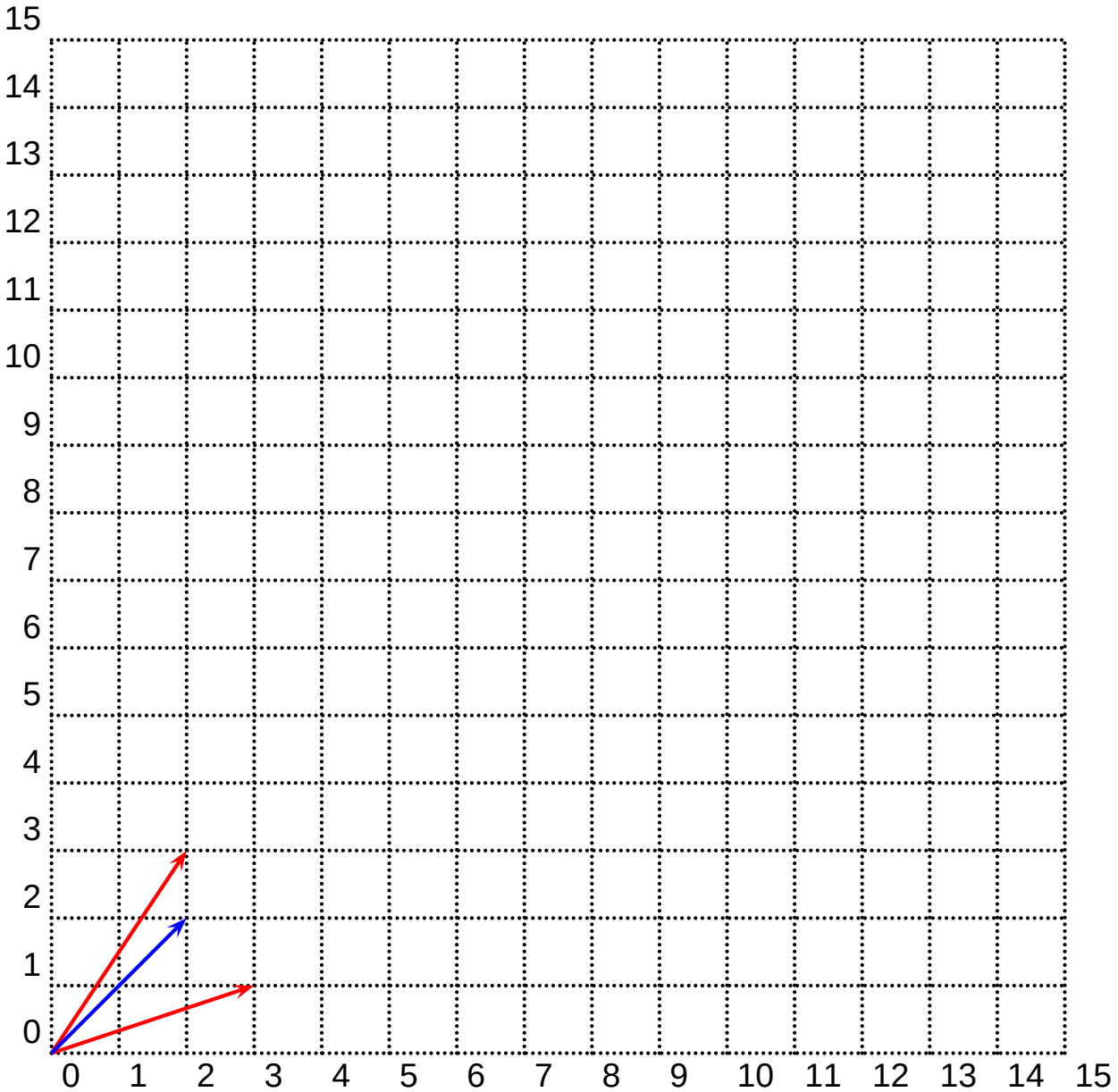
Use dynamic programming to compute all $\mathbf{w}_i$'s

Intuition for dimension 2



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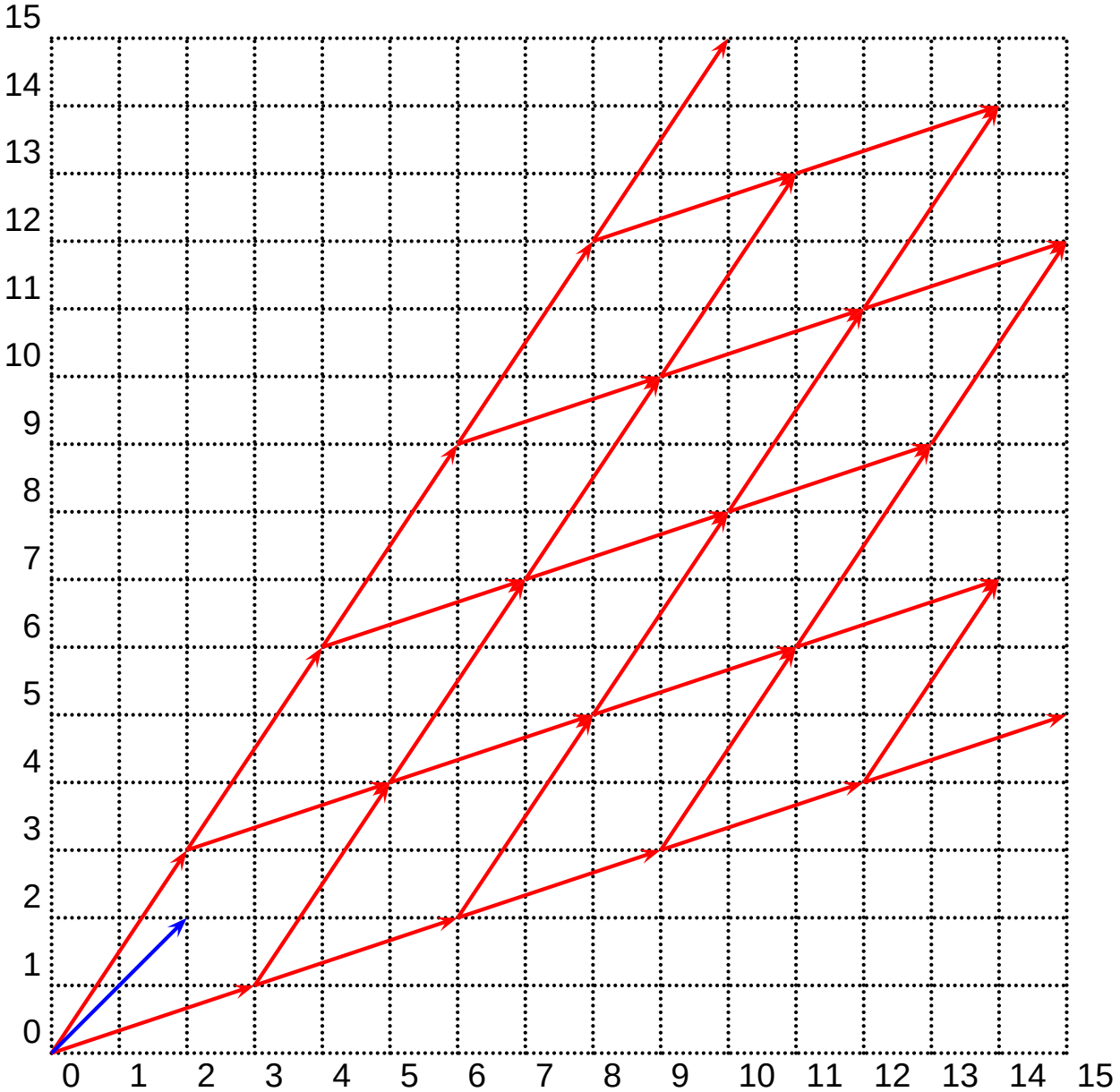

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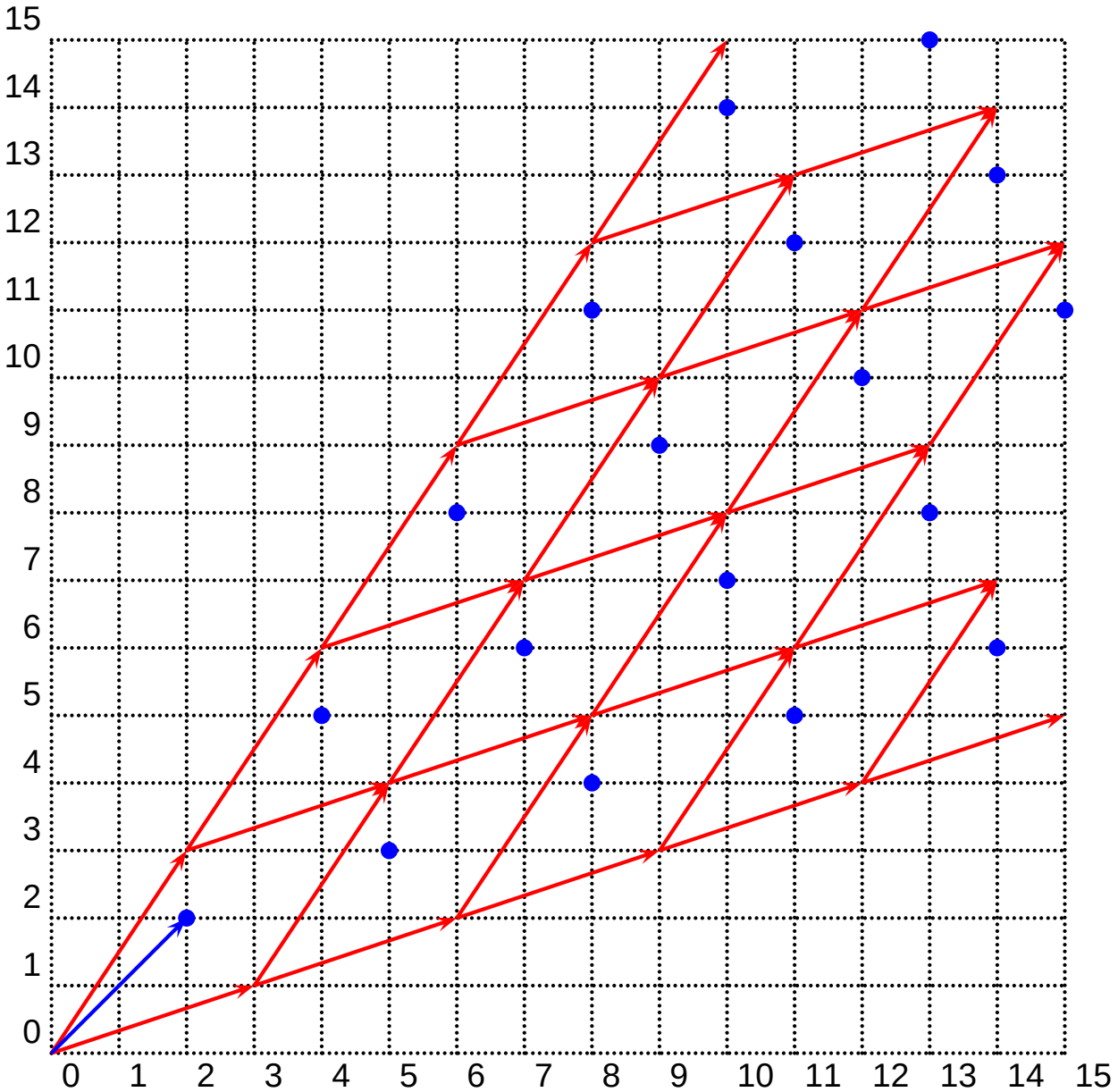

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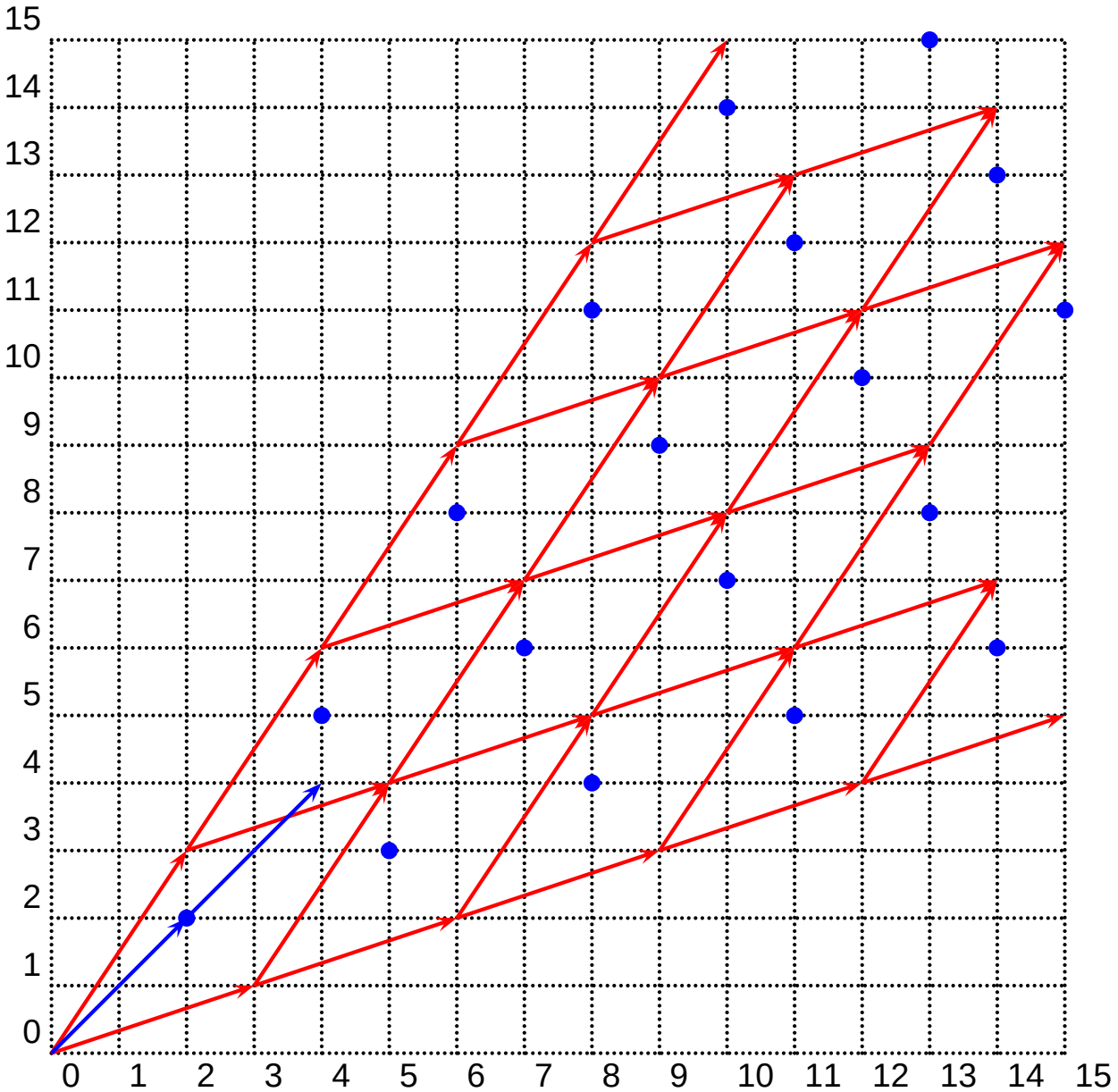

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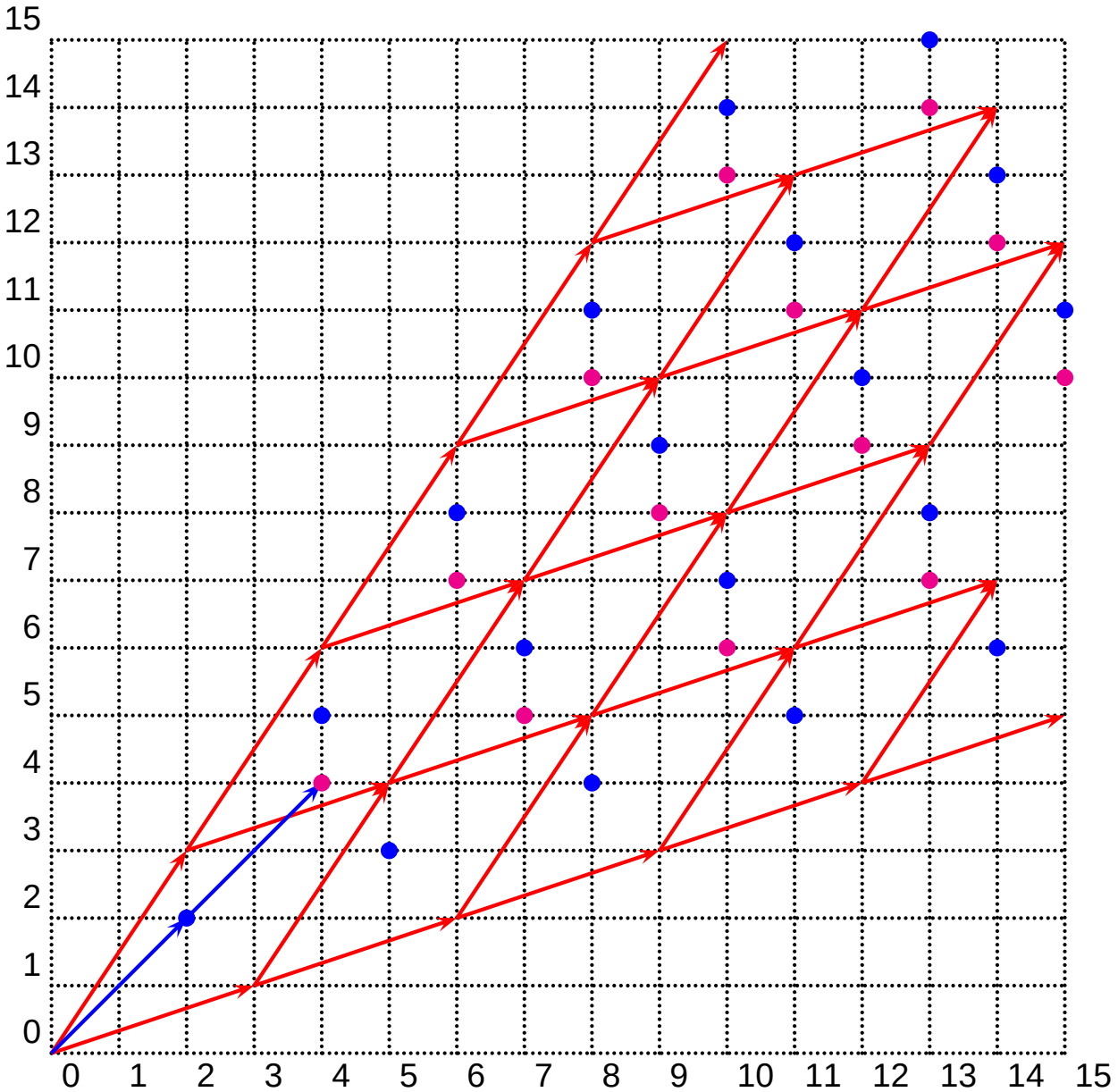

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- $|\mathcal{M}|$ and all numbers in $\mathcal{M}$ are "not big".

Case of dimension > 2

- Need to use linear sets with different periods (unlike $d = 2$)
- Replace finding two outermost vectors with Caratheodory's
- The rest are similar:
  - Dynamic programming,
  - Bounds from integer programming

Normal Form Theorem for Parikh Images of NFA

Statement of the Theorem

Let $\Sigma := \{a_1, \ldots, a_k\}$ for fixed $k \in \mathbb{Z}_{>0}$.

Theorem (Kopczynski & Lin, LICS 2010): Descriptional and computational complexity of Parikh Images of NFAs are polynomial.

- poly many union of linear sets with at most k periods with poly-bounded numbers
- Complexities are exponential in k .
- Generalizes Chrobak-Martinez Theorem (case $k = 1$).

Proof idea: Path types

Notation: $Q = \{q_0, \ldots, q_{n-1}\}$ with initial state q_0 and final state q_{n-1}.

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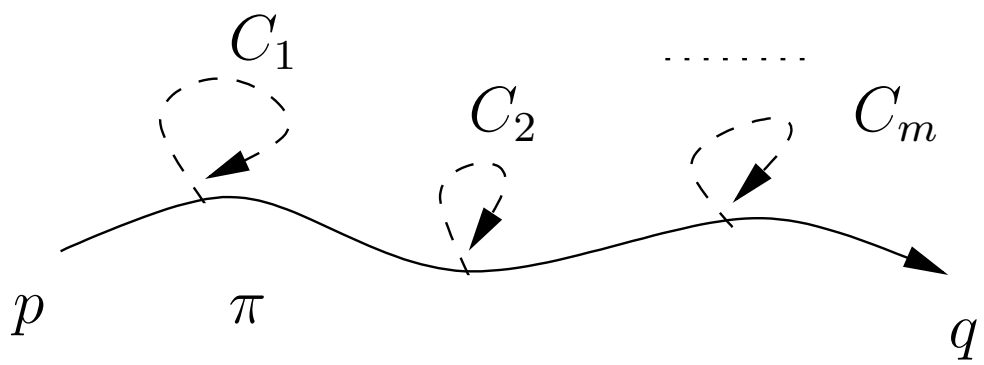
Given a path π and simple cycles $C_1, \ldots, C_m$ meeting with π:



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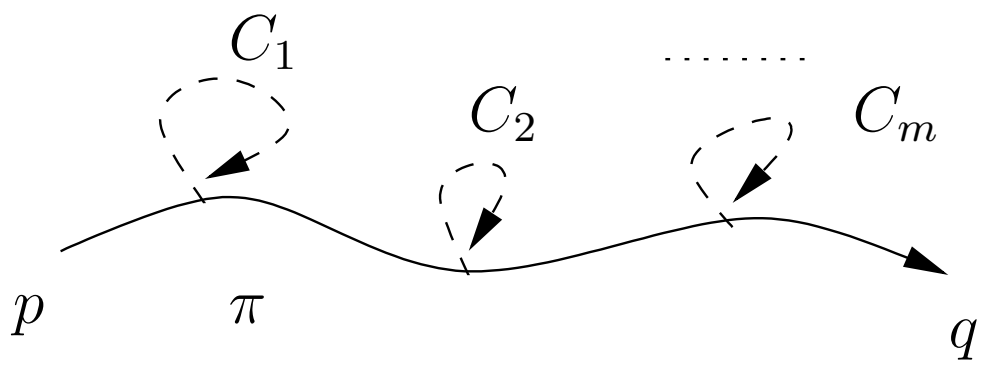

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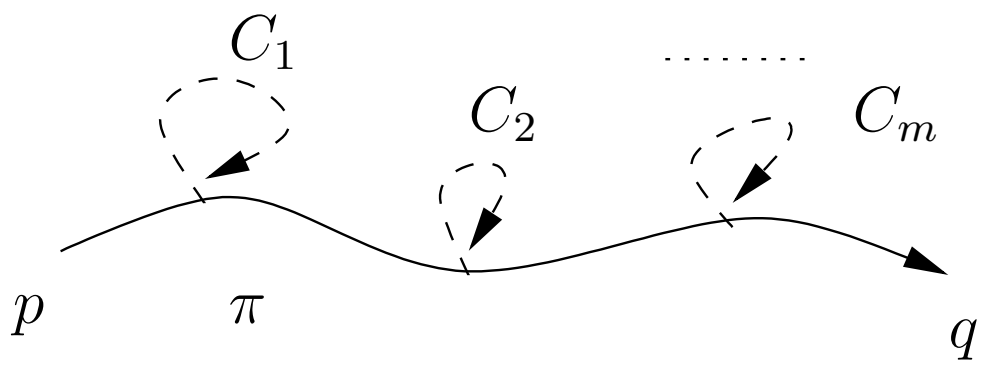

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There are at most $n^{k^{O(1)}}$ many periods.

Proof idea

Characterization of Parikh images of $L(\mathcal{A})$

Lemma: $\mathcal{P}(L(\mathcal{A})) = \bigcup_\pi T_\pi$, where π ranges over paths from initial to final state of length at most $O(n^2)$.

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Lemma: $\mathcal{P}(L(\mathcal{A})) = \bigcup_\pi T_\pi$, where π ranges over paths from initial to final state of length at most $O(n^2)$.

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BUT: can apply Caratheodory-like theorem on each T_π:

Corollary: $\mathcal{P}(L(\mathcal{A}))$ coincides with a union of polynomially many linear sets with polynomially large offsets and at most k polynomially large periods.

Computational complexity

Theorem: $\mathcal{P}(L(\mathcal{A}))$ coincides with a union of polynomially many linear sets with polynomially large offsets and at most k polynomially large periods.

A naive implementation takes exponential time.

Can improve to PTIME using dynamic programming!

Parikh images of extensions of NFA

Linear grammars

$$S \rightarrow A \tag{5}$$
$$A \rightarrow aBb \tag{6}$$
$$B \rightarrow aAb \tag{7}$$
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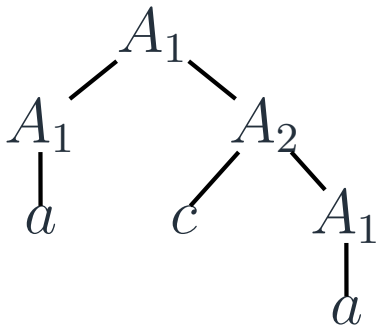
Proof: Linear grammars are essentially NFA ...

CFG of fixed “dimensions”



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Dimensions measure how many times “doubling tricks” possibly get used in a CFG.

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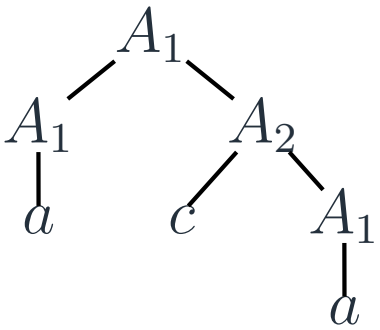

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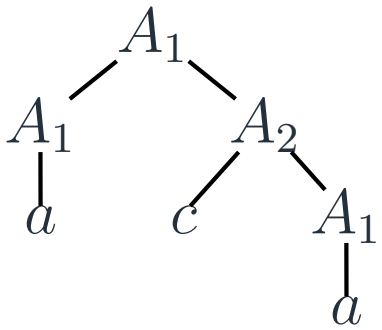

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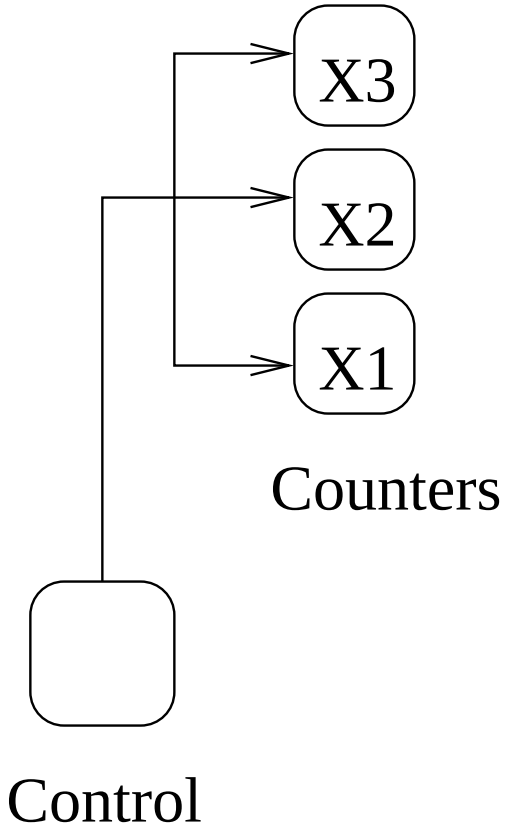
Proof: Compute an equivalent NFA of polynomial size and use Generalised Chrobak-Martinez Theorem.

Reversal-bounded counter automata

Minsky's counter automata



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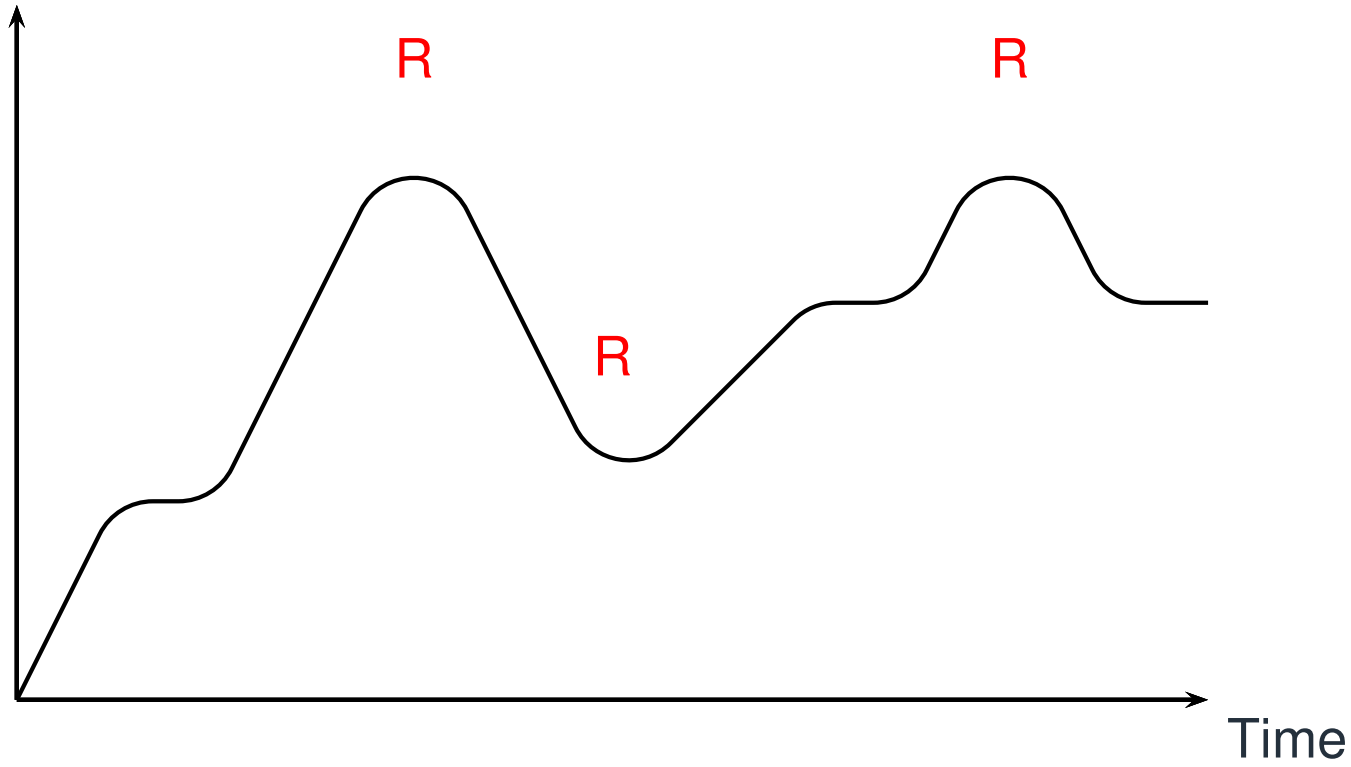

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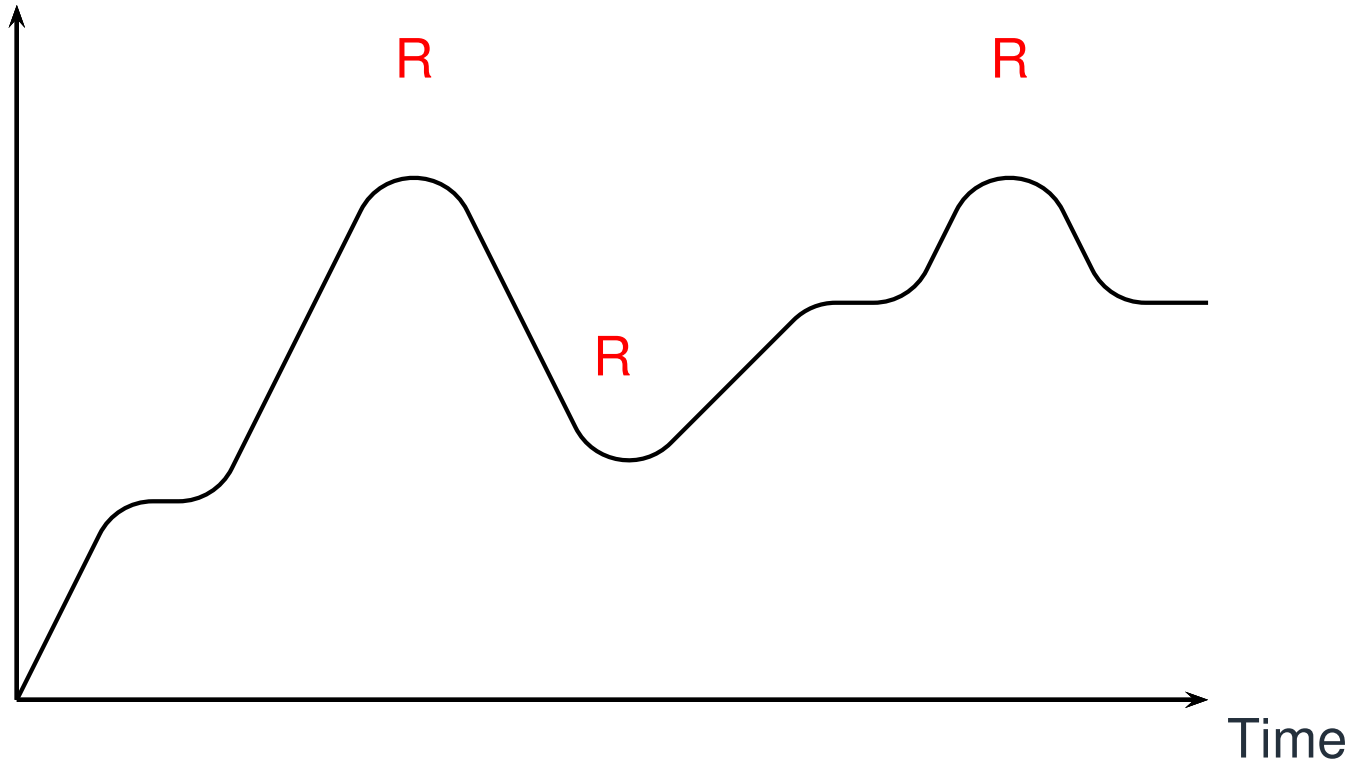

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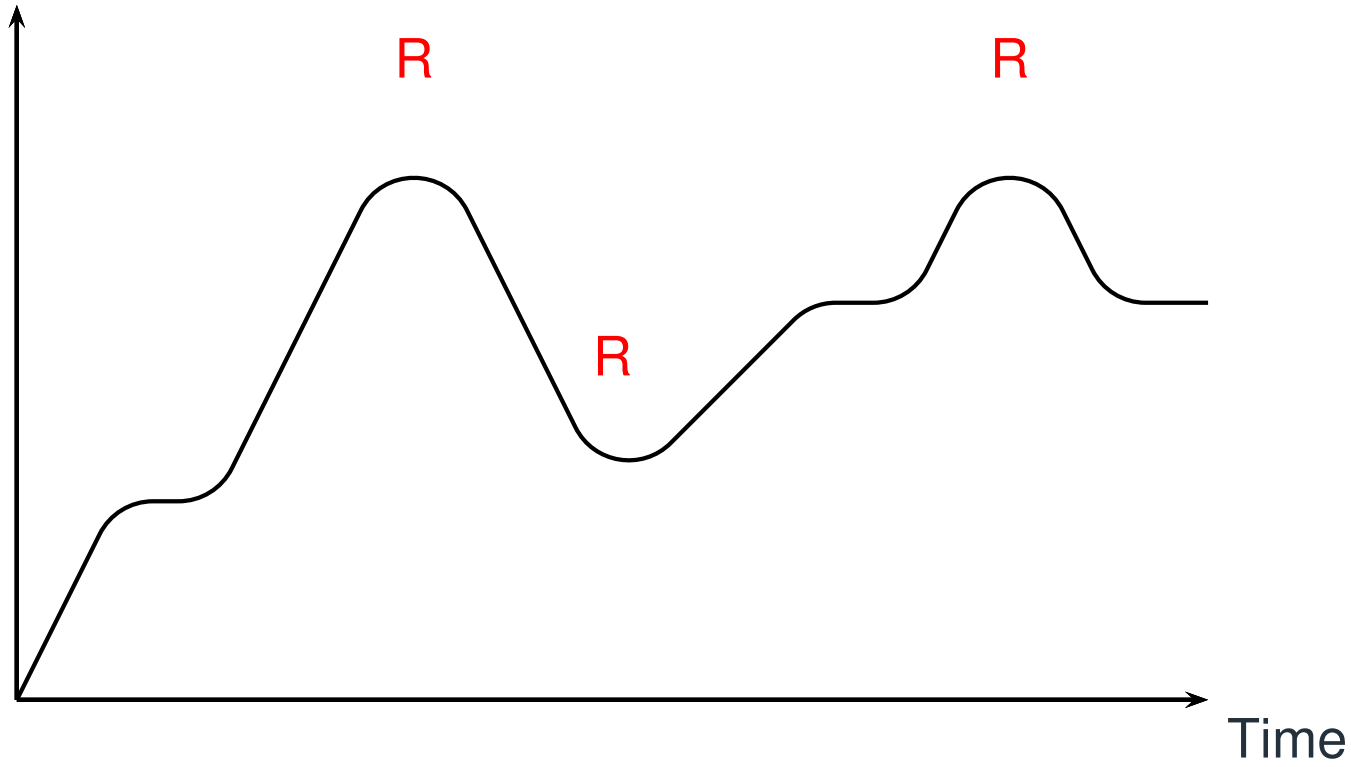

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No finite bounds on length of 0-reversal-bounded paths!

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Theorem (Hague-Lin'11): This does not work if # reversals/# counters are non-fixed, but you can compute an existential Presburger formula in polynomial time (even if a pushdown stack is added).

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Study the complexity of decision problems (e.g. membership, universality, ...) over different models.

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Can be proven by first constructing Parikh images (as semilinear set representation or existential Presburger formulas).

Conclusion

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THANKS!!