Hyper-Ackermannian Bounds for Pushdown Vector Addition Systems

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Table of Contents

1. Pushdown Vector Addition Systems
2. Reduced Reachability Tree for Pushdown VAS
3. Worst-Case Size of the Reduced Reachability Tree
4. Conclusion
Table of Contents

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Pushdown Vector Addition Systems — Model

(-1), push(B)  (-1), push(A)  (+2), pop(A)

(+5) → (+1) → (+1)

\( p \) \( q \) \( r \)

pop(B)

VAS

\( d \) (implicit) counters over \( \mathbb{N} \)

counter actions

- syntax: \( a \in \mathbb{Z}^d \)
- semantics: \( v \in \mathbb{N}^d \rightarrow v + a \in \mathbb{N}^d \)

Stack

finite stack alphabet

push and pop

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Hyper-Ack. Bounds for Pushdown VAS

ACTS 2015
Pushdown Vector Addition Systems — Model

$(-1), \text{push}(B)$

$(-1), \text{push}(A)$

$(+2), \text{pop}(A)$

$(+5)$

$(+1)$

$\text{pop}(B)$

$(p, 5, \varepsilon)$

$(p, 4, B)$

$(p, 3, B B)$

$(q, 4, B B)$

$(q, 3, B B A)$

$(q, 0, B B A A A A)$

$(r, 0, B B A A A A A)$

$(r, 8, B B)$

$(q, 8, B)$
Pushdown Vector Addition Systems — Motivations

VASS
\[\vdash\]
VAS
\[\vdash\]
Petri net

\Rightarrow Richer model for the verification of concurrent systems
- Multi-threaded recursive programs
- One recursive server + unboundedly many finite-state clients
Pushdown Vector Addition Systems — Motivations

- Richer model for the verification of concurrent systems
  - Multi-threaded recursive programs
  - One recursive server + unboundedly many finite-state clients

- Is the model too powerful?
## Brief State of the Art

<table>
<thead>
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<th>Reachability</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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[1] Lipton 1976

Subclasses of pushdown VAS with decidable reachability

- Multiset pushdown systems [Sen, Viswanathan 2006]
- \( \text{VAS} \cap \text{CFL of finite index} \) [Atig, Ganty 2011]
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Subclasses of pushdown VAS with decidable reachability

- Multiset pushdown systems [Sen, Viswanathan 2006]
- VAS $\cap$ CFL of finite index [Atig, Ganty 2011]
Our Contribution

- Boundedness is decidable for pushdown VAS
  - Reduced reachability tree: adaptation of the VAS case

- Worst-case complexity of the algorithm: hyper-Ackermannian
  - Bound the length of bad nested sequences over \((\mathbb{N}^d, \leq)\)
  - Weak computation of an hyper-Ackermannian function
  - Inspired from recent results on bad sequences for various wqo’s
    - [Figueira, Figueira, Schmitz, Schnoebelen 2011]
    - [Schmitz, Schnoebelen 2011]
    - ...
Table of Contents

1 Pushdown Vector Addition Systems

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Reachability Tree of a Pushdown VAS

Reachability Tree of a Pushdown VAS

- Exhaustive and enumerative forward exploration from $(v_{\text{init}}, \varepsilon)$
- Potentially infinite, need to truncate
Truncation rule:

\[ \forall v, v' \leq v \rightarrow v < v' \leq v' < \cdots \]

\[ v_{\text{init}} \]

\[ v \quad \text{if } v \leq v' \]
Reduced Reachability Tree for VAS [Karp, Miller 1969]

Truncation rule:

For every VAS, ≤ and < are simulation relations

Truncation entails that

- \( v_{init} \rightarrow^* v \rightarrow^* v' \rightarrow^* v'' \rightarrow^* v''' \cdots \)
- If \( v < v' \) then \( v < v' < v'' < v''' < \cdots \)
Theorem ([Karp, Miller 1969])

The reduced reachability tree of a VAS $A$ is finite.

Proof. König’s Lemma $+$ Dickson’s Lemma

Theorem ([Karp, Miller 1969])

A VAS $A$ is unbounded if, and only if, its reduced reachability tree contains a leaf that is strictly larger than one of its ancestors.
RRT-based Algorithm for VAS Boundedness

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A VAS $A$ is unbounded if, and only if, its reduced reachability tree contains a leaf that is strictly larger than one of its ancestors.

Theorem ([McAloon 1984], [Figueira et al. 2011])

The size of the reduced reachability tree of a VAS $A$ is at most

- primitive-recursive in $|A|$ when the dimension $d$ is fixed,
- Ackermannian in $|A|$ when the dimension is part of the input.
Consider a run

\[(v_{init}, \varepsilon) \rightarrow^* (v, \sigma) \rightarrow^* (v', \sigma')\]

such that

\[v \leq v' \quad \text{and} \quad \sigma \leq_{\text{suffix}} \sigma'\]

Then \[(v_{init}, \varepsilon) \rightarrow^* (v, \sigma) \rightarrow^* (v', \sigma') \rightarrow^* (v'', \sigma'') \rightarrow^* (v''', \sigma''') \ldots\]
Consider a run
\[(v_{\text{init}}, \varepsilon) \rightarrow^* (v, \sigma) \rightarrow^* (v', \sigma')\]
such that
\[v \leq v' \quad \text{and} \quad \sigma \leq_{\text{suffix}} \sigma'\]
Then
\[(v_{\text{init}}, \varepsilon) \rightarrow^* (v, \sigma) \rightarrow^* (v', \sigma') \rightarrow^* (v'', \sigma'') \rightarrow^* (v''', \sigma''') \cdots\]
But:
\[(p, \varepsilon) \downarrow (q, A) \downarrow (q, A B) \downarrow (q, A B B) \downarrow \cdots\]
\[\text{push}(A) \quad \text{push}(B)\]
No truncation, infinite branch!
Truncation rule:

\[ (v_{\text{init}}, \varepsilon) \xrightarrow{*} (v, \sigma) \xrightarrow{*} (v', \sigma') \xrightarrow{*} (v'', \sigma'') \xrightarrow{*} (v''', \sigma''') \cdots \]

If \( v < v' \) then \( v < v' < v'' < v''' < \cdots \)

If \( \sigma \prec_{\text{prefix}} \sigma' \) then \( \sigma \prec_{\text{prefix}} \sigma' \prec_{\text{prefix}} \sigma'' \prec_{\text{prefix}} \sigma''' \prec_{\text{prefix}} \cdots \)
Theorem

The reduced reachability tree of a pushdown VAS $A$ is finite.

Proof. By contradiction, assume that it is infinite. The tree is finitely branching. So, by König’s Lemma, there is an infinite branch

$$(v_{\text{init}}, \varepsilon) \to (v_1, \sigma_1) \to (v_2, \sigma_2) \cdots$$
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$$\cdots \cdots \cdots \cdots$$

$$v \quad v' \geq v$$
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Finiteness of the Reduced Reachability Tree

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![Diagram showing the tree structure with a proof symbol](image)
Finiteness of the Reduced Reachability Tree

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$$(v_{\text{init}}, \varepsilon) \rightarrow (v_1, \sigma_1) \rightarrow (v_2, \sigma_2) \cdots$$
A pushdown VAS $A$ is unbounded if, and only if, its reduced reachability tree contains a path

$$((v, \sigma) \rightarrow (v', \sigma'))$$

such that $v \leq v'$ and $\sigma \leq_{\text{prefix}} \sigma'$, and at least one of these inequalities is strict.

How big can the reduced reachability tree be?
Table of Contents

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Fast Growing Functions

Functions $F_\alpha : \mathbb{N} \rightarrow \mathbb{N}$, indexed by ordinals $\alpha \leq \omega^\omega$

\[
F_0(n) = n + 1,
\]
\[
F_{\alpha+1}(n) = F_\alpha^{n+1}(n)
\]
\[
F_\lambda(n) = F_{\lambda_n}(n) \quad \text{if } \lambda \text{ is a limit ordinal}
\]

$F_1$ : linear, $F_2$ exponential, $F_3$ tower of exponentials

$F_\omega$ is an Ackermannian function

$F_{\omega^\omega}$ is an hyper-Ackermannian function

Example

\[
F_{\omega^\omega}(2) = F_{\omega^3}(2) = F_{\omega^{2.3}}(2)
\]
\[
= F_{\omega^{2.2+\omega.3}}(2)
\]
\[
= F_{\omega^{2.2+\omega.2+3}}(2)
\]
\[
= F_{\omega^{2.2+\omega.2+2}(F_{\omega^{2.2+\omega.2+2}}(2))}
\]
Theorem

The height of the reduced reachability tree of a pushdown VAS $A$ is at most $F_{\omega(d+1)}(|A|)$ where $d$ is the dimension of $A$.

Corollary

The size of the reduced reachability tree of a pushdown VAS $A$ is at most

- multiply-recursive in $|A|$ when the dimension $d$ is fixed,
- hyper-Ackermannian in $|A|$ when the dimension is part of the input.

Theorem

For all $n \in \mathbb{N}$, there exists a pushdown VAS $A_n$, of size quadratic in $n$, such that the reduced reachability tree of $A_n$ has at least $F_{\omega^\omega}(n)$ nodes.
Lower Bound

Weak computation of \( F_{\omega^d}(n) \) by a bounded pushdown VASS \( A_d(n) \)

- Use the stack to implement recursive calls

- Maintain \( n \) in 2 counters \( r \) and \( \overline{r} \) such that \( r + \overline{r} = n + 1 \)

- Maintain \( \alpha = \omega^d \cdot c_d + \cdots + \omega^0 \cdot c_0 \) in \( d + 1 \) counters

- Implement the inductive definition of \( F_\alpha \) by pushdown VAS rules
Lower Bound

Weak computation of $F_{\omega^d}(n)$ by a bounded pushdown VASS $A_d(n)$

- Use the stack to implement recursive calls
  But we cannot store the calling context $\alpha$!

- Maintain $n$ in 2 counters $r$ and $\bar{r}$ such that $r + \bar{r} = n + 1$

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- Implement the inductive definition of $F_{\alpha}$ by pushdown VAS rules

---

Trick to restore the calling context $\alpha$ of pending recursive calls

- Push the operations (increments and decrements) that are being performed on $c_0, \ldots, c_d$

- Revert them when popping
Each branch of the RRT is a bad sequence over \((\mathbb{N}^d, \leq)\)

\[\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\downarrow \\
\not\mathbf{v} \\
\end{array}\]

Bad sequences are finite, but can be arbitrarily long

A sequence \(v_0, v_1, \ldots\) is \(n\)-controlled if \(\|v_i\|_\infty \leq n + i\) for all \(i \geq 0\)

Given \(S \subseteq \mathbb{N}^d\), define \(L_S(n)\) to be the maximum length of \(n\)-controlled bad sequences over \(S\)
Each branch of the RRT is a *bad sequence* over \((\mathbb{N}^d, \leq)\)

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\n\vdots \\
\bullet \\
\end{array}
\]

\[
\n \not\geq \ n
\]

Bad sequences are finite, but can be arbitrarily long

A sequence \(v_0, v_1, \ldots\) is *\(n\)-controlled* if \(\|v_i\|_{\infty} \leq n + i\) for all \(i \geq 0\)

Given \(S \subseteq \mathbb{N}^d\), define \(L_S(n)\) to be the maximum length of \(n\)-controlled bad sequences over \(S\)

\[
L_S(n) = \max_{v \in S, \|v\|_{\infty} \leq n} 1 + L_{S/v}(n+1)
\]
Each branch of the RRT is a bad nested sequence over \((\mathbb{N}^d, \leq)\)

Define the maximum length of \(n\)-controlled bad nested sequences in the same way as non-nested ones.
Each branch of the RRT is a **bad nested sequence** over \((\mathbb{N}^d, \leq)\)

Define the maximum length of \(n\)-controlled bad nested sequences in the same way as non-nested ones

\[
L_S(n) = \max_{\mathbf{v} \in S, \|\mathbf{v}\|_{\infty} \leq n} 1 + L_{S/\mathbf{v}}(n+1) + L_{S/\mathbf{v}}(n+1 + L_{S/\mathbf{v}}(n+1))
\]
Summary

- Extension of the reduced reachability tree from VAS to pushdown VAS
  - In the paper: extension to well-structured pushdown systems

- Boundedness and termination are decidable for pushdown VAS

- Hyper-Ackermannian ($F_{\omega \omega}$) worst-case running time
  - The reduced reachability tree of a pushdown VAS $\mathcal{A}$ has at most $F_{\omega \omega}(|\mathcal{A}|)$ nodes
  - This bound is tight

- Bounds on the reachability set when it is finite
Open Problems

 Complexity of the boundedness problem for pushdown VAS
  - Lower bound: tower of exponentials ($F_3$) from [Lazić 2012]
  - Upper bound: hyper-Ackermann ($F_{\omega \omega}$)

 Decidability of coverability / reachability for Pushdown VAS
  - Coverability decidable for 1-dim VASS + stack [Submitted]
  - Reachability open even for 1-dim VASS + stack

 Complexity of these problems for VAS + full counter
  - Coverability for this model is harder than reachability for VAS
Thank You
A pushdown vector addition system is a triple $\langle \mathbf{v}_{\text{init}}, \Gamma, \Delta \rangle$ where

- $\mathbf{v}_{\text{init}} \in \mathbb{N}^d$: initial vector
- $\Gamma$: finite stack alphabet
- $\Delta \subseteq (\mathbb{Z}^d \times 0p)$: finite set of actions, with

$$0p = \{\varepsilon\} \cup \{\text{push}(\gamma), \text{pop}(\gamma) \mid \gamma \in \Gamma\}$$
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- \( \mathbf{v}_{\text{init}} \in \mathbb{N}^d \): initial vector
- \( \Gamma \): finite stack alphabet
- \( \Delta \subseteq (\mathbb{Z}^d \times 0\mathbb{p}) \): finite set of actions, with

\[
0\mathbb{p} = \{\varepsilon\} \cup \{\text{push}(\gamma), \text{pop}(\gamma) \mid \gamma \in \Gamma\}
\]
The semantics of a pushdown VAS \( \langle \mathbf{v}_{\text{init}}, \Gamma, \Delta \rangle \) is the transition system \( \langle \mathbb{N}^d \times \Gamma^*, \langle \mathbf{v}_{\text{init}}, \varepsilon \rangle, \rightarrow \rangle \) whose transition relation \( \rightarrow \) is given by

\[
\begin{align*}
(a, \varepsilon) \in \Delta & \land \mathbf{v}' = \mathbf{v} + a \geq 0 \\
\quad \therefore (\mathbf{v}, \sigma) \rightarrow (\mathbf{v}', \sigma)
\end{align*}
\]

\[
\begin{align*}
(a, \text{push}(\gamma)) \in \Delta & \land \mathbf{v}' = \mathbf{v} + a \geq 0 \\
\quad \therefore (\mathbf{v}, \sigma) \rightarrow (\mathbf{v}', \sigma \cdot \gamma)
\end{align*}
\]

\[
\begin{align*}
(a, \text{pop}(\gamma)) \in \Delta & \land \mathbf{v}' = \mathbf{v} + a \geq 0 \\
\quad \therefore (\mathbf{v}, \sigma \cdot \gamma) \rightarrow (\mathbf{v}', \sigma)
\end{align*}
\]
VASs $\simeq$ Petri nets $\simeq$ VASSs

**Additional Feature of Petri nets**

Test $x \geq cst$ without modifying $x$

- $d := d + 2$
- $|Q| := |T| + 1$
- $d := d + 3$
- $\subseteq$

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Hyper-Ack. Bounds for Pushdown VAS

ACTS 2015 24 / 24
Pushdown VASS $A_d(n)$ for the Lower Bound

$0 \leq i < d$

$A^i_{\lambda}$

$A_{\alpha+1}$

$r \downarrow = r^{--}, r^{++}$

$r \uparrow = r^{++}, r^{--}$