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Conformally invariant path integral formulation of the Wess–Zumino–Witten → Liouville reduction

L. O'Raifeartaigh, V.V. Sreedhar

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10, Burlington Road, Dublin 4, Ireland

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Abstract

The path integral description of the Wess–Zumino–Witten → Liouville reduction is formulated in a manner that exhibits the conformal invariance explicitly at each stage of the reduction process. The description requires a conformally invariant generalisation of the phase-space path integral methods of Batalin, Fradkin, and Vilkovisky for systems with first class constraints. The conformal anomaly is incorporated in a natural way and a generalisation of the Fradkin–Vilkovisky theorem regarding gauge independence is proved. This generalised formalism should apply to all conformally invariant reductions in all dimensions. A previous problem concerning the gauge dependence of the centre of the Virasoro algebra of the reduced theory is solved.
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1. Introduction and the statement of the problem

In the course of the past decade the classical Hamiltonian reduction of Wess–Zumino–Witten (WZW) theories to Toda theories using first class constraints, and the concomitant reduction of Kac–Moody algebras to W-algebras, has been formulated in considerable detail [1]. The quantized version of the reduction process has also been considered, but mainly within the framework of canonical quantisation [2]. The elegance of the classical reduction process suggests, however, that the most natural framework for quantisation is through the functional integral. Accordingly, in this paper, we present the functional integral formulation for the quantisation of the simplest WZW → Toda reduction, namely the reduction of the $SL(2,\mathbb{R})$ WZW theory to the Liouville theory. More general cases may be dealt with in an analogous fashion and will be considered

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later. It turns out that a suitable refinement of the Batalin–Fradkin–Vilkovisky (BFV) formalism for constrained systems [3] introduced in this paper, does indeed allow the functional integral reduction to proceed in an elegant manner.

The setting up of the functional integral reduction process presents a few subtleties that make it worthwhile to present our results in some detail. The main point is that the WZW $\rightarrow$ Liouville reduction should be conformally invariant but neither the usual Faddeev–Popov–BRST formalism, nor its BFV generalisation guarantees this. These formulations are primarily concerned with gauge invariance and to make them conformally invariant as well requires a non-trivial extension, especially in view of the conformal anomaly. We find such an extension, and within this generalised formalism, prove a conformally invariant generalisation of the Fradkin–Vilkovisky theorem regarding gauge independence. An important point, although we have not pursued it in this paper, is that the same procedure should be valid for any conformally invariant reduction and for any number of dimensions.

The failure of the straightforward Faddeev–Popov–BRST method was pointed out in an earlier paper [4], where it took the form of a discrepancy between the values of the Virasoro centres obtained in two different gauges. More precisely, it turned out that while the two centres had the same functional form, the arguments were k and $k-2$, where k is proportional to the WZW coupling constant κ. Such a discrepancy suggests, of course, the existence of a conformal anomaly. But a straightforward application of the Faddeev–Popov method, or even the usual BFV method, produces no such anomaly. The generalised formalism that is developed in this paper, which allows us to keep track of both gauge and conformal invariances at each stage of the reduction, resolves this problem.

There are a number of other novel situations that arise in the WZW $\rightarrow$ Liouville reduction. First, we note that although, as usual, the original WZW Hamiltonian is not bounded below because it is based on a non-compact group, the Hamiltonian for the reduced theory is positive definite and is thus physically acceptable. More importantly, the fact that it is not possible to choose configurations such that both the kinetic term and the potential of the Liouville action are simultaneously finite on a non-compact base-space means that the base-space must be compact. As a consequence of this, one has to beware of zero-modes when gauge fixing. In fact it is the gauge-invariant zero-modes which actually produce the Liouville interaction term in the reduced theory.

This paper is organised in the following manner. In Section 2 we review the Hamiltonian formalism of the $SL(2,\mathbb{R})$ WZW model and sketch the essential ingredients of the classical reduction procedure. As the BFV formalism is the natural one to use for the path integral approach, the basic structure of this formalism is presented in Section 3. The heart of the paper is contained in Section 4 in which we formulate the conformally invariant generalisation of the BFV formalism, and establish its gauge invariance by proving the analogue of the Fradkin–Vilkovisky theorem. In Section 5 we compare the path integral reduction process as formulated in this paper with earlier attempts in this direction. In Section 6 we give a summary of our results.

2. The classical $SL(2,\mathbb{R})$ WZW $\longrightarrow$ Liouville reduction

The WZW model is defined on a two-dimensional manifold $\partial\Sigma$ by the action [5]

$$S_{\rm WZW} = \frac{\kappa}{2\pi}\int_{\partial\Sigma} {\rm Tr}(g^{-1}dg)\cdot(g^{-1}dg) - \frac{\kappa}{3\pi}\int_{\Sigma} {\rm Tr}(g^{-1}dg)\wedge(g^{-1}dg)\wedge(g^{-1}dg). \tag{2.1}$$

In the above $g \in G \equiv SL(2,\mathbb{R})$. In what follows we shall set the coupling constant κ/π equal to one, for convenience, and restore it when it becomes of interest in Section 4. The two-dimensional manifold is parametrized by the light-cone coordinates z_r and z_l defined by

$$z_r = \frac{z_0 + z_1}{2}, \qquad z_l = \frac{z_0 - z_1}{2}. \tag{2.2}$$

The action is invariant under

$$g \longrightarrow gu(z_r), \qquad g \longrightarrow v(z_l)g, \tag{2.3}$$

where $u(z_r), v(z_l) \in G$. The conserved Noether currents which generate the above transformations are given by

$$J_r = (\partial_r g)g^{-1}, \qquad J_l = g^{-1}(\partial_l g) \tag{2.4}$$

and take their values in the infinite-dimensional Lie algebra of the model. In order to set up the Hamiltonian formalism, let us introduce the Gauss decomposition for the group-valued field g,

$$g = \exp(\alpha\sigma_+)\exp(\beta\sigma_3)\exp(\gamma\sigma_-), \tag{2.5}$$

where $\sigma_\pm$ and σ_3 are the generators of the $SL(2,\mathbb{R})$ Lie algebra,

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.6}$$

As is well known, the Gauss decomposition is not valid globally. This issue has been dealt with in detail in [6]. For simplicity, we restrict our present considerations to the coordinate patch that contains the identity. Similar results hold for the other patches. In terms of the local coordinates α, β, γ on the group manifold the action can be rewritten as

$$S_{\rm WZW} = \int d^2z\ \left[(\partial_\mu\beta)(\partial^\mu\beta) + (\partial_l\alpha)(\partial_r\gamma)\exp(-2\beta)\right]. \tag{2.7}$$

The momenta canonically conjugate to α, β, γ respectively are defined, as usual, by

$$\pi_\alpha = \frac{\delta\mathcal{L}}{\delta(\partial_0\alpha)} = (\partial_r\gamma)\exp(-2\beta), \tag{2.8a}$$

$$\pi_\gamma = \frac{\delta \mathcal{L}}{\delta(\partial_0 \gamma)} = (\partial_l \alpha)\exp(-2\beta), \tag{2.8b}$$

$$\pi_\beta = \frac{\delta \mathcal{L}}{\delta(\partial_0 \beta)} = 2\partial_0 \beta. \tag{2.8c}$$

The canonical Hamiltonian density $H_{\rm WZW}$ is

$$H_{\rm WZW} = \tfrac{1}{4}\pi_\beta^2 + (\beta')^2 + \pi_\alpha \pi_\gamma \exp(2\beta) + \pi_\alpha \alpha' - \pi_\gamma \gamma'. \tag{2.9}$$

The currents can be expanded in the basis of the Lie algebra and the various components can be read off from the following equations:

$$\begin{pmatrix} J_r^+ \\ J_r^3 \\ J_r^- \end{pmatrix} = \begin{pmatrix} 1 & -2\alpha & -\alpha^2 \exp(-2\beta) \\ 0 & 1 & \alpha \exp(-2\beta) \\ 0 & 0 & \exp(-2\beta) \end{pmatrix} \begin{pmatrix} \partial_r \alpha \\ \partial_r \beta \\ \partial_r \gamma \end{pmatrix}, \tag{2.10a}$$

$$\begin{pmatrix} J_l^+ \\ J_l^3 \\ J_l^- \end{pmatrix} = \begin{pmatrix} \exp(-2\beta) & 0 & 0 \\ \gamma \exp(-2\beta) & 1 & 0 \\ -\gamma^2 \exp(-2\beta) & -2\gamma & 1 \end{pmatrix} \begin{pmatrix} \partial_l \alpha \\ \partial_l \beta \\ \partial_l \gamma \end{pmatrix}. \tag{2.10b}$$

The currents may also be expressed completely in terms of the phase-space variables α, β, γ and their conjugate momenta using the relations in Eq. (2.8). Further, by using canonical Poisson brackets for the phase-space variables, viz.

$$\{\alpha(z), \pi_\alpha(z')\} = \{\beta(z), \pi_\beta(z')\} = \{\gamma(z), \pi_\gamma(z')\} = \delta(z - z'), \tag{2.11}$$

the rest being zero, we can check explicitly that the currents satisfy two independent copies of the infinite-dimensional Kac-Moody algebra

$$\begin{aligned} &\{J_r^3(z_r), J_r^\pm(z_r')\} = \pm J_r^\pm \delta(z_r - z_r'), \\ &\{J_r^3(z_r), J_r^3(z_r')\} = \partial_{z_r} \delta(z_r - z_r'), \\ &\{J_r^+(z_r), J_r^-(z_r')\} = 2(J_r^3 - \partial_{z_r'})\delta(z_r - z_r'). \end{aligned} \tag{2.12}$$

Similar equations are valid for the left currents. In terms of the currents, the Hamiltonian density H can be written in the Sugawara form

$$H_{\rm WZW} = \tfrac{1}{2}\{J_r^+ J_r^- + (J_r^3)^2 + J_l^+ J_l^- + (J_l^3)^2\}. \tag{2.13}$$

The constraints we want to impose are

$$\phi_r \equiv J_r^- - m_r \approx 0, \qquad \phi_l \equiv J_l^+ - m_l \approx 0, \tag{2.14a}$$

or equivalently,

$$\phi_r \equiv \pi_\alpha - m_r \approx 0, \qquad \phi_l \equiv \pi_\gamma - m_l \approx 0, \tag{2.14b}$$

where m_r and m_l are constants. However, these constraints are not consistent with the conformal invariance defined by the two Sugawara Virasoro operators

$$\mathcal{T}_r = \tfrac{1}{2}\{J_r^+ J_r^- + (J_r^3)^2\}, \qquad \mathcal{T}_l = \tfrac{1}{2}\{J_l^+ J_l^- + (J_l^3)^2\} \tag{2.15}$$

because, as is well known, the currents J_r^- and J_l^+ are not conformal scalars, but spin-1 fields. However, taking advantage of the fact that the Virasoros for Kac–Moody algebras are unique only up to the addition of a diagonalisable element of the algebra or its first derivative, we modify the Sugawara Virasoros above to define the components of the so-called improved energy momentum tensor, namely T_r and T_l,

$$T_r = \tfrac{1}{2}\{J_r^+ J_r^- + (J_r^3)^2 - 2\partial_r J_r^3\}, \tag{2.16a}$$

$$T_l = \tfrac{1}{2}\{J_l^+ J_l^- + (J_l^3)^2 + 2\partial_l J_l^3\}. \tag{2.16b}$$

The physical meaning of the additional terms is that they are just the 'improvement' terms necessary to make the energy-momentum tensor of the reduced theory traceless. From Eqs. (2.13) and (2.16), it is clear that the above modification is tantamount to adding only total derivative terms to the Hamiltonian density. Hence this modification leaves the Hamiltonian, and consequently the dynamics of the theory, invariant. With respect to the conformal group generated by the Virasoros (2.16), the currents J_r^- and J_l^+ are conformal scalars i.e. they now have conformal weights, denoted by ω, as follows:

$$\omega(J_r^-) = \omega(J_l^+) = (0,0). \tag{2.17}$$

The constraints in Eq. (2.14) are, therefore, compatible with this conformal group.

The currents J_r^+ and J_l^- now have conformal weights $(0,2)$ and $(2,0)$ respectively. The phase-space variables α and γ become primary fields of conformal weights $(0,1)$ and $(1,0)$, respectively, the field β becomes a conformal connection, while $e^{2\beta}$ becomes a primary field of weight $(1,1)$, i.e. it has the opposite conformal weight to the volume element d^2z in the two-dimensional space.

Upon imposing the constraints (2.14b) on the classical Hamiltonian density (2.9) of the $SL(2,\mathbb{R})$ WZW model, we get, apart from boundary terms,

$$H_{\text{reduced}} = \tfrac{1}{4}\pi_\beta^2 + (\beta')^2 + m_r m_l e^{2\beta}. \tag{2.18}$$

This is easily recognised as the expression for the Hamiltonian density of the classical Liouville theory. Since the constraints we impose are linear in the momenta, it is natural to use the phase-space path integral, rather than the configuration space path integral, for setting up the functional integral formulation of the above classical reduction. The next section, therefore, prepares us for the quantisation of this reduction using phase-space path integral methods.

3. The Batalin–Fradkin–Vilkovisky path integral

As mentioned in Section I, our aim is to establish a functional integral formulation for the WZW $\longrightarrow$ Liouville reduction. But since one of the gauges we are interested in is the WZW gauge in which the Lagrange multipliers are set equal to zero (the analogue of the temporal gauge in QED), the standard Faddeev–Popov [FP] method does not quite suffice. A more general method for quantizing constrained systems, namely the BFV procedure, needs to be used. Hence we begin by recalling the basics of the BFV procedure. Let

$$Z = \int d(pq) \exp\left[- \int dx\, dt [p\dot{q} - H(p,q)] \right], \tag{3.1}$$

where p and q are any set of canonically conjugate variables, be the phase-space path integral which is to be reduced by a set of first class constraints $\Phi(q,p)$. Let A be a set of Lagrange multipliers, B their canonically conjugate momenta, and $b, \bar{c}$ and $c, \bar{b}$ be conjugate ghost pairs. Then define the BRST charge by

$$\Omega = \int dx\ [c\Phi + bB] + \dots, \tag{3.2a}$$

where the dots refer to terms which involve higher order ghosts which occur in the general case but do not occur in the WZW → Liouville reduction. The BRST charge Ω also satisfies the nilpotency condition

$$\{\Omega, \Omega\} = 0. \tag{3.2b}$$

A minimal gauge-fixing fermion $\bar{\Psi}$ is then defined as

$$\bar{\Psi} = \bar{c}\chi + \bar{b}A, \tag{3.3}$$

where $\chi(p,q,A,B)$ is a set of gauge-fixing conditions. The BFV procedure now consists of inserting the following reduction factor:

$$F = \int d(ABb\bar{b}c\bar{c}) \exp\left[- \int dx\, dt\ [\bar{b}\dot{c} + \{\Omega, \bar{\Psi}\}] \right] \tag{3.4}$$

into the path integral in Eq. (3.1).

From the non-zero Poisson brackets for the variables

$$\{q(x), p(x')\} = \{A(x), B(x')\} = \{b(x), \bar{c}(x')\} = \{c(x), \bar{b}(x')\} = \delta(x - x'), \tag{3.5}$$

we see that the gauge variations of the fields are

$$\{\Omega, f(q,p)\} = c\{\Phi, f(q,p)\}, \tag{3.6a}$$

$$\{\Omega, A\} = -b, \qquad \{\Omega, B\} = 0, \tag{3.6b}$$

$$\{\Omega, \bar{b}\} = \Phi, \qquad \{\Omega, \bar{c}\} = B, \tag{3.6c}$$

$$\{\Omega, b\} = \{\Omega, c\} = 0, \tag{3.6d}$$

where $f(q,p)$ is an arbitrary function of the phase-space variables. It follows from the above equations that

$$\{\Omega, \bar{\Psi}\} = (A\Phi + B\chi) + (-\bar{b}b + \bar{c}[\text{FP}]c + \bar{c}[\text{BFV}]b), \tag{3.7}$$

where the FP and BFV terms are defined by

$$\{\Phi(x), \chi(x')\} = [\text{FP}]\delta(x - x'), \qquad \{B(x), \chi(x')\} = [\text{BFV}]\delta(x - x'). \tag{3.8}$$

Note that in the definition of the reduction factor above, it is not necessary to include the term $B\dot{A} + \dot{\bar{c}}b$ in the action. This is because such a term can always be generated by letting $\chi \to \chi + \bar{c}\dot{A}$. By virtue of the Fradkin–Vilkovisky theorem, which says that the reduced functional integral Z_R is independent of the choice of the gauge-fixing fermion $\bar{\Psi}$, the above definition of the functional integral produces the correct quantum theory. Substituting for $\{\Omega, \bar{\Psi}\}$ in F and doing the $\bar{b}b$ integrations yields

$$F = \int d(AB\bar{c}c) \exp\left[-\int dx\, dt\, [A\Phi + B\chi + \bar{c}\{[\text{FP}] + [\text{BFV}]\partial_t\}c]\right]. \tag{3.9}$$

Inserting this factor into Eq. (3.1) we get, for the reduced path integral,

$$Z_R = \int d(pq)d(AB)d(c\bar{c}) \times \exp\left[-\int dx\, dt\, \left[p\dot{q} - H(p,q) + A\Phi + B\chi + \bar{c}\{[\text{FP}] + [\text{BFV}]\partial_t\}c\right]\right]. \tag{3.10}$$

Since, in general, χ may depend on A as well as p and q, the above expression can be used to specialise to either the temporal gauge, for which $[\text{FP}] = 0$, or to gauges which do not depend on the Lagrange multipliers, for which $[\text{BFV}] = 0$, with equal facility. In the latter case, we can integrate over A, B, and the remaining ghosts $\bar{c}$ and c to obtain the standard Faddeev–Popov result [7], viz.

$$Z_R = \int d(pq)\delta(\Phi)\delta(\chi)||[\text{FP}]||\exp\left[-\int dx\, dt\, [p\dot{q} - H(p,q)]\right]. \tag{3.11}$$

In contrast, for temporal (or ghost-free) gauges, $\chi \equiv A \approx 0$ and $\chi \equiv \dot{A} \approx 0$ we obtain, if we ignore intricacies regarding zero-modes,

$$Z_R = \int d(pq)||(\partial_t)||\exp\left[-\int dx\, dt\, [p\dot{q} - H(p,q)]\right]. \tag{3.12}$$

Thus, in this case, we obtain the unconstrained phase-space path integral modified by the determinant for a free field. The purpose of the more general formula for the path integral reduction factor in Eq. (3.10) is thus clear. It allows us to treat the WZW gauge for which $A = 0$ on the same footing as other gauges which do not involve the Lagrange multipliers. In the following we shall generalise the above results to the case at hand, viz. the WZW→ Liouville reduction.

4. The path integral reduction procedure

Armed with the basic details about the classical WZW $\rightarrow$ Liouville reduction and the Batalin–Fradkin–Vilkovisky formalism for quantizing constrained systems, from the previous sections, we may now return to the problem of constructing the corresponding quantum reduction in terms of the phase-space path integral. However, our application of the BFV formalism to the present problem differs from the standard approach reviewed in the last section in two respects. First, because we are dealing with independent left-handed and right-handed constraints, it is convenient to replace the standard BFV formalism by a light-cone BFV formalism. This is done by replacing the space and time directions by the two branches of the light-cone parametrised by the light-cone coordinates defined in Eq. (2.2). It is important, however, to state that since we use the Euclidean space formulation of the path integral, these light-cone coordinates actually get converted into holomorphic and anti-holomorphic coordinates. As a consequence of this all the fields in the theory will be functions of the latter complex coordinates and any function which depends only on z_r or z_l will be a holomorphic or anti-holomorphic function. This fact will have important repercussions in the next section. Second because the straightforward BFV formalism does not respect conformal invariance it has to be modified. We shall modify it in such a way that the conformal invariance is manifest at each stage.

We begin by noting that the correct phase-space path integral measure for the unconstrained WZW model is the symplectic measure $d(\alpha\beta\gamma\pi_\alpha\pi_\beta\pi_\gamma)$. This is because an integration over the momenta with this measure produces the configuration space path integral with the correct group-invariant measure $d(e^{-2\beta}\alpha\beta\gamma)$,

$$I_{\rm WZW}(j) = \int d(\alpha\beta\gamma\pi_\alpha\pi_\beta\pi_\gamma) \exp\left[-\int d^2z\ [\pi_\alpha\dot{\alpha} + \pi_\beta\dot{\beta} + \pi_\gamma\dot{\gamma} - H_{\rm WZW} + j\beta]\right]$$
$$= \int d(e^{-2\beta}\alpha\beta\gamma) \exp\left[-\int d^2z\ [L_{\rm WZW} + j\beta]\right]. \tag{4.1}$$

In the above formula for the Schwinger functional, $L_{\rm WZW}$ stands for the Wess–Zumino–Witten Lagrangian density and j, as usual, stands for an external source. The source is attached only to β on account of the proposed reduction.

As discussed in detail in the previous section, the imposition of the constraints, by means of the BFV formalism, will bring into the phase space path integral a reduction factor which involves the Lagrange multipliers, the ghosts, and their conjugate momenta. In the following we shall proceed to construct this factor. As a first step towards constructing the reduction factor, we write down the expression for the nilpotent BRS charge Ω, following the usual BFV prescription, namely

$$\Omega \equiv \Omega_r + \{r \leftrightarrow l\}, \qquad \Omega_r = \int dz_r\ \Omega_r(z), \tag{4.2}$$

where

$$\Omega_r(z) = c_r(z)\phi_r(z) + b_r(z)B_r(z). \tag{4.3}$$

In the above expression, c_r and b_r are ghost fields and B_r is the momentum conjugate to the Lagrange multiplier field A_r to be introduced shortly. The exact splitting of the BRS charge into left and right sectors is to be expected because the constraints we are imposing are completely independent of each other. For the same reason, the gauge-fixing fermion also splits into left and right parts. The expression for the right part, $\bar{\Psi}_r$ is given by

$$\bar{\Psi}_r(z) = \bar{b}_r(z)A_r(z) + \bar{c}_r(z)\chi_r(z). \tag{4.4}$$

A similar expression holds for the left part $\bar{\Psi}_l$. χ_r in the above equation is the gauge-fixing condition for the constraint ϕ_r. As a consequence of the left–right splitting, the reduction factor F factorises

$$F = F_r F_l, \tag{4.5}$$

F_r and F_l being the corresponding factors for the right and left reductions respectively. We shall therefore restrict our attention henceforth to one of the sectors. Identical considerations apply naturally to the other sector. Notice that there are no higher order terms in the ghosts in the expression for the BRS charge. This is because the constraints have exactly vanishing Poisson brackets. It is straightforward to check that the Poisson bracket of the improved Hamiltonian density with the BRS charge given above is identically zero, i.e. the Hamiltonian is gauge invariant. The reduction factor F_r can be written as

$$F_r = \int d(B_r A_r \bar{c}_r c_r \bar{b}_r b_r) \exp\left[-\int d^2z \; [\bar{b}_r \partial_l c_r + \{\Omega_r, \bar{\Psi}_r\}]\right]. \tag{4.6}$$

This factor differs from the standard BFV one only in the replacement of $\dot{c}_r$ by $\partial_l c_r$ due to the fact that ∂_l and ∂_r play the role of the time derivative in the right-hand and left-hand sectors, respectively.

As mentioned earlier, the straightforward application of the above BFV formalism is not expected to respect conformal invariance. This can be seen as follows. The physical (Liouville) gauge is defined by the condition $\chi_r \equiv \alpha \approx 0$. The important point to note is that a derivative of α would not suffice to fix the gauge completely. Accordingly, the natural conformal weight for χ_r is

$$\omega(\chi_r) = (0,1). \tag{4.7}$$

We shall now show that it is not possible to satisfy this condition without making some modifications. Since Ω_r generates gauge transformations, it is required to be a conformal scalar. And since the action is a scalar, the gauge-fixing fermion $\bar{\Psi}_r$ is required to have a conformal weight $(1,1)$. Using the fact that the constraint ϕ_r is a conformal scalar, the above two requirements translate into the following equations for the weights of the various fields respectively

$$\omega(c_r) = \omega(b_r) + \omega(B_r) = (0,1) \tag{4.8a}$$

and

$$\omega(\bar{c}_r) + \omega(\chi_r) = \omega(\bar{b}_r) + \omega(A_r) = (1,1). \tag{4.8b}$$

On the other hand, the conventional Poisson brackets for the fields,

$$\{A_r(z), B_r(z')\} = \{b_r(z), \bar{c}_r(z')\} = \{c_r(z), \bar{b}_r(z')\} = \delta(z_r - z_r'), \tag{4.9}$$

imply that the conformal weights for the fields satisfy the following equations:

$$\omega(A_r) + \omega(B_r) = \omega(b_r) + \omega(\bar{c}_r) = \omega(c_r) + \omega(\bar{b}_r) = (0,1). \tag{4.10}$$

It is easy to see that the set of Eqs. (4.8) and (4.10) is not compatible with Eq. (4.7). It is in this sense that the BFV formalism does not automatically incorporate conformal invariance.

The way in which we propose to overcome this difficulty is to introduce invertible auxiliary fields e_r and e_l with conformal weights

$$\omega(e_r) = (0,1), \qquad \omega(e_l) = (1,0). \tag{4.11}$$

At this stage the only purpose of these fields is to incorporate manifest conformal invariance but their significance will become clear later. We use these auxiliary fields to define new Poisson brackets

$$\begin{aligned} &\{A_r(z), B_r(z')\} = \{b_r(z), \bar{c}_r(z')\} = e_l\delta(z_r - z_r'), \\ &\{c_r(z), \bar{b}_r(z')\} = \delta(z_r - z_r'). \end{aligned} \tag{4.12}$$

Similar modifications apply on the left sector in which we introduce the right partner e_r. Upon using these new Poisson brackets, the requirement (4.10) is replaced by

$$\omega(A_r) + \omega(B_r) = \omega(b_r) + \omega(\bar{c}_r) = (1,1), \qquad \omega(c_r) + \omega(\bar{b}_r) = (0,1). \tag{4.13}$$

It is easy to check that the system of Eqs. (4.8) and (4.13) is compatible with Eq. (4.7). There is a certain amount of freedom in assigning weights to the fields so as to satisfy these equations but for later convenience we choose the following assignment:

α	ϕ_r	A_r	B_r	b_r	$\bar{b}_r$	c_r	$\bar{c}_r$
(0,1)	(0,0)	(1,1)	(0,0)	(0,1)	(0,0)	(0,1)	(1,0)

(4.14)

The modified reduction factor F_r is defined by

$$F_r = \int d\Gamma \exp\left[-\int d^2z\ [\bar{b}_r\partial_l c_r + \{\Omega_r, \bar{\Psi}_r\}]\right]. \tag{4.15}$$

In the above equation we have deliberately refrained from explicitly writing down the measure $d\Gamma$ at this stage as it will be constructed a little later taking into account the conformal properties of its constituent fields.

In passing, let us also mention that it is easy to verify that with the modified ghost algebra, the BRS charge satisfies the nilpotency condition

$$\{\Omega, \Omega\} = 0. \tag{4.16a}$$

It also generates the following gauge transformations:

$$\{\Omega_r, \alpha\} = -c_r, \qquad \{\Omega, A_r\} = -e_l b_r, \tag{4.16b}$$

$$\{\Omega, \bar{b}_r\} = \phi_r, \qquad \{\Omega, \bar{c}_r\} = B_r e_l, \tag{4.16c}$$

the rest of the brackets being zero. The consistency with respect to the conformal dimensionality of the above relations is easily verified. Since the right-hand sides of the ghost Poisson brackets now involve e_r and e_l which could, in principle, depend on the background field β, the generalised Jacobi identity involving π_β and the two ghost fields $b, \bar{c}$, or the Lagrange multipliers A, B impels π_β to have non-vanishing Poisson brackets with either the set (b, B) or $(\bar{c}, A)$. We choose the latter option as it automatically ensures that the above modifications in the algebra of the ghosts do not tamper with the gauge invariance of the improved Hamiltonian density. This therefore reconciles the requirements of conformal invariance with the standard ingredients of the BFV procedure in a consistent manner.

We may now readily evaluate the all important $\{\Omega_r, \bar{\Psi}_r\}$ term in F_r using the modified algebra for the ghosts given in (4.12), to find

$$\{\Omega_r, \bar{\Psi}_r\} = -e_l \bar{b}_r b_r + \bar{c}_r [\mathrm{FP}]_r c_r + \bar{c}_r [\mathrm{BFV}]_r b_r + e_l B_r \chi_r + A_r \phi_r, \tag{4.17}$$

where $[\mathrm{FP}]_r$ and $[\mathrm{BFV}]_r$ are conformal scalars defined by

$$\{\phi_r(z), \chi_r(z')\} = [\mathrm{FP}]_r \delta(z_r - z'_r), \qquad \{B_r(z), \chi_r(z')\} = [\mathrm{BFV}]_r \delta(z_r - z'_r). \tag{4.18}$$

As in Section 3, we now wish to perform the integration over the $\bar{b}b$ ghosts. Before we carry out these integrations, however, we have to define the correct phase-space path integral measure $d\Gamma$ for the Lagrange multipliers and their conjugate momenta as well as for all the ghosts. This is easily done from first principles.

Let $\phi(z)$ be a quasi-primary field with a conformal dimension $s = s_l + s_r$, s_l and s_r being the conformal weights corresponding to the left and right Virasoros, respectively. On an arbitrary manifold, we can expand the field as follows:

$$\phi(z) = \sum c_n \phi_n(z), \tag{4.19}$$

where c_n are constants and $\{\phi_n(z)\}$ constitute a complete set of orthonormal functions. The orthonormality condition is expressed in a coordinate-invariant way through the equation

$$(\phi_n, \phi_m) = \int dz_r \int dz_l \; (e_r e_l) e_r^{-2s_r} e_l^{-2s_l} \phi_n^* \phi_m. \tag{4.20}$$

Thus the correct fields which have the square integrability property in the usual sense are scaled by factors of $e_r^{(1/2)-s_r}e_l^{(1/2)-s_l}$. Accordingly, the correct functional measure for the fields is $d[e_r^{(1/2)-s_r}e_l^{(1/2)-s_l}\phi]$. Thus fields which have a conformal weight $(0,1)$ need a factor of $(e_l/e_r)^{1/2}$, fields which have a conformal weight $(1,0)$ need a factor of $(e_r/e_l)^{1/2}$, conformal scalars require a factor of $(e_re_l)^{1/2}$, and fields which have a conformal weight $(1,1)$ require a factor of $(e_re_l)^{-1/2}$.

Such being the general rule for constructing the conformally invariant measure, we are now in a position to write down the correct expression for $d\Gamma$. Taking into consideration the assignment of the weights in Eq. (4.14), we see that most of the contributions coming from the various fields cancel requiring us to modify the standard measure $d(B_rA_r\bar{b}_rb_r\bar{c}_rc_r)$ by just a factor of e_l. Thus we have for the reduction factor

$$F_r = \int d(e_lB_rA_r\bar{b}_rb_r\bar{c}_rc_r) \times \exp\left[-\int d^2z\left[\bar{b}_r\partial_lc_r - e_l\bar{b}_rb_r + \bar{c}_r[\mathrm{FP}]_rc_r + \bar{c}_r[\mathrm{BFV}]_rb_r + e_lB_r\chi_r + A_r\phi_r\right]\right].$$

Integrating over the $\bar{b}b$ fields now gives

$$F_r = \int d(B_rA_r\bar{c}_rc_r) \times \exp\left[-\int d^2z\left[\bar{c}_r[\mathrm{FP}]_rc_r + \bar{c}_r[\mathrm{BFV}]_re_l^{-1}\partial_lc_r + e_lB_r\chi_r + A_r\phi_r\right]\right]. \quad (4.21)$$

All the results we have obtained above are equally valid in the left sector of the reduction and can be obtained simply by exchanging the suffixes r and l and interchanging the two entries corresponding to the left and right Virasoros in the conformal weights of the fields. We therefore have for F_l

$$F_l = \int d(B_lA_l\bar{c}_lc_l) \times \exp\left[-\int d^2z\left[\bar{c}_l[\mathrm{FP}]_lc_l + \bar{c}_l[\mathrm{BFV}]_le_r^{-1}\partial_rc_l + e_rB_l\chi_l + A_l\phi_l\right]\right]. \quad (4.22)$$

The full reduction factor that needs to be introduced into the WZW path integral is therefore

$$F = \int d(B_rB_lA_rA_l\bar{c}_rc_r\bar{c}_lc_l) \times \exp\left[-\int d^2z\left[\bar{c}_r[\mathrm{FP}]_rc_r + \bar{c}_r[\mathrm{BFV}]_re_l^{-1}\partial_lc_r + e_lB_r\chi_r + A_r\phi_r + \{r\leftrightarrow l\}\right]\right]. \quad (4.23)$$

This expression is the conformally invariant generalisation of the standard BFV reduction factor in Eq. (3.9) for the present theory. Since this generalisation introduces non-trivial modifications to the standard BFV formalism, we have to prove that these modifications are indeed consistent. We do this by proving an analogue of the Fradkin–Vilkovisky theorem for the gauge independence of the path integral of the reduced theory within our generalised formalism.

Theorem 1. If the reduction factor $F(\bar{\Psi})$ is defined as in Eq. (4.23), and the gauge-fixing functions χ_r and χ_l are independent of the fields B_r and B_l, as is usually the case, the reduced path integral

$$I_{\bar{\Psi}}(j) = \int d(\alpha\beta\gamma\pi_\alpha\pi_\beta\pi_\gamma) \times \exp\left[-\int d^2z\ [\pi_\alpha\dot{\alpha} + \pi_\beta\dot{\beta} + \pi_\gamma\dot{\gamma} - H_{\rm WZW} + j\beta]\right] F(\bar{\Psi}) \tag{4.24}$$

is independent of $\bar{\Psi}$. In fact,

$$I_{\bar{\Psi}}(j) = \int d(e_r^{-1}e_l^{-1}\beta) e^{-\int d^2z[(\partial_\mu\beta)(\partial^\mu\beta)+m_r m_l e^{2\beta}+j\beta]} \tag{4.25}$$

which is manifestly independent of $\bar{\Psi}$.

Proof. Since the gauge-fixing functions χ_r, χ_l are independent of B_r and B_l, we may integrate over the B fields in Eq. (4.23) to get

$$F = \int d(e_r^{-1}e_l^{-1}A_rA_l\bar{c}_rc_r\bar{c}_lc_l)\delta(\chi_r)\delta(\chi_l) \times \exp\left[-\int d^2z\left[\bar{c}_r[\mathrm{FP}]_rc_r + \bar{c}_r[\mathrm{BFV}]_re_l^{-1}\partial_lc_r + A_r\phi_r + \{r\leftrightarrow l\}\right]\right]. \tag{4.26}$$

Using the fact that the constraints ϕ_r and ϕ_l are expressible in terms of the momenta π_α and π_γ through Eq. (2.14b), we can introduce the above reduction factor into Eq. (4.24) and integrate over the momenta $\pi_\alpha, \pi_\beta, \pi_\gamma$ to get the gauged WZW model

$$I_{\bar{\Psi}}(j) = \int d(e_r^{-1}e_l^{-1}\alpha\beta\gamma A_rA_le^{-2\beta})\delta(\chi_r)\delta(\chi_l) \times \exp\left[-\int d^2z\ [L_{\rm GWZW} + j\beta]\right] \times G, \tag{4.27a}$$

where $L_{\rm GWZW}$ stands for the Lagrangian density of the gauged Wess–Zumino–Witten model and is given by

$$L_{\rm GWZW} = (\partial_\mu\beta)(\partial^\mu\beta) + e^{-2\beta}(\partial_r\gamma + A_l)(\partial_l\alpha + A_r) - A_lm_l - A_rm_r \tag{4.27b}$$

and

$$G = \int d(\bar{c}_rc_r\bar{c}_lc_l)\exp\left[-\int d^2z\ \left[\bar{c}_r[\mathrm{FP}]_rc_r + \bar{c}_r[\mathrm{BFV}]_re_l^{-1}\partial_lc_r + \{l\leftrightarrow r\}\right]\right]$$
$$= \int d(\bar{c}_rc_r\bar{c}_lc_l)\exp\left[-\int d^2z\ \left[\bar{c}_r[\frac{\partial\chi_r}{\partial\alpha} - \frac{\partial\chi_r}{\partial A_r}\partial_l]c_r + \{r\leftrightarrow l\}\right]\right] \tag{4.27c}$$

stands for the ghost factor. Notice that this expression differs from what one might naively expect for the gauged WZW path integral because of the appearance of the auxiliary fields e_r and e_l in the measure. But, as is amply clear from the foregoing, these are precisely the factors that enable us to carry out the reduction in a conformally invariant fashion. We now define the shifted fields

$$A_l \longrightarrow \bar{A}_l = A_l + \partial_r\gamma, \qquad A_r \longrightarrow \bar{A}_r = A_r + \partial_l\alpha. \tag{4.28}$$

Notice that $\partial_l\alpha$ and $\partial_r\gamma$ can always be absorbed by a redefinition of the A fields as above, although the presence of zero-modes may not always allow us to completely eliminate the A fields themselves by shifting α and γ appropriately. In terms of the shifted fields, Eqs. (4.27) become

$$I_{\bar{\Psi}}(j) = \int d(e_r^{-1}e_l^{-1}\alpha\beta\gamma\bar{A}_r\bar{A}_l e^{-2\beta})\delta(\chi_r)\delta(\chi_l) \times \exp\left[-\int d^2z\ [L_{\text{GWZW}} + j\beta]\right] \times G \tag{4.29a}$$

and

$$L_{\text{GWZW}} = (\partial_\mu\beta)(\partial^\mu\beta) + e^{-2\beta}\bar{A}_r\bar{A}_l - \bar{A}_l m_l - \bar{A}_r m_r \tag{4.29b}$$

respectively, where we have dropped total derivative terms that appear in shifting the m_r and m_l dependent terms. The ghost factor G has the following nice interpretation in terms of the shifted fields. Recall that the gauge-fixing condition χ_r is, in general, a function of both α and the Lagrange multiplier A_r which are independent of each other. If we work in terms of the shifted fields defined above, this is no longer true and we have

$$\left[\frac{\partial\chi_r}{\partial\alpha}\right]_{\bar{A}_r} = \left[\frac{\partial\chi_r}{\partial\alpha}\right]_{A_r} - \left[\frac{\partial\chi_r}{\partial A_r}\right]_{\alpha}\partial_l, \tag{4.30}$$

where the partial derivatives in the above equation are to be taken keeping the fields appearing as subscripts fixed. Notice that the right-hand side of the above equation is just the argument in the determinant that results from performing the ghost integrations in Eq. (4.27c). Taking this into account, the measure in Eq. (4.29a) becomes

$$d(e_r^{-1}e_l^{-1}\alpha\beta\gamma\bar{A}_r\bar{A}_l e^{-2\beta})\delta(\chi_r)\delta(\chi_l)\left\|\left[\frac{\partial\chi_r}{\partial\alpha}\right]_{\bar{A}_r}\left[\frac{\partial\chi_l}{\partial\gamma}\right]_{\bar{A}_l}\right\| = d(e_r^{-1}e_l^{-1}\alpha\beta\gamma\bar{A}_r\bar{A}_l e^{-2\beta})\delta(\alpha)\delta(\gamma). \tag{4.31}$$

The α and γ integrations now drop out to yield

$$I_{\bar{\Psi}}(j) = \int d(e_r^{-1}e_l^{-1}\beta\bar{A}_r\bar{A}_l e^{-2\beta}) \exp\left[-\int d^2z\ L_{\text{GWZW}}\right]. \tag{4.32a}$$

Carrying out the gaussian integration over the $\bar{A}$ fields we then obtain

$$I_{\bar{\Psi}}(j) = \int d(e_r^{-1}e_l^{-1}\beta)\exp\left[-\int d^2z\ \left[(\partial_\mu\beta)(\partial^\mu\beta) - m_r m_l e^{2\beta} + j\beta\right]\right], \tag{4.32b}$$

as required. We have therefore proved that the Fradkin–Vilkovisky theorem can be generalized to include conformal invariance. Although this was done within the context of the WZW $\rightarrow$ Liouville reduction, it is clear that the principle is sufficiently general to transcend the domains of the present theory and should apply to all conformally invariant gauge theories.

We will now discuss the role played by the auxiliary fields. The crucial point to note is that they appear in the final result and because they have non-zero conformal weights, they can not be set equal to unity without breaking conformal invariance. Thus they are an intrinsic part of the reduction.

On the other hand they appear only in the measure and only in the form of the product $e_r e_l$ which has a conformal weight (1,1). It is this fact that allows us to use them without introducing any new dynamics since the conformal weights allow us to make the following natural identification:

$$e_r e_l \equiv e^{2\beta}. \tag{4.33}$$

Moreover, if we regard $e^{2\beta}$ as $\sqrt{g}$, where g is the determinant of a two-dimensional metric, Eq. (4.33) allows us to immediately recognise the e_r and e_l fields as the two components of a zweibein. It is interesting to note that had the reduction not been left–right symmetric, other combinations of the components of the zweibein would have occurred in the final results and these would have corresponded to genuine external fields. Using Eq. (4.33) in Eq. (4.32b), and reintroducing the WZW coupling constant κ we get

$$I(j) = \int d(e^{-2\beta}\beta)\exp\left[-\frac{\kappa}{\pi}\int d^2z\left[(\partial_\mu\beta)(\partial^\mu\beta) - m_r m_l e^{2\beta} + j\beta\right]\right], \tag{4.34}$$

where the allusion to $\bar{\Psi}$ has been dropped for obvious reasons.

As is well known [8], the exponential factor in the measure of Eq. (4.34) corresponds to the conformal anomaly and can be removed by making a suitable shift in the WZW coupling constant to yield

$$I(j) = \int d\beta \exp\left[-\frac{(\kappa-2)}{\pi}\int d^2z\left[\partial_\mu\beta\partial^\mu\beta - m_r m_l e^{2\beta} + j\beta\right]\right]. \tag{4.35}$$

This then is the Liouville theory that is the result of the reduction. It is well known [9] that the Virasoro centre for the above theory has the form

$$c = \hbar + 6\left[\sqrt{k-2\hbar} + \frac{\hbar}{\sqrt{k-2\hbar}}\right]^2, \tag{4.36}$$

where $k = \kappa/2\pi$. In the next section, we shall give a simple interpretation of this formula.

5. Comparison with earlier path-integral results

Although most quantized treatments of WZW → Liouville reductions use the canonical formalism [2], the functional integral formalism was considered in Refs. [4,10]. In these references the Faddeev–Popov method was used and led to a Liouville theory. These references study the path integral in two special gauges, namely the Liouville gauge and the WZW gauge. It would therefore be reassuring to redo our analysis in these gauges in order to compare our results with these earlier works. In fact these gauges highlight the roles of the anomaly and the zero-modes respectively. We first examine the Liouville gauge.

5.1. Liouville gauge

In this gauge we have

$$\chi_r \equiv \alpha \approx 0, \qquad \chi_l \equiv \gamma \approx 0 \tag{5.1}$$

and hence it follows from Eqs. (4.18) that

$$[\mathrm{FP}]_{r,l} = -1, \qquad [\mathrm{BFV}]_{r,l} = 0. \tag{5.2}$$

Substituting the above equalities into our expression for the generalised reduction factor Eq. (4.26), we get

$$F = \int d(e_r^{-1} e_l^{-1} A_r A_l \bar{c}_r c_r \bar{c}_l c_l)\, \delta(\alpha)\delta(\gamma) \times \exp\left[-\int d^2z\ [\bar{c}_r c_r + A_r \phi_r + \{r \leftrightarrow l\}] \right]. \tag{5.3}$$

Doing the A integrations and the trivial ghost integrations we find that the BFV reduction factor is just

$$F = \det||e_r^{-1} e_l^{-1}||\,\delta(\phi_r)\delta(\phi_l)\delta(\alpha)\delta(\gamma). \tag{5.4}$$

Inserting this factor into the unconstrained WZW phase-space path integral and carrying out the various delta function integrations as well as the gaussian π_β integration we get, as expected, Eq. (4.32b). This result differs from the result of earlier path integral formulations of the problem by the appearance of the factor $||e_r^{-1} e_l^{-1}||$ in the measure. Since $(e_r e_l)^{-1} = e^{-2\beta}$ according to Eq. (4.33), the use of the zweibein changes the Liouville measure from $d(\beta)$ to $d(e^{-2\beta}\beta)$ and thus produces a conformal anomaly. As already mentioned, the insertion of this factor in the measure is equivalent to a change of k to $k-2$ in the exponent and thus leads to a change

$$\hbar + 6\Big(\sqrt{k} + \frac{\hbar}{\sqrt{k}}\Big)^2 \rightarrow \hbar + 6\Big(\sqrt{k-2\hbar} + \frac{\hbar}{\sqrt{k-2\hbar}}\Big)^2 \tag{5.5}$$

in the Virasoro centre. The difference between the two expressions in Eq. (5.5) is the discrepancy that was mentioned in the Introduction and can now be seen to be due to the fact that the zweibein was not used in the earlier papers.

5.2. WZW gauge

This gauge is the analogue of the temporal gauge in QED and is defined by setting the Lagrange multipliers equal to zero, modulo zero modes. On a compact 2-space (whose compactness, we recall, is necessitated by the Liouville potential, which in turn is present because of the non-zero constants m_r and m_l) there is just one zero-mode for each A. To see this let us consider A_r, for example, and decompose it according to

$$A_r = A_r^0 + \hat{A}_r, \qquad \hat{A}_r = \partial_l \lambda_r, \qquad \int d^2 z \ (e_r e_l)^{-1} A_r^0 \hat{A}_r = 0, \tag{5.6}$$

i.e. into a part $\hat{A}_r$ that can be gauged away and its orthogonal complement A_r^0. In the above equation the gauge transformation parameter λ_r has a conformal weight $\omega(\lambda_r) = (0,1)$. The factor $(e_r e_l)^{-1}$ in the integral comes from the requirement that the orthogonality condition be conformally invariant. Since the orthogonality must hold for arbitrary λ_r, it follows from a simple partial integration that

$$\partial_l(e_r^{-1} e_l^{-1} A_r^0) = 0 \quad \text{or } A_r^0 = e_r e_l f(z_r), \tag{5.7}$$

where $f(z_r)$ is an arbitrary holomorphic function. However, since there are no holomorphic functions on a compact Riemann surface except the constant functions [11], we see that $f(z_r)$ must be constant and thus the only normalised zero-mode is

$$A_r^0 = \frac{e_r e_l}{\sqrt{V}}, \qquad \text{where } V = \int d^2 z \, e_r e_l = \int d^2 z \, e^{2\beta}. \tag{5.8}$$

A similar expression holds for A_l^0. Thus the WZW gauge is

$$\begin{aligned} \chi_r &\equiv e_l^{-1} \hat{A}_r \approx 0, \qquad & A_r^0 &= \mu_r \frac{e_r e_l}{\sqrt{V}}, \\ \chi_l &\equiv e_r^{-1} \hat{A}_l \approx 0, \qquad & A_l^0 &= \mu_l \frac{e_r e_l}{\sqrt{V}}, \end{aligned} \tag{5.9}$$

where the μ's are arbitrary constants. Notice that this is a complete gauge fixing because it determines the gauge parameter λ_r up to a function $\lambda(z_r)$ and the only such function is a constant which must be zero because λ_r has a conformal weight $(0,1)$. Similar considerations apply for λ_l. The measure for the Lagrange multipliers now becomes

$$d(e_r^{-1} e_l^{-1} A_r A_l) = d(\mu_r \mu_l) d(e_r^{-1} e_l^{-1} \hat{A}_r \hat{A}_l). \tag{5.10}$$

The expressions for the χ's imply that

$$[\mathrm{FP}]_{r,l} = 0, \qquad [\mathrm{BFV}]_{r,l} = -1. \tag{5.11}$$

Substituting the above results in Eq. (4.27) and doing the ghost integrations yields

$$I(j) = \int d(e_r^{-1}e_l^{-1}\alpha\beta\gamma\mu_r\mu_l\hat{A}_r\hat{A}_l e^{-2\beta})\delta(e_l^{-1}\hat{A}_r)\delta(e_r^{-1}\hat{A}_l)$$
$$\times||e_l^{-1}\partial_l e_r^{-1}\partial_r||\exp\left[-\int d^2z\ [L_{\mathrm{GWZW}} + j\beta]\right]. \qquad (5.12)$$

The integration over the $\hat{A}$ fields can now be performed using the gauge-fixing delta functions to yield

$$I(j) = \int d(\alpha\beta\gamma e^{-2\beta}\mu_r\mu_l)||e_l^{-1}\partial_l e_r^{-1}\partial_r||\exp\left[-\int d^2z\ [L^0_{\mathrm{GWZW}} + j\beta]\right], \qquad (5.13a)$$

where

$$L^0_{\mathrm{GWZW}} = (\partial_\mu\beta)(\partial^\mu\beta) + e^{-2\beta}(\partial_l\alpha + A^0_r)(\partial_r\gamma + A^0_l) - m_r A^0_r - m_l A^0_l$$
$$= (\partial_\mu\beta)(\partial^\mu\beta) + e^{-2\beta}(\partial_l\alpha)(\partial_r\gamma) + \mu_r\mu_l - \frac{\mu_r m_r}{\sqrt{V}}e_r e_l - \frac{\mu_l m_l}{\sqrt{V}}e_r e_l \qquad (5.13b)$$

In arriving at the above equation we have used the expressions for A^0's in Eq. (5.9) and the equality $e_r e_l = e^{2\beta}$ in Eq. (4.33). The integration over the α and γ fields produces a factor that exactly cancels the $e^{-2\beta}$ factor in the measure and the $||\partial_r\partial_l||$ factor in the fermionic determinant. The path integral therefore reduces to

$$I(j) = \int d(e_r^{-1}e_l^{-1}\beta\mu_r\mu_l)\exp\left\{-\int d^2z\ \left[(\partial_\mu\beta)(\partial^\mu\beta) + j\beta\right.\right.$$
$$\left.\left. + e_r e_l\left(\frac{\mu_r\mu_l}{V} - \frac{\mu_l}{\sqrt{V}}m_r - \frac{\mu_r}{\sqrt{V}}m_l\right)\right]\right\}. \qquad (5.14)$$

The zero-modes can now be integrated without further ado to produce

$$I(j) = \int d(e_r^{-1}e_l^{-1}\beta)\exp\left[-\int d^2z\ [(\partial_\mu\beta)(\partial^\mu\beta) - m_r m_l e^{2\beta} + j\beta]\right], \qquad (5.15)$$

where we have used Eq. (4.33) to set $e_r e_l = e^{2\beta}$. We have thus verified that the WZW gauge produces the same result as the Liouville gauge. From the foregoing discussion it is clear that the Liouville potential is actually due to the zero-modes.

However, our interest here is not in verifying that the WZW gauge leads to the correct result but in comparing the final results with the expressions obtained in the previous papers [4,10]. In those papers the zero-modes were neglected and the WZW gauge was defined as $A_r = A_l \approx 0$. As a result, the final expression in the WZW gauge was the same as in Eq. (5.15) but without the Liouville potential. The omission of the Liouville potential actually made no difference to the final results because the purpose of those papers was to compute the Virasoro centre; and for that purpose the only role of the Liouville potential is to require the use of the improved Virasoro operators. Since the improved Virasoros were used in any case, the result obtained for the centre in those papers was the correct one.

The fact that the earlier computations of the Virasoro centre in the WZW gauge are still valid allows us to draw two interesting conclusions. First, since the expression for the Virasoro centre is independent of m_r and m_l, there is a smooth transition for the Virasoro algebra to the case $m_l = m_r = 0$ even though the reduced system in the latter case does not require the 2-space to be compact. Second, since the earlier WZW-gauge computations are valid, they provide an interesting interpretation of the formula for the Virasoro centre in the Liouville theory which is not at all obvious in the context of the Liouville theory itself. In fact, they show that if the Liouville theory formula for the centre is expanded according to

$$C = \hbar + 6\left(\sqrt{k-2\hbar} + \frac{\hbar}{\sqrt{k-2\hbar}}\right)^2 = \frac{3k\hbar}{k-2\hbar} - 2\hbar + 6k, \tag{5.16}$$

it is just the sum of three independent centres, namely the centre for the $SL(2,\mathbb{R})$ WZW model, the centre for the ghosts, and the centre for the classical improvement term. The results of [4,10] show that a similar interpretation exists for Toda theories.

6. Summary and conclusions

We have introduced a generalisation of the Batalin–Fradkin–Vilkovisky formalism which allows us to incorporate conformal invariance into the usual procedure for the path integral quantisation of systems with first-class constraints. Although we have done this only for WZW → Liouville reduction in two dimensions it is clear that the procedure should apply to all conformally invariant reductions and should be independent of the dimension. In later papers we hope to apply it to WZW → Toda and Goddard–Olive reductions the latter of which will require a further generalisation of our analysis to include second class constraints. An essential feature of our procedure is the introduction of a zweibein which makes the conformal invariance manifest at each stage of the reduction. The two components of this zweibein appear in the final theory only as products of the form $e_r e_l = e^{2\beta}$ where β is the Liouville field, and thus introduce non-trivial modifications of the reduction (actually a conformal anomaly) without introducing new fields. Our main result is that, in spite of the conformal anomaly, an analogue of the Fradkin–Vilkovisky theorem is still valid.

An interesting feature of the WZW → Liouville (or indeed Toda) reductions is that the first-class constraints are obtained by setting the momenta not equal to zero but to constants m_l and m_r. When these constants are not zero the gauge fields (Lagrange multipliers) have zero-modes and it is precisely these zero-modes that produce the exponential Liouville interaction.

Earlier papers, in which the straightforward Faddeev–Popov formalism was used, did not produce the conformal anomaly in the Liouville gauge, which led to a discrepancy in the expression for the Virasoro centre in the Liouville and WZW gauges. Our analysis traces the origin of this discrepancy to the fact that the standard Faddeev–Popov formalism, in spite of its appearance, is not conformally invariant. A modification using a zweibein produces a formalism which is both gauge and conformal invariant.

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