The maximal invariance group of Newton's equations for a free point particle[a)]

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(Received 22 February 2001; accepted 9 April 2001)

The maximal invariance group of Newton's equations for a free nonrelativistic point particle is shown to be larger than the Galilei group. It is a semidirect product of the static (nine-parameter) Galilei group and an $SL(2,R)$ group containing time translations, dilations, and a one-parameter group of time-dependent scalings called *expansions*. This group was first discovered by Niederer in the context of the free Schrödinger equation. We also provide a road map from the free nonrelativistic point particle to the equations of fluid mechanics to which the symmetry carries over. The hitherto unnoticed $SL(2,R)$ part of the symmetry group for fluid mechanics gives a theoretical explanation for an observed similarity between numerical simulations of supernova explosions and numerical simulations of experiments involving laser-induced implosions in inertial confinement plasmas. We also give examples of interacting many-body systems of point particles which have this symmetry group.
[DOI: 10.1119/1.1379736]

Almost all introductory books on the special theory of relativity mention, at least in passing, that Newton's equations of motion for a classical free nonrelativistic point particle are invariant under Galilei transformations. Probably not many eyebrows would be raised if we jumped from this fact to the conclusion that the Galilei transformations are the most general coordinate changes under which Newton's equations retain their form. It would therefore come as a considerable surprise to learn that there are other transformations which do the same. A simple example of such transformations may be given by noting that a freely moving point particle, with initial position $\mathbf{x}_0$ and velocity $\mathbf{v}_0$, traverses a straight line $\mathbf{x}(t)=\mathbf{x}_0+\mathbf{v}_0 t$. This equation may be rewritten as $\mathbf{x}/t=\mathbf{v}_0+\mathbf{x}_0/t$. In this representation t is replaced by $1/t$, lengths are scaled by a factor proportional to time, and the initial position and velocity are interchanged; but the important point is that the trajectory remains a straight line. Hence, the new variables $\mathbf{x}/t$, $1/t$ satisfy the same equations as the old ones $\mathbf{x}$, t.[1] Therefore the following question naturally arises: What is the maximal invariance group of Newton's equations of motion for a classical free nonrelativistic point particle? The answer to this question reveals that the maximal invariance group is a twelve-parameter group consisting of the usual ten-parameter Galilei group, the one-parameter group of dilations, and a one-parameter group of time-dependent scalings called *expansions*, which are nonrelativistic analogues of special conformal transformations. The existence of these transformations is not merely of academic interest. As explained in Refs. 1 and 2, such transformations provide a theoretical explanation for the plausibility of simulating astrophysical systems like supernova explosions by performing laser-induced plasma implosions.

In order to find the maximal invariance group of an equation, one has to find the set of all space–time transformations which leave it form invariant up to a factor.[3] This condition is equivalent to the requirement that the action be invariant. The action S for a free point particle of mass m, in d space dimensions, at position $x_i(t)$ is given by

$$S=\frac{m}{2}\int dt\left(\frac{dx_i}{dt}\right)^2, \tag{1}$$

where $i=1\cdots d$ and the sum over the index i has been suppressed. Let us now consider a transformation to new coordinates ξ, τ,

$$\xi_i=f_i(x,t), \quad \tau=h(x,t). \tag{2}$$

We wish to find the most general functions f_i, h which leave the action (1) form invariant:

$$\int dt\left(\frac{dx_i}{dt}\right)^2=\int d\tau\left(\frac{d\xi_i}{d\tau}\right)^2. \tag{3}$$

This is achieved by requiring

$$\frac{\left(\frac{\partial f_i}{\partial x_j}\frac{dx_j}{dt}+\frac{\partial f_i}{\partial t}\right)^2}{\left(\frac{\partial h}{\partial x_k}\frac{dx_k}{dt}+\frac{\partial h}{\partial t}\right)}=\left(\frac{dx_i}{dt}\right)^2+\frac{d}{dt}F(x,t) \tag{4}$$

for arbitrary functions $x_i(t)$ where $F(x,t)$ is an arbitrary boundary term and

$$\frac{d}{dt}F(x,t)=\left(\frac{\partial F}{\partial x_i}\frac{dx_i}{dt}+\frac{\partial F}{\partial t}\right). \tag{5}$$

Comparing powers of dx_i/dt on both sides of (4), we get

$$\frac{\partial h}{\partial x_i}=0\Rightarrow\tau=h(t), \tag{6}$$

$$\left(\frac{\partial f_i}{\partial x_j}\right)\left(\frac{\partial f_i}{\partial x_k}\right)=\frac{\partial h}{\partial t}\delta_{jk}, \tag{7}$$

$$2\frac{\partial f_i}{\partial x_j}\frac{\partial f_i}{\partial t}=\frac{\partial F}{\partial x_j}\frac{\partial h}{\partial t}, \tag{8}$$

$$\left(\frac{\partial f_i}{\partial t}\right)^2=\frac{\partial F}{\partial t}\frac{\partial h}{\partial t}. \tag{9}$$

From (6) and (7) it follows that

$$T_{ljk}+T_{lkj}=0,$$

where

$$T_{ljk}=\frac{\partial^2 f_i}{\partial x_l \partial x_j}\frac{\partial f_i}{\partial x_k}. \tag{10}$$

Subtracting $T_{kjl}+T_{klj}$ from the first equation in (10) and using the symmetry of T_{ljk} under a permutation of the first two indices, it may be shown that T_{ljk} is a totally symmetric tensor. Therefore $T_{ijk}=0$, which implies that f_i is linear in x,

$$f_i(x,t)=l(t)R_{ij}(t)x_j+m_i(t). \tag{11}$$

It follows from (7) that R can be chosen to be orthogonal, $R^T(t)R(t)=1$. Further, differentiating (8) with respect to x_k and using the explicit expression for f from (11) we get

$$(l\dot{l})\,\delta_{kj}+l^2(R^T\dot{R})_{kj}=\frac{1}{2}\frac{\partial^2 F}{\partial x_k \partial x_j}\dot{h}, \tag{12}$$

where the dot refers to a derivative with respect to t. Since the right-hand side and the first term on the left-hand side are symmetric in k and j, whereas the second term on the left-hand side is antisymmetric, by virtue of being in the Lie algebra of the rotation group; we have $\dot{R}=0$ and thus, R is a constant (rigid) rotation matrix. It then follows from (7) that

$$\dot{h}=l^2. \tag{13}$$

Eliminating F from (8) and (9) we get

$$\frac{\partial^2 f_i}{\partial t^2}\frac{\partial h}{\partial t}=\frac{\partial f_i}{\partial t}\frac{\partial^2 h}{\partial t^2}. \tag{14}$$

Substituting (11) into this equation and comparing powers of x we find

$$l\ddot{l}=2(\dot{l})^2, \tag{15}$$

$$l\ddot{m}_i=2(\dot{m}_i\dot{l}). \tag{16}$$

Using (13), Eq. (15) can be rewritten in terms of $h(t)$ and takes the form

$$\frac{\dddot{h}}{\dot{h}}-\frac{3}{2}\left(\frac{\ddot{h}}{\dot{h}}\right)^2=0. \tag{17}$$

The left-hand side of Eq. (17) is called the Schwarzian derivative of h and a standard result of complex analysis—although only the real part is relevant here—is that the solution of the above differential equation is

$$h(t)=\frac{\alpha t+\beta}{\gamma t+\delta}, \quad \text{where} \quad \alpha\delta-\beta\gamma=1, \tag{18}$$

where α, β, γ, δ are real. These transformations go by various names: fractional linear, projective, and global conformal. They form the group $SL(2,R)$. Substituting the above result in (13), and solving (16) for $m_i(t)$ we get

$$l(t)=\frac{1}{\gamma t+\delta}, \quad m_i(t)=\frac{b_i}{\gamma t+\delta}+d_i, \tag{19}$$

where b_i and d_i are integration constants. From (11) and (18) we then have, for the most general transformations that leave Newton's equations invariant,

$$\xi_i=\frac{R_{ij}x_j+a_i+v_i t}{\gamma t+\delta}, \quad \tau=\frac{\alpha t+\beta}{\gamma t+\delta},$$
$$\text{where} \quad \alpha\delta-\beta\gamma=1, \quad R^TR=1, \tag{20}$$

where a_i and v_i are constants expressible in terms of b_i and d_i. It is useful to consider the following two special cases.

(I) $\beta=\gamma=0$, $\alpha=\delta=1$: *Connected, Static Galilei Group* G: In this case, we have

$$g: \quad \tau=t, \quad \xi=R\mathbf{x}+\mathbf{a}+\mathbf{v}t. \tag{21}$$

These equations describe connected, static Galilei transformations which exclude parity and time reversal. It is clear from (21) that this is a nine-parameter group.

(II) $\mathbf{a}=\mathbf{v}=\mathbf{0}$, $R=1$: $SL(2,R)$ *Transformations*: In this case, we have

$$\sigma: \quad \tau=\frac{\alpha t+\beta}{\gamma t+\delta}, \quad \xi=\frac{\mathbf{x}}{\gamma t+\delta}, \quad \alpha\delta-\beta\gamma=1. \tag{22}$$

These are the $SL(2,R)$ generalizations of the inversion transformations presented in Ref. 1 and include time translations $(\gamma=0,\alpha=\delta=1)$, dilations $(\beta=\gamma=0)$, and a one-parameter group of time-dependent scalings called *expansions* $(\alpha=\delta=1,\beta=0)$. Since the parameters are constrained by the condition $\alpha\delta-\beta\gamma=1, SL(2,R)$ is a three-parameter group.

To understand the structure of the group, we study the relationship between the $SL(2,R)$ group and the connected static Galilei group G. Let us first consider a conjugation of a $g\in G$ by a $\sigma\in SL(2,R)$. By making three successive transformations of x and t we find that

$$\sigma^{-1}(\alpha,\beta,\gamma,\delta)g(R,\mathbf{a},\mathbf{v})\sigma(\alpha,\beta,\gamma,\delta)=g(R,\mathbf{a}_\sigma,\mathbf{v}_\sigma), \tag{23}$$

where

$$\begin{pmatrix}\mathbf{v}_\sigma\\ \mathbf{a}_\sigma\end{pmatrix}=\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{a}\end{pmatrix}, \tag{24}$$

which shows that G is an invariant subgroup. This result can be used to determine the product of two general elements σg and $\sigma' g'$ of the full group,

$$\sigma g\sigma' g'=\sigma\sigma'\tilde{g}g', \quad \text{where} \quad \tilde{g}=\sigma'^{-1}g\sigma'\in G. \tag{25}$$

This shows that the full group is not a direct, but only a semidirect product

$$\mathcal{G}=SL(2,R)\wedge G. \tag{26}$$

As is apparent from (25), the two factors in the semidirect product are on a different footing: While G is an invariant subgroup, $SL(2,R)$ is not. Furthermore, recall that G itself takes the form

$$G=R\wedge(T\otimes B), \tag{27}$$

where R is the rotation group and T and B are translation and boost groups with parameters $\mathbf{a}$ and $\mathbf{v}$, respectively. Since $SL(2,R)$ commutes with R, $\mathcal{G}$ can be expressed as a single semidirect product

$$\mathcal{G}=(SL(2,R)\otimes R)\wedge(T\otimes B). \tag{28}$$

Further it may be noted that the inversion Σ considered in Ref. 1 is the special element of $SL(2,R)$ for which $(\alpha,\beta,\gamma,\delta)=(0,-1,1,0)$. Note that $\Sigma^2=P$, where P is the parity transformation. This observation can be used to give a novel interpretation to a Galilei transformation. To see this we consider the coset elements $g_\Sigma(R,\mathbf{a},\mathbf{v})\equiv\Sigma g(R,\mathbf{a},\mathbf{v})$, where $g\in G$, we have

$$g_\Sigma(R',\mathbf{a}',\mathbf{v}')g_\Sigma(R,\mathbf{a},\mathbf{v})=g_P(R'R,R'\mathbf{a}-\mathbf{v}',R'\mathbf{v}+\mathbf{a}')$$
$$\Rightarrow g_\Sigma^2(R,\mathbf{a},\mathbf{v})$$
$$=g_P(R^2,R\mathbf{a}-\mathbf{v},R\mathbf{v}+\mathbf{a}), \tag{29}$$

where we have used the obvious notation $g_P(R,\mathbf{a},\mathbf{v})=Pg(R,\mathbf{a},\mathbf{v})$. Since every pair of vectors can be expressed as $R\mathbf{a}-\mathbf{v}$ and $R\mathbf{v}+\mathbf{a}$ for suitable $\mathbf{a}$ and $\mathbf{v}$, this shows that every parity reflected static Galilei transformation is the square of a coset transformation g_Σ. Therefore every connected static Galilei transformation is the fourth power of a coset transformation.

As is well known, according to Noether's theorem, there exists a conserved quantity corresponding to every continuous symmetry. The conserved quantities for the usual Galilean transformations are standard and will not be repeated here. The conserved quantities for the $SL(2,R)$ symmetry can be derived as follows: The invariance of the action implies

$$\delta L(x,\dot{x},t)=\frac{d\mathcal{F}}{dt}. \tag{30}$$

For time-independent Lagrangians $L(x_i,\dot{x}_i)$, we have

$$\delta L=\frac{\partial L}{\partial x_i}\delta x_i+\frac{\partial L}{\partial \dot{x}_i}\delta\dot{x}_i=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right), \tag{31}$$

where the Euler–Lagrange equations have been used in the second equality. Combining the two expressions for δL, we get a conservation law

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i-\mathcal{F}\right)=0. \tag{32}$$

The $SL(2,R)$ transformations are

$$t\rightarrow\tau=\frac{\alpha t+\beta}{\gamma t+\delta},$$
$$x(t)\rightarrow\xi(\tau)=\frac{x(t)}{\gamma t+\delta}=(\alpha-\gamma\tau)x\left(\frac{\delta\tau-\beta}{-\gamma\tau+\alpha}\right). \tag{33}$$

For infinitesimal transformations, $\alpha=1+\epsilon$ and $\delta=1-\epsilon$ (to ensure $\alpha\delta-\beta\gamma=1$) with infinitesimal β, γ, ϵ,

$$\delta t=\beta+2\epsilon t-\gamma t^2,$$
$$\delta x(t)=(\epsilon-\gamma t)x(t)-(\beta+2\epsilon t-\gamma t^2)\dot{x}(t). \tag{34}$$

The change of $L=(m/2)\dot{x}^2$ is given by

$$\delta L=m\dot{x}\,\delta\dot{x}=m\dot{x}(-\gamma x-(\epsilon-\gamma t)\dot{x}-(\beta+2\epsilon t-\gamma t^2)\ddot{x}). \tag{35}$$

It is easily seen to be the total time derivative of

$$\mathcal{F}=-(\beta+2\epsilon t-\gamma t^2)\frac{m\dot{x}^2}{2}-\gamma\frac{mx^2}{2}. \tag{36}$$

Therefore, we obtain the conserved quantity

$$X=m\dot{x}\,\delta x-\mathcal{F}=-(\beta+2\epsilon t-\gamma t^2)\frac{m\dot{x}^2}{2}+(\epsilon-\gamma t)mx\dot{x}$$
$$+\gamma\frac{mx^2}{2}. \tag{37}$$

Extracting the coefficients of β, ϵ, and γ we get

$$X=-\beta H+\epsilon D+\gamma A,\quad H=\frac{p^2}{2m},$$
$$D=p\left(x-\frac{tp}{m}\right),\quad A=\frac{(tp-mx)^2}{2m}, \tag{38}$$

where $p=m\dot{x}$ and H, D, A are the conserved quantities related to time translations, dilatations, and expansions, respectively. The following interesting observation about the conserved quantities can now be made: Noether's theorem can also be used to show that the conserved quantities corresponding to the usual translation and boost symmetries are $p_i=m\dot{x}_i$ and $K_i=tp_i-mx_i$, respectively; hence it follows that $A=K^2/2m$ is related to K in the same way as the Hamiltonian is related to the momenta. Mathematically, H, A, and $D=-pK/m$ form the adjoint representation of the $SL(2,R)$, while p and K transform as a doublet, and rotations are invariant. This concludes the discussion of the symmetries of the classical nonrelativistic point particle.

We shall now consider the quantum mechanical generalization of the above results. For this it is convenient to think of the wave function of the particle as a nonrelativistic field. For the usual ten-parameter Galilei group G, it is well known that, in the field theoretic realization, there is a one-parameter mass group M that commutes with G. This is called the central extension and has the effect of modifying the Lie algebra of G in a nontrivial manner, while at the same time preserving the Jacobi identity. It turns out that a similar feature holds for the twelve-parameter group $\mathcal{G}$.

In the field representation the conserved quantities of $\mathcal{G}$ are

$$\mathbf{P}=-i\boldsymbol{\nabla},\quad \mathbf{J}=-i\mathbf{x}\times\boldsymbol{\nabla},\quad \mathbf{K}=-it\boldsymbol{\nabla}-m\mathbf{x}, \tag{39a}$$

$$D=i\left(2t\frac{\partial}{\partial t}+\mathbf{x}\cdot\boldsymbol{\nabla}+\frac{3}{2}\right),$$
$$A=-i\left(t^2\frac{\partial}{\partial t}-\frac{m}{2}\mathbf{x}^2+t\mathbf{x}\cdot\boldsymbol{\nabla}+\frac{3}{2}t\right),\quad H=i\frac{\partial}{\partial t}, \tag{39b}$$

where the factors of 3/2 appear because we have Weyl ordered products of position and momentum in the quantum theory and set $\hbar=1$. $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ generate translations, rotations, and boosts, respectively, which constitute the connected static Galilei group (21). H, D, and A produce time translations, dilations, and *expansions* which together constitute the $SL(2,R)$ group. The central extension of the standard Lie algebra of the former,

$$[J_i,J_k]=i\epsilon_{ikl}J_l,\quad [J_i,P_k]=i\epsilon_{ikl}P_l,\quad [P_i,P_k]=0,$$

$$[K_i,K_k]=0,\quad [J_i,K_k]=i\epsilon_{ikr}K_r,\quad [K_i,P_k]=-(im\delta_{ik}),$$

$$[P_i,H]=0,\quad [K_i,H]=-iP_i,\quad [J_i,H]=0,$$

is augmented by the additional relations involving the $SL(2,R)$ generators

$$[D,J_i]=0, \quad [D,K_i]=iK_i, \quad [D,P_i]=-iP_i,$$
$$[A,J_i]=0, \quad [A,K_i]=0, \quad [A,P_i]=iK_i,$$
$$[D,H]=-2iH, \quad [A,H]=iD, \quad [D,A]=2iA$$

to give the Lie algebra of the full group $\mathcal{G}$. The bracketed term in the $[K_i,P_j]$ commutator is the nontrivial modification that the central extension brings about in the Lie algebra with m—to be physically identified with the mass of the particle—standing for the value that the generator of M takes in the given representation.

Since the existence of the central extension implies that

$$[K_i,P_j]=-(im\delta_{ij}) \tag{40}$$

the groups of translations $T(\mathbf{a})=e^{i\mathbf{a}\cdot\mathbf{P}}$ and boosts $B(\mathbf{v})=e^{i\mathbf{v}\cdot\mathbf{K}}$ no longer form a direct product, but a Heisenberg–Weyl group defined by the relation

$$T(\mathbf{a})B(\mathbf{v})=B(\mathbf{v})T(\mathbf{a})e^{im\mathbf{a}\cdot\mathbf{v}}. \tag{41}$$

The action (24) of $SL(2,R)$ on $\mathbf{a}$ and $\mathbf{v}$ induces the transformations

$$\sigma^{-1}\begin{pmatrix}\mathbf{K}\\ \mathbf{P}\end{pmatrix}\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\mathbf{K}\\ \mathbf{P}\end{pmatrix} \tag{42}$$

of the generators $\mathbf{K}$ and $\mathbf{P}$. The commutator (40) does not change under these transformations because of the condition $\alpha\delta-\beta\gamma=1$. Thus the central extension is compatible with $SL(2,R)$ and the full invariance group of the quantized system is the central extension of $\mathcal{G}$.

Indeed it was in the quantum theory that the group $\mathcal{G}$ was first discovered by Niederer, who showed that it is the maximal kinematical invariance group of the free particle Schrödinger equation.[4] Since the invariance under the Galilei group is well known, it suffices to verify that the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2m}\frac{\partial^2\psi}{\partial x_i^2}=0 \tag{43}$$

remains form invariant under $SL(2,R)$. It is easily checked that this is accomplished by the following transformation of the wave function that is generated by A, H, and D of the $SL(2,R)$ group:

$$\psi(\mathbf{x},t)\propto(\gamma\tau-\alpha)^{3/2}\exp\left(i\frac{F}{2}\right)\psi(\xi,\tau), \tag{44}$$

where F is determined through (8) and (9).

As promised, we now sketch a road map to an astrophysical application of the results of this paper. As is explained in detail in Ref. 2, recent experimental programs try to simulate astrophysical systems like supernova explosions in the laboratory by creating implosions in inertial confinement plasmas. This research is inspired by a remarkable similarity which was observed in the results of numerical models of 1987A supernova observations and results of numerical simulations of experiments involving plasma implosions. Referring the reader to Ref. 2 for further details of this research program, we note that this is a puzzling observation because the former system involves very large length and time scales whereas the latter involves very small scales. A theoretical explanation of this intriguing similarity was given in Ref. 1 and can be traced to the symmetry properties of the fluid dynamic equations[5,6] that describe both stellar structure and the plasma state. As conjectured in Ref. 1, this symmetry has its origin in the symmetries of the free point particle. In particular the symmetry responsible for mapping supernova explosions to plasma implosions is the fluid mechanical analogue of the *expansion* transformations discussed in this paper. To make this connection more precise, we note that the analysis of the single free point particle can be carried over to the noninteracting many particle case in a straightforward manner. Indeed, using the expression for the Hamiltonian of an ensemble of free point particles labeled by I, and the expression for the momenta $\mathbf{p}_I$,

$$H=\sum_I\frac{\mathbf{p}_I^2}{2m}, \quad \mathbf{p}_I=m\dot{\mathbf{x}}_I, \tag{45}$$

it is easy to see that the corresponding Liouville equation stating the invariance of the density of particles ρ along the flow

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\{\rho,H\}=0 \tag{46}$$

also has a maximal invariance group given by (26). As is well known, the Liouville equation can be converted into the Boltzmann equation by expressing all the momenta in terms of the velocities. Therefore, the symmetry group carries over to the collisionless Boltzmann equation. Further, in the continuum limit, one can use the standard procedure of deriving the various fluid dynamic equations as moments of the Boltzmann equation.[7] The simplest example is the set of Euler equations[8]

$$D\rho=-\rho\boldsymbol{\nabla}\cdot\mathbf{u}, \tag{47a}$$

$$\rho D\mathbf{u}=-\boldsymbol{\nabla}p, \tag{47b}$$

$$\mathbf{D}\epsilon=-(\epsilon+p)\boldsymbol{\nabla}\cdot\mathbf{u}, \tag{47c}$$

where ρ, $\mathbf{u}$, p, and ϵ stand for the density, the velocity vector field, the pressure, and the energy density of the fluid, respectively. The convective derivative D in the above equations is defined by $D=\partial/\partial t+\mathbf{u}\cdot\boldsymbol{\nabla}$. The above differential equations of fluid flow are usually augmented by an algebraic condition called the polytropic equation of state which relates the pressure to the energy density as follows:

$$p=(\gamma_0-1)\epsilon, \tag{47d}$$

where γ_0 is a constant called the polytropic exponent. For the ensemble of free nonrelativistic point particles being considered here, γ_0 takes the value 5/3. The maximal symmetry group $\mathcal{G}$ therefore extends to the fluid dynamic equations, which explains the observed similarity between supernova explosions and plasma implosions.

Although we have considered free particles so far, it is interesting to note that the symmetry group discussed in this paper extends to an interesting class of interacting many-body problems namely, those for which the potential is an inverse square of the coordinate differences. In one dimension these include the so-called Calogero–Moser models[9] with Hamiltonian

$$H=\frac{1}{2}\left(\sum_{I=1}^{N}p_I^2+g^2\sum_{I\neq J}\frac{1}{(x_I-x_J)^2}\right). \tag{48}$$

These models are integrable and admit *exclusion statistics*—an exciting area of current research.[10] In two dimensions it may be shown that models with Hamiltonians of the form

$$H=\frac{1}{2}\left(\sum_{I=1}^{N}(\mathbf{p}_I-e\mathbf{A}(\mathbf{x}_I))^2\right),$$

where

$$A_k(\mathbf{x}_I)\propto\sum_{J\neq I}\frac{\epsilon_{kl}(\mathbf{x}_I-\mathbf{x}_J)_l}{|x_I-x_J|^2}, \tag{49}$$

which describe a gas of anyons—particles having arbitrary spin and statistics—have potentials which admit the symmetry group $\mathcal{G}$. As is well known, anyons are of interest because they appear as excitations in fractional quantum Hall systems.[11]

To conclude, we have investigated the maximal kinematical invariance group of a free nonrelativistic point particle and have found that it is bigger than the Galilei group. It is a semidirect product of the form $SL(2,R)\wedge G$, where G is the static Galilei group. As shown in this paper, this group is in fact the maximal invariance group of a host of interesting systems in which the physics content is captured by the quintessential free nonrelativistic point particle. However there exists a class of interacting many-particle systems—of which the well-known Calogero—Moser models and the anyon model are of particular interest—for which this is also true.

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