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ON THE TOPOLOGICAL ORIGIN OF ENTANGLEMENT IN ISING SPIN GLASSES

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The origin of entanglement in a class of three-dimensional spin models, at low momenta, is traced to topological reasons. The establishment of the result is facilitated by the gauge principle which, in conjunction with the duality mapping of the spin models, enables us to recast them as lattice Chern–Simons theories. The entanglement measures are expressed in terms of the correlators of Wilson lines, loops, and their generalisations. For continuous spins, these yield the invariants of knots and links. For Ising-like models, they can be expressed in terms of three-manifold invariants obtained from finite group cohomology — the so-called Dijkgraaf–Witten invariants.

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The history of modern physics is replete with stories of the progress that followed every time an appropriate choice of gauge fields was made to account for the fundamental forces of nature. Inspired by these successes, we shall use the idea of gauge invariance as a *tour de force* to gain insight into the origin of entanglement.

The need to understand the origin of quantum entanglement arises from the recognition that it is the theoretical bedrock that supports the emerging revolution in storing, processing, and retrieving information.[1] A fundamental question that naturally arises in this context is: What is the origin of entanglement in quantum systems? We shall show that the gauge principle holds the key to unlock this mystery.

The applicability of the gauge principle in this context follows from the realisation that purely quantum features with no classical analogues are usually associated with non-trivial topology and are most elegantly captured by the gauge principle. A staggeringly large set of examples which illustrate the role of topology in quantum theory can be found in a variety of physical contexts.[2–4]

We mention that an intriguing analogy between quantum entanglement and classical topology was publicised by Arvind,[5] following his serendipitous discovery of a similarity between the entanglement properties of the GHZ state[6] and the curious linking properties of Borromean and Hopf rings, and further developed by Kauffman and Lomonaco.[7] The hope that the breakthrough in finding new topological invariants of knots and links[8] may be used to characterise/classify entangled quantum states, gave rise to some excitement. Progress in this programme was hindered primarily because the rules to associate a closed loop to a quantum state, and the snipping of a loop to a measurement, remain vague. In this work we circumvent this problem by approaching it from Heisenberg's point of view. We thus deal only with physically observable quantities which allows us to directly focus on the intrinsic topological content of entanglement in a prototypical spin system namely, the 3D Ising model. In the process, we discover unambiguous relations between the usual measures of entanglement and the correlators of gauge-invariant observables in a lattice Chern–Simons theory. The latter produce knot and link invariants when the gauge group is continuous.[9] For finite gauge groups, the appropriate invariants, following from finite group cohomology, are called the Dijkgraaf–Witten invariants.[10]

The Ising model is defined by the Hamiltonian $H = -J\sum_{\langle ij\rangle} S_i S_j$, where $J > 0$ and $S_i = \pm 1$. Here i, j label the sites of a 3D cubical lattice and the $\langle\,\rangle$ parentheses indicate that the summation is over different, but nearest neighbour sites. The positivity of J implies that the ferromagnetic state minimises the energy. For the opposite sign of J, the antiferromagnetic state is favoured. As already motivated, we proceed to replace the above Hamiltonian which couples spins at different sites, by

$$H_U = -J\sum_{\langle ij\rangle} S_i U_{ij} S_j \tag{1}$$

in which the interaction between spatially separated spins is mediated by the gauge field U. The U_{ij} live on the links connecting sites i and j, are Z_2-valued, and hence equal to ± 1. The gauged Ising model accommodates either a ferromagnetic or an antiferromagnetic bond between various nearest neighbour sites. Such models play a crucial role in understanding the behaviour of disordered systems and are called *spin glasses*.[11] In this paper, however, a spin glass is merely an expedient to replace the action at a distance (of the order of the lattice spacing) between Ising spins by an interaction mediated by the gauge fields U_{ij}.

We define the partition function for the above model in the usual way as the trace over the Gibbs' measure, i.e. $Z = \mathrm{Tr}\, e^{-\beta H_U}$ where $\beta = 1/k_{\mathrm{B}}T$, k_{B} being the Boltzmann constant, and T, the temperature. In the first step, called *quenching* in the language of spin glasses, we sum over all the spin degrees of freedom to get[12]

$$Z = \cosh J \sum_{U,V} e^{-S_V - S_{\mathrm{CS}}}\,, \tag{2}$$

where

$$S_V = -\tilde{J}\beta \sum_{\square} \prod_{\square} V_{ij}\,, \quad S_{\rm CS} = \beta \sum_{\langle ij \rangle} i\frac{\pi}{4}\left(1 - \prod_{\square} V\right)(1-U) \;\; \text{and} \;\; \tanh \tilde{J} = e^{-2J}. \tag{3}$$

The V_{ij}, like the U_{ij}, are Z_2-valued fields, but live on the links of the dual lattice. The product of the dual gauge variables V_{ij} around an elementary plaquette on the dual lattice is indicated by the $\square$ under the product symbol. When the $\square$ appears under the summation symbol, it is an instruction to sum over all such elementary plaquettes. The S_V term is recognised as the standard Wilsonian action for V. The $S_{\rm CS}$ term is a measure of the flux passing through a dual plaquette perpendicular to a given link U on the primary lattice. It is a lattice Chern–Simons action. Two crucial steps in arriving at the above result consist of an expansion of Z in the characters of the Z_2 group, and the introduction of the dual lattice variables. The details can be found in Refs. 12 and 13. The generalisation to other finite abelian groups Z_p follows along the same lines,[14] as does the limiting case $p \to \infty$. The latter corresponds to the familiar lattice U(1) Chern–Simons theory.[15]

At this stage, a few remarks are in order. First, from the continuum perspective, for the case of a continuous group, this result is easily anticipated. It is well known from Ref. 16 that a derivative expansion of the 3D fermionic determinant produces, at the lowest two orders, the Chern–Simons and Maxwell terms. The result in (2) is a lattice realisation of the above continuum result. Second, from a lattice point of view, it is well known that the 3D Ising model is equivalent to a Z_2 gauge theory on the dual lattice.[17] The Ising spins with nearest neighbour interactions on the primary lattice have thus been traded for the V-fields with a Wilson action that appears in (3). In the duality transformation, the U fields introduced by the gauge principle are spectators and hence we arrive at a set of two gauge fields. Moreover, since there is an inextricable linkage between the primary and dual lattices, each link on the primary lattice pierces a plaquette of the dual lattice (and *vice versa*), and the Chern–Simons action is a measure of this flux. Third, the noticeable difference between the continuum and lattice realisations of what is essentially the same result is reminiscent of the lattice fermion doubling problem and has been discussed before in attempts to discretise Chern–Simons terms.[18] From a physical point of view, this is a consequence of the fact that the Chern–Simons term couples matter fields to the magnetic flux. In the present case, the Ising spins residing on the lattice sites are the matter fields. The magnetic flux, in lattice gauge theory, is defined by the plaquettes. There is no natural way of coupling these two objects without doing violence to the structure of the lattice. This very fact was used by Kantor and Susskind for one of the early models for anyons.[19]

Finally, we mention that anyonic models of quantum computation hold a lot of promise because they are topological.[20–22] Quite independently, remarkable progress has been achieved by using statistical mechanical techniques to study spin glasses in theories of information processing.[11,23] Interestingly, (2) establishes the equivalence of the above two seemingly different approaches to the subject of fault-tolerant quantum computation. More importantly, since the conceptual roots of the former approach lie in topology, while those of the latter lie in gauge invariance, it reinforces the idea anticipated at the beginning of this paper.

To proceed further, we note that although (3) does not treat U and V on the same footing, the exponential of $-S_{\rm CS}$ which appears in (2) is invariant under an exchange of U and V.[12] However, (2) itself is lop-sided because it does not have a Wilson term associated with the U field. This suggests that in the next step, called *configurational averaging*, we choose the weights such that the $U \leftrightarrow V$ symmetry is restored. This is tantamount to using $H_U + S_U$ where $S_U = -K\sum_\Box \prod_\Box U_{ij}$ instead of H_U. It is clear from the context that, in this case, the plaquettes under consideration belong to the primary lattice. If we now introduce the two-component vector $\Omega = (U, V)$ and the matrices $M = \sigma_x$ and $N = K(1+\sigma_z)/2 + \tilde{J}(1-\sigma_z)/2$, the partition function can be rewritten in the neat form

$$Z = \cosh J \sum_\Omega e^{-S} \quad \text{where} \quad S = \frac{\beta}{2}\sum_{\langle ij\rangle} i\frac{\pi}{4}\left(1 - \prod_\Box \Omega\right) M(1-\Omega)\,. \tag{4}$$

The $\langle ij\rangle$ in the above equation refers to, as before, nearest neighbour sites, but these could now be either on the primary or dual lattice. Also, a term of the form $-\beta N \sum_\Box \prod_\Box \Omega_{ij}$ has been dropped by taking the infrared (low momentum) limit. Equation (4) is recognised as the lattice Chern–Simons theory — also referred to as the BF-theory.

We are now in a position to examine the entanglement properties of Ising spins in the above system. The object of central interest in studying entanglement is the (reduced) density matrix. Once it is known, a relatively straightforward calculation yields the von Neumann entropy S of the system through the standard formula $S = -\mathrm{Tr}\,\rho \ln \rho$. The single particle reduced density matrix ρ_i, obtained by tracing over all the spins except the ith spin, can be expanded as $\rho_i = \frac{1}{2}\sum_{\alpha=0}^{3} c_\alpha \sigma_i^\alpha$ where $\sigma^0 = \mathbf{1}$ and $\sigma_i^{\alpha\neq 0}$ are the Pauli matrices at site i. A general expression for the coefficients c_α reads $c_\alpha = Z^{-1}\langle \sigma_i^\alpha \mathcal{P} \prod_{\Gamma(i,\infty)} U_{ij}\rangle$. The correlation function has been modified by the path-ordered insertion of a string Γ of links connecting the point i to ∞ making it path-dependent; the modification ensures that the density matrix is gauge-invariant. In a similar fashion, we can obtain the two-particle reduced density matrix by expanding it in terms of the tensor product of Pauli matrices at the two sites under consideration: $\rho_{ij} = \frac{1}{4}\sum_{\alpha,\beta=0}^{3} c_{\alpha\beta}\sigma_i^\alpha \otimes \sigma_j^\beta$. The coefficients of the expansion $c_{\alpha\beta}$ are the connected correlators of two Wilson lines. The two-particle reduced density matrix is useful in calculating the amount of entanglement *localisable* between the two chosen spins.[24] Similar results hold, in general, for n-particle entanglement. In each of these cases, the density matrix is expressed in terms of correlators of gauge-invariant observables in a topological field theory, i.e. topological invariants. The difference between two choices of the path Γ is a measure of the frustrations (plaquettes with an odd number of (anti-)ferromagnetic bonds) enclosed by the loop formed by the two paths. There is another way to obtain a closed loop from an open path namely, by imposing periodic boundary conditions. If we choose this option, c_α and $c_{\alpha\beta}$ are just the correlators of Wilson loop observables in the lattice Chern–Simons theory.

Although the Wilson loop observables are gauge-invariant, they are not the only interesting observables. Notice that the path connecting two points on the primary lattice pierces one plaquette on the dual lattice with every step it advances, accumulating a unit of flux in the process. Let us therefore consider the gauge-invariant operator $C = V^{-1}UV$ obtained by dressing (conjugating) the string U_{ij} by the group-valued fields V. Labelling the dual lattice sites by barred coordinates, if V runs from site $\bar{i}$ to site $\bar{j}$, V^{-1} runs in the opposite direction circumnavigating the link U. We can use this operator instead of the Wilson loop to define the reduced density matrices. It may be mentioned that because of duality, the above conjugation operation simply corresponds to local unitary transformations of nearest neighbour spins on the dual lattice. Physically the operator C represents a tube of dual plaquettes whose axis lies on the primary lattice with fixed end-points.

So far there is nothing quantum about the discussion of entanglement. Indeed, the Ising spins we have considered take values ± 1, much like classical bits; they are not allowed to be in a superposition state. The density matrices we considered are purely thermal in nature, obtained, as they are, from the Gibbs' measure. This kind of entanglement is called thermal entanglement.[25] By using the standard Suzuki–Trotter[11] method, however, we can map the 3D Ising spin glass to a 2D Ising spin glass in a transverse magnetic field. The presence of the transverse magnetic field allows for transitions between the two classical states of the Ising spins and makes the system quantum mechanical. Care must be exercised in defining the spin-flip operation: σ_z and σ_x behave differently under gauge transformations because of their non-commutativity. Furthermore, the situation here is slightly more complicated because the Suzuki–Trotter mapping from a d-dimensional classical statistical mechanical system to the $(d-1)$-dimensional quantum system, requires the classical system to have different couplings along the missing (replica) dimension and the remaining dimensions. This makes the duality transformation technically more involved. The results in so far as the entanglement (now truly quantum) are concerned, follow the same pattern. The correlation functions that appear in this case are those of the quantum spins on a 2D square sub-lattice of the original 3D lattice whose third dimension acts as the discretised time direction. To summarise, the reduced density matrices that one is interested in, both for thermal and quantum entanglement, are expressed in terms of topological invariants.

It is difficult to obtain a ready physical insight into the correlation functions that appear above. Let us therefore consider a simpler case. Recall that similar results hold for all Abelian groups Z_p. In particular for $p \to \infty$, we have a U(1) Chern–Simons gauge theory on the lattice. In this case, it is well known that the correlation function of a pair of Wilson loops is the Gauss's linking number. The operator C

is the lattice generalisation of the operator introduced in the continuum BF theory in Ref. 26, and gives it a nice physical interpretation as a tube of dual plaquettes. Its correlation function gives the Alexander–Conway polynomial of the (possibly) knotted axis of a plaquette-tube, with fixed endpoints on the primary lattice. The partition function Z is also a topological invariant, namely, the Reidemeister torsion of the three-manifold defined by the boundary conditions.[28] Curiously enough, the entanglement properties of the GHZ-state which has a non-zero tripartite entanglement, but a zero bipartite entanglement when a measurement is made along the $\hat{z}$ direction on one of the qubits, cannot be accounted for by the simple linking number invariants. This property is exactly like the corresponding property of the Borromean rings which, however, are known to be distinguished from a disjoint union of unlinked rings by a higher order topological invariant, namely the Massey triple product.[27] An Abelian topological theory cannot produce this invariant. It is therefore necessary to treat the spins as genuinely non-Abelian objects, like one is forced to in the presence of a transverse magnetic field. On the other hand, if the measurement is along the $\hat{x}$ direction, the entanglement between the remaining qubits can be easily accounted for by the linking number invariants that describe the Hopf rings. In general, therefore, one needs to associate a whole class of links to a given quantum state. The parallels between the entanglement properties of such a quantum state and the topological entanglement of the class of appropriately chosen braids and links[5] are then not so intriguing, when viewed in terms of the invariants that appear naturally in the form of correlators of gauge-invariant operators in a topological field theory.

Intuitively, topological invariants are insensitive to the presence, or changes, in length scales; this being the only distinction between a lattice and the continuum, we could borrow the relevant topological invariants from the simple continuum U(1) theory in the above discussion. The topological invariants in that case are easy to visualise. This luxury is lost if the group under consideration is finite, e.g. Z_2 in the case of the Ising model. Unlike classical gauge symmetries which come from continuous local invariances, finite groups usually appear as remnants of a continuous symmetry group which is spontaneously broken. In such instances, these groups are known to give rise to cocycles in quantum field theory. This is a reflection of nontrivial group cohomology. In the present case, the finite group appears because we are dealing with spin systems. The construction of topological theories with finite groups follows from a deep result due to Dijkgraaf and Witten,[10] who showed that the Chern–Simons actions for a finite group H are in one-to-one correspondence with the elements of the cohomology group $H^4(BH, Z)$, BH being the classifying space of H. The isomorphism $H^4(BH, Z) \approx H^3(H, \mathrm{U}(1))$ further implies that these actions are algebraic 3-cocycles, $\omega \in H^3(H, \mathrm{U}(1))$, which take values in U(1). The presence of non-trivial group cohomology in general leads to non-trivial G bundles and the path integral involves a summation over all possible bundles. Unlike in the U(1) theory, in which a determinantal expression can be derived for the partition function,[28] the resulting invariants in the case of finite groups — called the

Dijkgraaf–Witten invariants — are not expressible in terms of determinants. Analogues of linking numbers in topological gauge theories with finite gauge groups have also been worked out by Ferguson.[29]

As the first closing remark, we mention that the details omitted in this paper can be found in a longer publication under preparation. Second, we wish to point out that many interesting problems remain. The connections between quantum entanglement and more sophisticated topological invariants like the Jones polynomial, require a non-Abelian generalization of the results of this paper. Similar investigations in two and four dimensions should produce interesting connections between quantum entanglement and the intersection theory on the moduli space of Riemann surfaces, and Donaldson's invariants respectively. Finally, it is not an exaggeration to say that this letter offers a mere glimpse of a new vista which is beginning to unfold, on the relevance of finite group cohomology in the studies of entanglement in spin glasses. We hope to dilate on these issues, in the near future.

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