World Scientific
www.worldscientific.com

A general method for deriving vector potentials produced by knotted solenoids

V. V. Sreedhar

Chennai Mathematical Institute, Plot H1, SIPCOT IT Park,
Siruseri, Kelambakkam Post, Chennai 603103, India
sreedhar@cmi.ac.in

A general method for deriving exact expressions for vector potentials produced by arbitrarily knotted solenoids is presented. It consists of using simple physics ideas from magnetostatics to evaluate the magnetic field in a surrogate problem. The latter is obtained by modeling the knot with wire segments carrying steady currents on a cubical lattice. The expressions for a 3_1 (trefoil) and a 4_1 (figure-eight) knot are explicitly worked out. The results are of some importance in the study of the Aharonov–Bohm effect generalized to a situation in which charged particles moving through force-free regions are scattered by fluxes confined to the interior of knotted impenetrable tubes.

The physical importance of the vector potential experienced by a charged particle traveling through force-free regions was first stressed in a celebrated paper by Aharonov and Bohm;[1] although it was anticipated much earlier by Ehrenberg and Siday.[2] Landmark experiments by Chambers[3] and later by Tonomura *et al.*[4] established the results by studying the shifts in the electron interference patterns produced by solenoidal and toroidal windings, respectively. It is expected that in the foreseeable future, these experiments may be repeated with more complicated current distributions, namely knotted solenoids.[5,6]

In a recent paper, the author has presented an exact expression for a flat connection on the complement of a torus knot.[7] This was accomplished by computing the vector potential produced by an infinitesimally thin knotted solenoid which effectively confines the magnetic flux to the interior of an ϵ-neighborhood of the knot. The choice of toroidal coordinates greatly facilitated the calculation since

arbitrary torus knots can be regarded as closed loops winding, a relatively prime number of times, around the two inequivalent cycles of a putative torus.

Although the ideas of Ref. 7 were born out of a different motivation, they are fairly general. Besides, the choice of toroidal coordinates limits their applicability to torus knots. To apply the ideas fruitfully to more general knots, we need to develop a different technique. Towards this end, we replace the putative torus mentioned above by a putative cubical lattice. The knot is then defined as a self-avoiding polygon with vertices at prescribed lattice sites. This way of defining a knot has several advantages. First, it is tailor-made for testing important results using computers. Second, the lattice spacing introduces a natural cut-off which allows us to distinguish between physical knots (which have a finite thickness) and mathematical knots (which have a vanishing thickness).[8] Third, it can be used to discuss any knot — not just a torus knot. Finally, it allows us to use Cartesian coordinates.

In what follows, we use the aforementioned motivations to derive a general expression for the vector potential produced in the force-free region by a solenoidal winding around an arbitrary knot on a cubical lattice. As special cases, we then explicitly work out the exact expressions for the 3_1 (trefoil) and the 4_1 (figure-eight) knots. In principle, this is a straightforward exercise and consists in putting together the contributions to the vector potential coming from the various finite-length segments of the polygonal knot on the cubical lattice. As is well-known, the vector potential produced by a finite-length solenoid is given by a complicated expression involving elliptic integrals.[9] So putting together such expressions coming from all the line segments is an arduous task.

The paper relies on the following thesis: Given a knot, consider a small ϵ-neighborhood around it. Imagine a closely-spaced winding of a wire carrying uniform current around this tube. In the limit of small radius of cross-section of the tube, the current distribution reduces to a collection of magnetic dipoles (tiny magnets) which are lined up along the knot. Since the vector potential produced by a magnetic dipole at a given point is given by a well-known formula, the coveted answer for the vector potential produced by the knot is obtained by integrating the dipole result over the length of the knot. This expression has the Biot–Savart form. As already mentioned in Ref. 7, this is a consequence of the fact that, in the magnetostatic limit, the magnetic field due to a current wire and the vector potential due to an infinitesimally thin solenoid of the same shape and size satisfy the same equations and hence have the same solutions in the region of interest viz. the source-free knot complement. In view of this, it is sufficient to solve the surrogate problem of finding the magnetic field produced by a steady filamentary current, in the shape of the knot under consideration, and to formally identify the answer with the vector potential produced by the corresponding knot tube. The holonomy viz. the line integral of the vector potential around a closed loop in the latter problem is then formally equal to the line integral of the magnetic field along the same closed loop in the surrogate problem which, as a consequence of Stokes' law, equals the current passing through the knot and measures the linking number

of the loop with the knot under consideration. In other words, the role of the flux in the original problem is played by the current in the surrogate problem.

We note that, on the cubical lattice, a polygonal knot (tube) is modeled by a collection of finite-length solenoids. Hence, solving the surrogate problem amounts to putting together contributions coming from finite-length wire segments which make up the knot, each of which has its own length and is parallel to one of the coordinate axes.

A lattice realization for an arbitrary knot can be obtained by starting at a given site on a cubical lattice and setting up a self-avoiding random walk. This allows a hypothetical particle at a given site to move to a neighboring site at random, so long as such a site was hitherto not visited; till it finally returns to the starting point, which is the only site that can be visited twice.[8] The (closed) trajectory of the particle then describes the knot.

A more systematic and instructive way of constructing lattice knots comes from the connection between knots and braids.[10] Consider, for example, the trefoil knot. As is well-known, it can be obtained by end-on-end closure of the element $(\sigma_1)^3$ in $\mathbf{B}_2$ — the braid group on two strings. Now, instead of looking at the two-dimensional projection of the braid, suppose we look at it from the side so that we can perceive the depth. From this perspective, over-passes and under-passes are not represented by continuous and broken arcs, but manifest themselves as windings between the strings which are separated by a distance. Flatten out the arcs into straight line segments and imagine that one string lies on the front face and the other on the back face of a putative cubical lattice. A crossing corresponds to the strings migrating from one face to the other. Now choose a grid such that the ends of the line segments lie on lattice sites. Finally, join the loose ends of the strings along trivial paths (those which do not introduce new crossings) on the lattice and put a bounding box to get a lattice trefoil knot. The lattice figure-eight knot can similarly be obtained by the closure of the element $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ in $\mathbf{B}_3$ — the braid group on three strings.

Clearly there are many ways to produce a lattice version of a knot by the above prescription. The most economical way is to minimize the stick number (number of line segments) and the step number (related to the size of the lattice) of the knot. Important theorems regarding the minimum stick number and minimum step number for popular knots have been proved recently.[11] The figures for 3_1 and 4_1 knots with minimum stick numbers, adapted from Ref. 11 are shown in Fig. 1.

Taking (220) as the fiducial point, we label the successive line segments in the direction of the current flow, by the letter α, starting with $\alpha = 1$. Clearly, for the trefoil, $\alpha = 1, 2, \ldots, 12$ and for the figure-eight knot, $\alpha = 1, 2, \ldots, 14$. We double the number of sites from the minimum suggested by the diagrams. This is indicated by the lighter dots in the pictures. While this will not increase the complexity of the computation in any way, it will be useful subsequently in checking the non-triviality of the flat connection. We denote by I, the uniform current passing through the circuits.
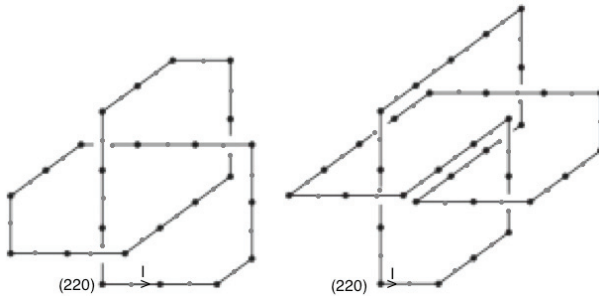
V. V. Sreedhar

Fig. 1. 3_1 (trefoil) and 4_1 (figure-eight) lattice knots with minimum stick numbers.

Before we compute the magnetic field **B**, due to a polygonal knot carrying uniform current I, at any point P with coordinates **r**, it is useful to recall the corresponding result for a finite straight wire carrying a steady current. As is well-known,[12] this is obtained by a straightforward application of the Biot–Savart law, and is given by following expression for a thin straight wire placed along the x-axis:

$$\mathbf{B}(\mathbf{r}) = \left(\frac{I}{c}\right)\left(\frac{\hat{\phi}}{r}\right)(\cos\theta_1 + \cos\theta_2)\,, \tag{1}$$

where r is the perpendicular distance of the point P from the wire, θ_1 and θ_2 are the interior angles subtended by the line joining the point P to the extremities of the wire and $\hat{\phi}$ is the unit vector in the azimuthal direction defined by the right-hand rule in the plane perpendicular to the wire. The field produced by the knot can now be written down easily by adding the contributions from the finite-length straight wire segments that make up the knot and is given by

$$\mathbf{B}(\mathbf{r}) = \sum_\alpha \mathbf{B}_\alpha \qquad \text{where} \quad \mathbf{B}_\alpha = \left(\frac{I}{c}\right)\left(\frac{\hat{\phi}_\alpha}{r_\alpha}\right)(\cos\theta_1^\alpha + \cos\theta_2^\alpha)\,. \tag{2}$$

In the above expression, we use the obvious notations: $\mathbf{B}_\alpha$ is the field produced by the αth segment, r_α is the perpendicular distance from the point P to the segment α, θ_1^α and θ_2^α are the interior angles subtended by the lines joining the point P to the extremities of the segment α, and $\hat{\phi}_\alpha$ is the unit vector in the azimuthal direction in the plane perpendicular to the segment α and is given by the right-hand rule. For example, if the segment is parallel to the z-axis, $\hat{\phi} = -y\hat{i} + x\hat{j}$, etc. Since a general knot will have segments parallel to all the three axes, the total magnetic field will have nonvanishing contributions along all the three Cartesian basis vectors. In view of this, we can re-express the magnetic field in (2) as follows

$$\mathbf{B}(x, y, z) = \sum_{i=1}^{3} B_i \hat{e}_i\,. \tag{3}$$

Comparing the above two equations, we get the following expression for the Cartesian components of the magnetic field

$$B_i = \frac{I}{c} \sum_\alpha \gamma_{\alpha i} \frac{(\cos \theta_1^\alpha + \cos \theta_2^\alpha)}{r_\alpha} \, , \tag{4}$$

where the weights $\gamma_{\alpha i}$ for a given α are given by $\gamma_{\alpha i} = \hat{\phi}_\alpha \cdot \hat{e}_i$.

Given a knot it is straightforward to evaluate the $\gamma_{\alpha i}$. As already mentioned, one may formally identify these expressions for the magnetic field produced by a knotted current wire with the expressions for the vector potential produced by solenoidal windings around the corresponding knots.

Substituting the $\gamma_{\alpha i}$ and summing over α we get, after suppressing a factor of I/c in all the equations, the following expressions for the magnetic field of the 3_1 knot:

$$
\begin{aligned}
B_x = &-\frac{z}{(6-x)^2 + z^2} \left[\frac{4-y}{\sqrt{(6-x)^2 + (4-y)^2 + z^2}} + \frac{2-y}{\sqrt{(6-x)^2 + (2-y)^2 + z^2}} \right] \\
&- \frac{(4-y)}{(6-x)^2 + (4-y)^2} \left[\frac{4-z}{\sqrt{(6-x)^2 + (4-y)^2 + (4-z)^2}} \right. \\
&\left. + \frac{z}{\sqrt{(6-x)^2 + (4-y)^2 + z^2}} \right] + \frac{(4-z)}{x^2 + (4-z)^2} \left[\frac{-y}{\sqrt{x^2 + y^2 + (4-z)^2}} \right. \\
&\left. - \frac{(4-y)}{\sqrt{x^2 + (4-y)^2 + (4-z)^2}} \right] + \frac{(2-z)}{(4-x)^2 + (2-z)^2} \\
&\times \left[\frac{(6-y)}{\sqrt{(4-x)^2 + (6-y)^2 + (2-z)^2}} + \frac{y}{\sqrt{(4-x)^2 + y^2 + (2-z)^2}} \right] \\
&- \frac{(6-y)}{(4-x)^2 + (6-y)^2} \left[\frac{(6-z)}{\sqrt{(4-x)^2 + (6-y)^2 + (6-z)^2}} \right. \\
&\left. - \frac{(2-z)}{\sqrt{(4-x)^2 + (6-y)^2 + (2-z)^2}} \right] + \frac{(6-z)}{(2-x)^2 + (6-z)^2} \\
&\times \left[\frac{(2-y)}{\sqrt{(2-x)^2 + (2-y)^2 + (6-z)^2}} - \frac{(6-y)}{\sqrt{(2-x)^2 + (6-y)^2 + (6-z)^2}} \right] \\
&- \frac{(2-y)}{(2-x)^2 + (2-y)^2} \left[\frac{(-z)}{\sqrt{(2-x)^2 + (2-y)^2 + z^2}} \right. \\
&\left. - \frac{(6-z)}{\sqrt{(2-x)^2 + (2-y)^2 + (6-z)^2}} \right] + \frac{y}{x^2 + y^2} \\
&\times \left[\frac{(2-z)}{\sqrt{x^2 + y^2 + (2-z)^2}} - \frac{(4-z)}{\sqrt{x^2 + y^2 + (4-z)^2}} \right] , \tag{5}
\end{aligned}
$$

$$B_y = \frac{z}{(2-y)^2 + z^2} \left[\frac{6-x}{\sqrt{(6-x)^2 + (2-y)^2 + z^2}} - \frac{2-x}{\sqrt{(2-x)^2 + (2-y)^2 + z^2}} \right]$$

$$+ \frac{(6-x)}{(6-x)^2 + (4-y)^2} \left[\frac{4-z}{\sqrt{(6-x)^2 + (4-y)^2 + (4-z)^2}} \right.$$

$$\left. + \frac{z}{\sqrt{(6-x)^2 + (4-y)^2 + z^2}} \right] - \frac{(4-z)}{(4-y)^2 + (4-z)^2}$$

$$\times \left[\frac{6-x}{\sqrt{(6-x)^2 + (4-y)^2 + (4-z)^2}} + \frac{x}{\sqrt{x^2 + (4-y)^2 + (4-z)^2}} \right]$$

$$+ \frac{x}{x^2 + y^2} \left[\frac{(2-z)}{\sqrt{x^2 + y^2 + (2-z)^2}} - \frac{(4-z)}{\sqrt{x^2 + y^2 + (4-z)^2}} \right]$$

$$- \frac{(2-z)}{y^2 + (2-z)^2} \left[\frac{(4-x)}{\sqrt{(4-x)^2 + y^2 + (2-z)^2}} + \frac{x}{\sqrt{x^2 + y^2 + (2-z)^2}} \right]$$

$$+ \frac{(4-x)}{(4-x)^2 + (6-y)^2} \left[\frac{(6-z)}{\sqrt{(4-x)^2 + (6-y)^2 + (6-z)^2}} \right.$$

$$\left. - \frac{(2-z)}{\sqrt{(4-x)^2 + (6-y)^2 + (2-z)^2}} \right] - \frac{(6-z)}{(6-y)^2 + (6-z)^2}$$

$$\times \left[\frac{(2-x)}{\sqrt{(2-x)^2 + (6-y)^2 + (6-z)^2}} - \frac{(4-x)}{\sqrt{(4-x)^2 + (6-y)^2 + (6-z)^2}} \right]$$

$$+ \frac{(2-x)}{(2-x)^2 + (2-y)^2} \left[\frac{-z}{\sqrt{(2-x)^2 + (2-y)^2 + z^2}} \right.$$

$$\left. - \frac{(6-z)}{\sqrt{(2-x)^2 + (2-y)^2 + (6-z)^2}} \right], \tag{6}$$

$$B_z = \frac{(2-y)}{(2-y)^2 + z^2} \left[\frac{6-x}{\sqrt{(6-x)^2 + (2-y)^2 + z^2}} - \frac{2-x}{\sqrt{(2-x)^2 + (2-y)^2 + z^2}} \right]$$

$$- \frac{(6-x)}{(6-x)^2 + z^2} \left[\frac{4-y}{\sqrt{(6-x)^2 + (4-y)^2 + z^2}} + \frac{(2-y)}{\sqrt{(6-x)^2 + (2-y)^2 + z^2}} \right]$$

$$- \frac{(4-y)}{(4-y)^2 + (4-z)^2} \left[\frac{6-x}{\sqrt{(6-x)^2 + (4-y)^2 + (4-z)^2}} \right.$$

$$+ \frac{x}{\sqrt{x^2 + (4-y)^2 + (4-z)^2}}\Bigg] - \frac{x}{x^2 + (4-z)^2}\Bigg[\frac{-y}{\sqrt{x^2 + y^2 + (4-z)^2}}$$

$$- \frac{(4-y)}{\sqrt{x^2 + (4-y)^2 + (4-z)^2}}\Bigg] - \frac{y}{y^2 + (2-z)^2}\Bigg[\frac{(4-x)}{\sqrt{(4-x)^2 + y^2 + (2-z)^2}}$$

$$+ \frac{x}{\sqrt{x^2 + y^2 + (2-z)^2}}\Bigg] - \frac{(4-x)}{(4-x)^2 + (2-z)^2}$$

$$\times \Bigg[\frac{(6-y)}{\sqrt{(4-x)^2 + (6-y)^2 + (2-z)^2}} + \frac{y}{\sqrt{(4-x)^2 + y^2 + (2-z)^2}}\Bigg]$$

$$+ \frac{(6-y)}{(6-y)^2 + (6-z)^2}\Bigg[\frac{(2-x)}{\sqrt{(2-x)^2 + (6-y)^2 + (6-z)^2}}$$

$$- \frac{(4-x)}{\sqrt{(4-x)^2 + (6-y)^2 + (6-z)^2}}\Bigg] - \frac{(2-x)}{(2-x)^2 + (6-z)^2}$$

$$\times \Bigg[\frac{(2-y)}{\sqrt{(2-x)^2 + (2-y)^2 + (6-z)^2}} - \frac{(6-y)}{\sqrt{(2-x)^2 + (6-y)^2 + (6-z)^2}}\Bigg]. \tag{7}$$

The corresponding expressions for the 4_1 (figure-eight) knot can be obtained exactly in the same manner and are given below:

$$B_x = -\frac{z}{(4-x)^2 + z^2}\Bigg[\frac{6-y}{\sqrt{(4-x)^2 + (6-y)^2 + z^2}} - \frac{2-y}{\sqrt{(4-x)^2 + (2-y)^2 + z^2}}\Bigg]$$

$$- \frac{(6-y)}{(4-x)^2 + (6-y)^2}\Bigg[\frac{4-z}{\sqrt{(4-x)^2 + (6-y)^2 + (4-z)^2}}$$

$$+ \frac{z}{\sqrt{(4-x)^2 + (6-y)^2 + z^2}}\Bigg] + \frac{(4-z)}{(4-x)^2 + (4-z)^2}$$

$$\times \Bigg[\frac{-y}{\sqrt{(4-x)^2 + y^2 + (4-z)^2}} - \frac{(6-y)}{\sqrt{(4-x)^2 + (6-y)^2 + (4-z)^2}}\Bigg]$$

$$+ \frac{(4-z)}{x^2 + (4-z)^2}\Bigg[\frac{(8-y)}{\sqrt{x^2 + (8-y)^2 + (4-z)^2}} + \frac{y}{\sqrt{x^2 + y^2 + (4-z)^2}}\Bigg]$$

$$- \frac{(8-y)}{(6-x)^2 + (8-y)^2}\Bigg[\frac{(2-z)}{\sqrt{(6-x)^2 + (8-y)^2 + (2-z)^2}}$$

$$- \frac{(4-z)}{\sqrt{(6-x)^2+(8-y)^2+(4-z)^2}} \Bigg] + \frac{(2-z)}{(6-x)^2+(2-z)^2}$$

$$\times \left[\frac{(8-y)}{\sqrt{(6-x)^2+(8-y)^2+(2-z)^2}} - \frac{(4-y)}{\sqrt{(6-x)^2+(4-y)^2+(2-z)^2}} \right]$$

$$+ \frac{(2-z)}{(2-x)^2+(2-z)^2} \Bigg[\frac{(8-y)}{\sqrt{(2-x)^2+(8-y)^2+(2-z)^2}}$$

$$- \frac{(4-y)}{\sqrt{(2-x)^2+(4-y)^2+(2-z)^2}} \Bigg] - \frac{(10-y)}{(2-x)^2+(10-y)^2}$$

$$\times \left[\frac{(6-z)}{\sqrt{(2-x)^2+(10-y)^2+(6-z)^2}} - \frac{(2-z)}{\sqrt{(2-x)^2+(10-y)^2+(2-z)^2}} \right]$$

$$+ \frac{(6-z)}{(2-x)^2+(6-z)^2} \Bigg[\frac{(10-y)}{\sqrt{(2-x)^2+(10-y)^2+(6-z)^2}}$$

$$- \frac{(2-y)}{\sqrt{(2-x)^2+(2-y)^2+(6-z)^2}} \Bigg] - \frac{(2-y)}{(2-x)^2+(2-y)^2}$$

$$\times \left[\frac{-z}{\sqrt{(2-x)^2+(2-y)^2+z^2}} - \frac{(6-z)}{\sqrt{(2-x)^2+(2-y)^2+(6-z)^2}} \right], \qquad (8)$$

$$B_y = - \frac{z}{(2-y)^2+z^2} \left[\frac{4-x}{\sqrt{(4-x)^2+(2-y)^2+z^2}} - \frac{2-x}{\sqrt{(2-x)^2+(2-y)^2+z^2}} \right]$$

$$+ \frac{(4-x)}{(4-x)^2+(6-y)^2} \Bigg[\frac{4-z}{\sqrt{(4-x)^2+(6-y)^2+(4-z)^2}}$$

$$+ \frac{z}{\sqrt{(4-x)^2+(6-y)^2+z^2}} \Bigg] - \frac{(4-z)}{y^2+(4-z)^2} \Bigg[\frac{-x}{\sqrt{x^2+y^2+(4-z)^2}}$$

$$- \frac{(4-x)}{\sqrt{(4-x)^2+y^2+(4-z)^2}} \Bigg] - \frac{(4-z)}{(8-y)^2+(4-z)^2}$$

$$\times \left[\frac{(6-x)}{\sqrt{(6-x)^2+(8-y)^2+(4-z)^2}} + \frac{x}{\sqrt{x^2+(8-y)^2+(4-z)^2}} \right]$$

$$+ \frac{(6-x)}{(6-x)^2+(8-y)^2} \Bigg[\frac{(2-z)}{\sqrt{(6-x)^2+(8-y)^2+(2-z)^2}}$$

$$
- \frac{(4-z)}{\sqrt{(6-x)^2+(8-y)^2+(4-z)^2}} \Bigg] - \frac{(2-z)}{(4-y)^2+(2-z)^2}
$$

$$
\times \left[\frac{(2-x)}{\sqrt{(2-x)^2+(4-y)^2+(2-z)^2}} - \frac{(6-x)}{\sqrt{(6-x)^2+(4-y)^2+(2-z)^2}} \right]
$$

$$
+ \frac{(2-x)}{(2-x)^2+(10-y)^2} \left[\frac{(6-z)}{\sqrt{(2-x)^2+(10-y)^2+(6-z)^2}} \right.
$$

$$
+ \frac{(2-z)}{\sqrt{(2-x)^2+(10-y)^2+(2-z)^2}} \Bigg] + \frac{(2-x)}{(2-x)^2+(2-y)^2}
$$

$$
\times \left[\frac{(6-z)}{\sqrt{(2-x)^2+(2-y)^2+(6-z)^2}} - \frac{(2-z)}{\sqrt{(2-x)^2+(2-y)^2+(2-z)^2}} \right],
$$

$$
\tag{9}
$$

$$
B_z = \frac{(2-y)}{(2-y)^2+z^2} \left[\frac{4-x}{\sqrt{(4-x)^2+(2-y)^2+z^2}} - \frac{2-x}{\sqrt{(2-x)^2+(2-y)^2+z^2}} \right]
$$

$$
- \frac{(4-x)}{(4-x)^2+z^2} \left[\frac{6-y}{\sqrt{(4-x)^2+(6-y)^2+z^2}} - \frac{(2-y)}{\sqrt{(6-x)^2+(2-y)^2+z^2}} \right]
$$

$$
- \frac{(4-x)}{(4-x)^2+(4-z)^2} \left[\frac{-y}{\sqrt{(4-x)^2+y^2+(4-z)^2}} \right.
$$

$$
- \frac{(6-y)}{\sqrt{(4-x)^2+(6-y)^2+(4-z)^2}} \Bigg] - \frac{y}{y^2+(4-z)^2} \left[\frac{-x}{\sqrt{x^2+y^2+(4-z)^2}} \right.
$$

$$
- \frac{(4-x)}{\sqrt{(4-x)^2+y^2+(4-z)^2}} \Bigg] + \frac{x}{x^2+(4-z)^2} \left[\frac{(8-y)}{\sqrt{x^2+(8-y)^2+(4-z)^2}} \right.
$$

$$
+ \frac{y}{\sqrt{x^2+y^2+(4-z)^2}} \Bigg] + \frac{(8-y)}{(8-y)^2+(4-z)^2}
$$

$$
\times \left[\frac{(6-x)}{\sqrt{(6-x)^2+(8-y)^2+(4-z)^2}} + \frac{x}{\sqrt{x^2+(8-y)^2+(4-z)^2}} \right]
$$

$$
- \frac{(6-x)}{(6-x)^2+(2-z)^2} \left[\frac{(4-y)}{\sqrt{(6-x)^2+(4-y)^2+(2-z)^2}} \right.
$$

$$
- \frac{(8-y)}{\sqrt{(6-x)^2+(8-y)^2+(2-z)^2}} \Bigg] + \frac{(4-y)}{(4-y)^2+(2-z)^2}
$$

V. V. Sreedhar

$$
\times \left[\frac{(2-x)}{\sqrt{(2-x)^2 + (4-y)^2 + (2-z)^2}} - \frac{(6-x)}{\sqrt{(6-x)^2 + (4-y)^2 + (2-z)^2}} \right]
$$

$$
- \frac{(2-x)}{(2-x)^2 + (2-z)^2} \left[\frac{(8-y)}{\sqrt{(2-x)^2 + (8-y)^2 + (2-z)^2}} \right.
$$

$$
\left. - \frac{(4-y)}{\sqrt{(2-x)^2 + (4-y)^2 + (2-z)^2}} \right] - \frac{(2-x)}{(2-x)^2 + (6-z)^2}
$$

$$
\times \left[\frac{(2-y)}{\sqrt{(2-x)^2 + (2-y)^2 + (6-z)^2}} - \frac{(10-y)}{\sqrt{(2-x)^2 + (10-y)^2 + (6-z)^2}} \right].
$$

$$
(10)
$$

It is obvious that the resulting expressions are cumbersome and not very illuminating. Hence, we turn our attention towards establishing the nontriviality of the connection by examining a holonomy. It is obtained by evaluating the line integral of the connection for the 3_1 knot above along the closed path (532)–(732)–(752)–(552)–(532). This path lies in the xy-plane. We expect the answer to furnish the value of the flux through the surface bounded by the loop, which in turn is the flux contained in the segment $\alpha = 3$, which is parallel to the z-axis, and passes through the center (321) of the surface. In the surrogate problem, this is just the current passing through the segment $\alpha = 3$. A painstaking sanity-check, using Mathematica, reveals that the answer for the holonomy is indeed what is expected. Similar tests can be carried out with various choices of the holonomy both for the 3_1 and 4_1 knots.

To summarize, we have worked out explicit expressions for the vector potentials produced by knotted solenoids. This was accomplished by working out the expressions for the magnetic fields produced by uniform currents running through corresponding knotted wire segments. The answers we obtain are nontrivial by construction, because they follow from a nontrivial solution of Maxwell's equations, namely, Biot–Savart law, for a given current distribution. The nontriviality of the potential is further directly verified by explicitly computing certain holonomies.

It is obvious that the technique presented in this paper is sufficiently general to be applied for an arbitrary knot. Besides being of interest in view of the fact that it yields exact expressions, it is hoped that it will be of some use in solving potential problems of the Aharonov–Bohm type[1] and its generalizations.[5,6]

References

1. Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
2. W. Ehrenberg and R. E. Siday, *Proc. Phys. Soc. B* **62**, 8 (1949).
3. R. G. Chambers, *Phys. Rev. Lett.* **5**, 3 (1960).
4. A. Tonomura, N. Osakabe, T. Matsuda, T. Kawasaki and J. Endo, *Phys. Rev. Lett.* **56**, 792 (1986).
5. J. C. Kimball and H. L. Frisch, *Phys. Rev. Lett.* **93**, 093001 (2004).
6. R. V. Buniy and T. W. Kephart, *Phys. Lett. A* **373**, 919 (2009).
7. V. V. Sreedhar, *J. Knot Theory Ramifications* **22**, 1350054 (2013).
8. B. Hayes, *Comput. Sci.* **85**, 506 (1997).
9. E. E. Callaghan and S. H. Maslen, The magnetic field of a finite solenoid, NASA Technical Note D-465 (1960).
10. J. Alexander, *Proc. Natl. Acad. Sci. USA* **9**, 93 (1928).
11. Y. Huh and S. Oh, *J. Knot Theory Ramifications* **14**, 859 (2005).
12. P. A. Tipler and G. Mosca, *Physics for Scientists and Engineers* (W. H. Freeman, 2007).