A FEW WRONG MEN IN A PARADISE FOR MATHEMATICIANS

(With Apologies to Gurudev Rabindranath Tagore)

V. V. Sreedhar, Chennai Mathematical Institute

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- $\vec{u} = \frac{1}{2}(y\hat{i} - x\hat{j}) \Rightarrow (\vec{\nabla} \times \vec{u})_z = \frac{1}{2}(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}) = 1$
  i.e. $(\vec{\nabla} \times \vec{u}) = \hat{k}$*: Two-dimensional vortex*

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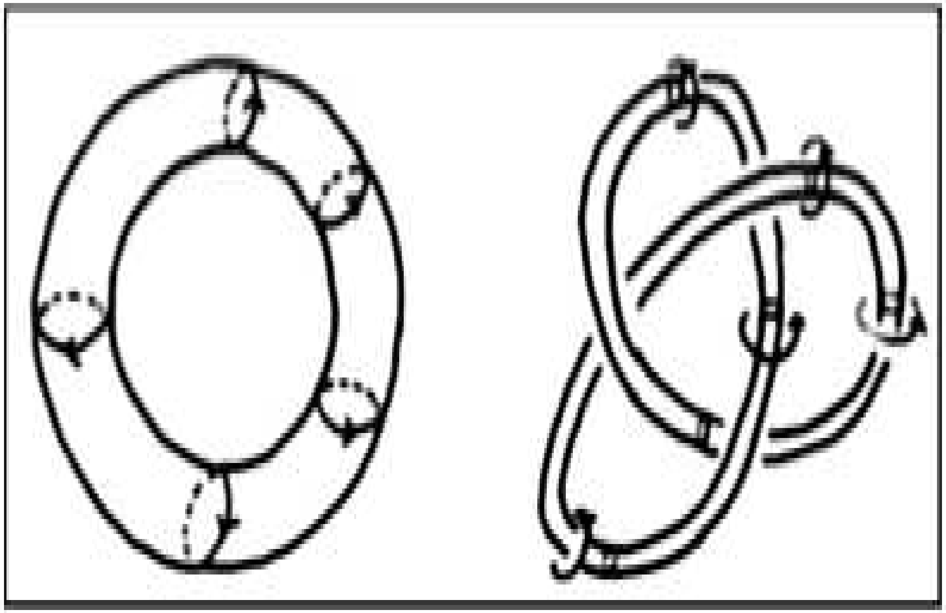
Water Twists



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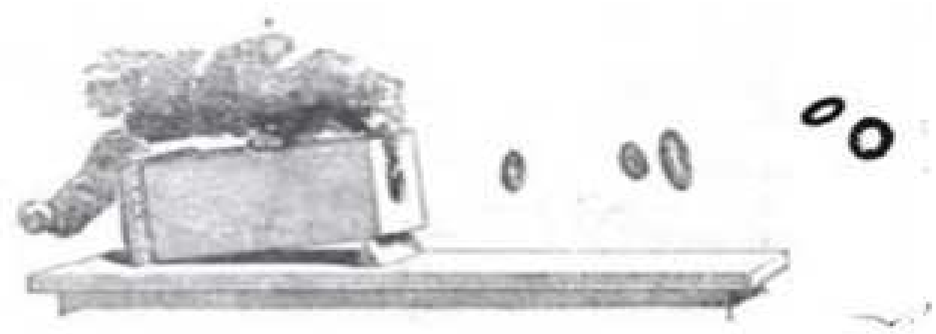

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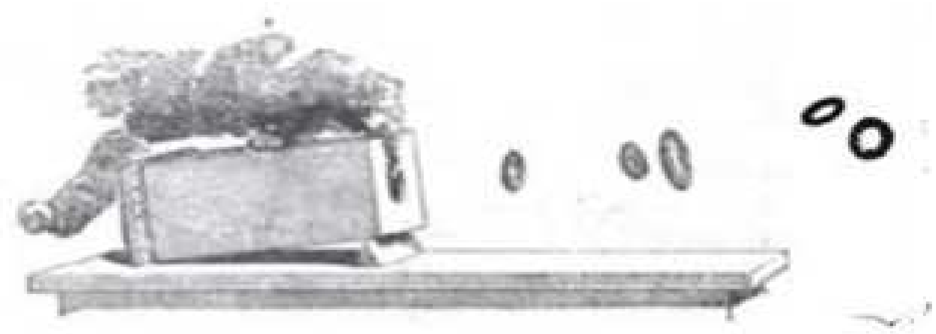

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- "*Never, I think, can there have been a more superb demonstrator. I have his burly figure before me. The small twinkling eyes had a fascinating gleam in them; he could concentrate them until they held the object looked at; when they flashed back around the room he seemed to have drawn a rapier. I have seen a man fall back in alarm under Tait's eyes, though there were a dozen benches between them*". *– J. M. Barrie (Author of Peter Pan)*

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- *They simply wriggle around a knife trying to cut them, and maintain their form.*

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- *Servant to his master: "You are, sir!"*

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- *Intellectual Support: J. C. Maxwell*

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Year	Class Prizes - Maxwell	Class Prizes - Tait
1842	No Mention	Dux 1st Class; 1st English reader, 2nd in English
1843	19th in 3rd class; 1st in Scripture Biography.	Dux 2nd Class; 2nd in English.
1844	? in 4th Class; 1st in Scripture Biography.	Dux 3rd Class; 4th in Arithmetic.
1845	11th in 5th Class; Silver Medal of Academical Club; 1st in English Verses.	Dux 4th Class; 1st in Latin Verses; 1st in Arithmetic.
1846	5th in 6th Class; 1st in English Verses and in English; 2nd in Mathematics.	Dux 5th Class; 2nd in English; 1st in French
1846 Academical Club Prizes	6th overall; 3rd in Mathematics, in History, in Geography and in Scripture Biography; 6th in English and in French.	3rd overall; 1st in Mathematics; 3rd in History, in Geography and in Scripture Biography; 5th in English and in French; 10th in Latin.
1847	1st in Mathematics in 7th Class; Silver Medal.	Dux 6th Class; 1st in Mathematics and in Latin Verses; 2nd in French and in Physical Science.
1847 Academical Club Prizes	2nd overall; 1st in Mathematics and in English; 2nd in Latin; 4th = in Greek;	3rd overall; 2nd in Mathematics; 3rd = in Latin; 4th = in English.

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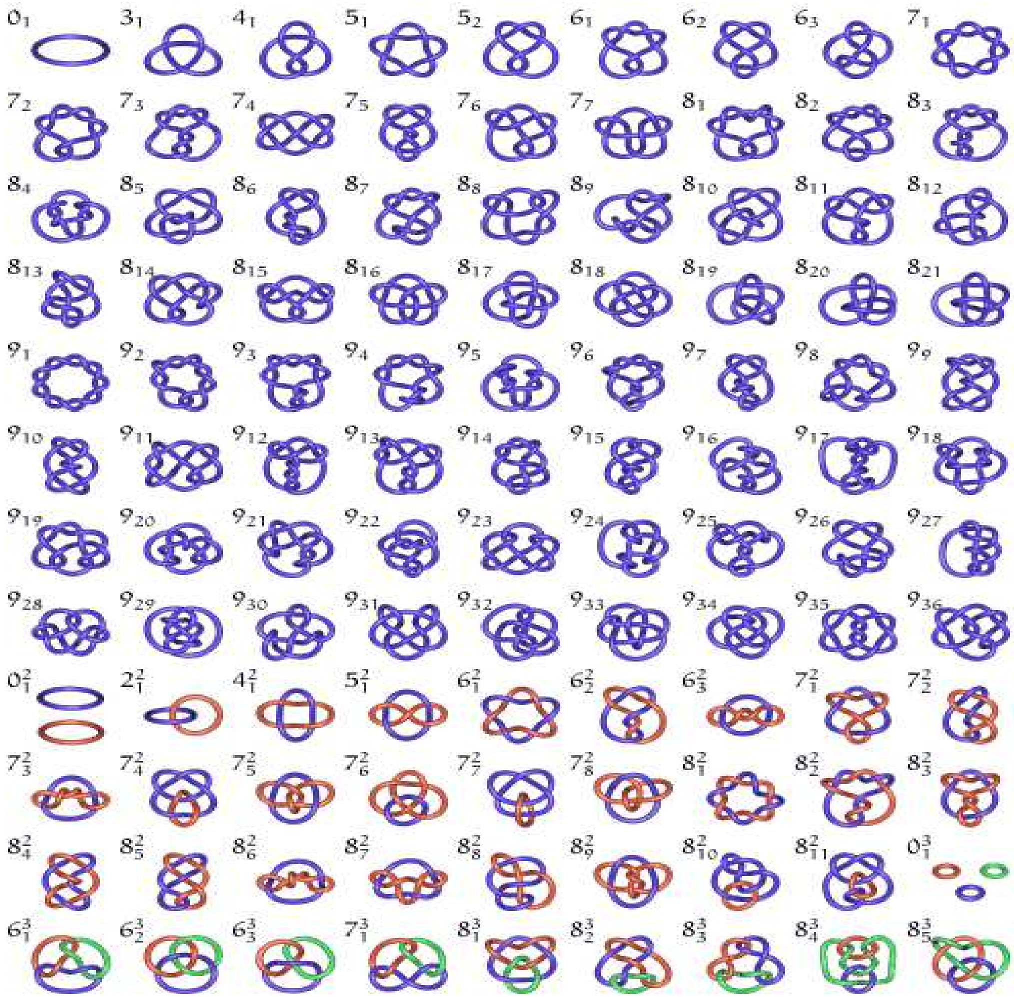
- *Code: T (Thomson), $T^{'}$ (Tait), $T^{''}$ (Tyndall), C (Clausius), H (Hamilton), H^2 (Helmholtz). (Decode: dp/dt)*

Tait’s Knot Table



Tait's Conjectures

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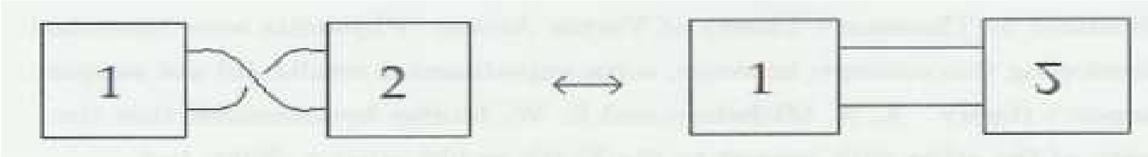

- *Two alternating diagrams, with no nugatory crossings, of the same oriented link have the same Tait (or writhe) number, i.e. the signed sum of all crossings of the diagram with the convention that an overpass being +1 and an underpass being -1*.

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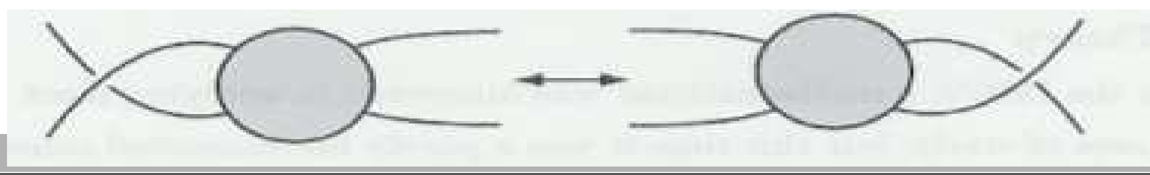

Knot Theory

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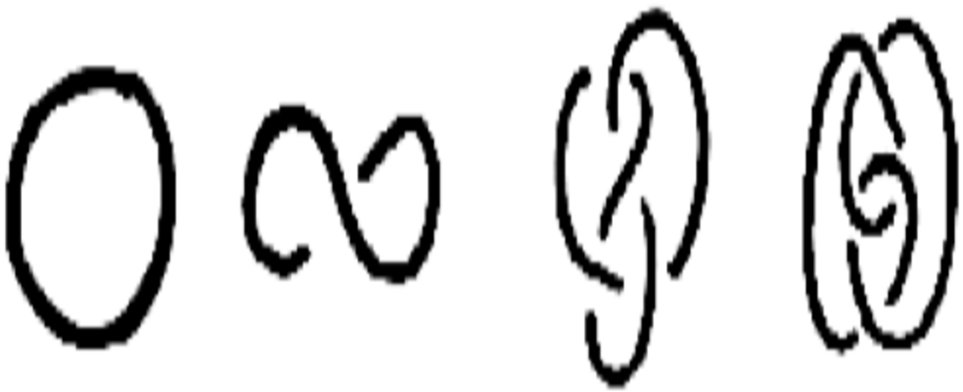

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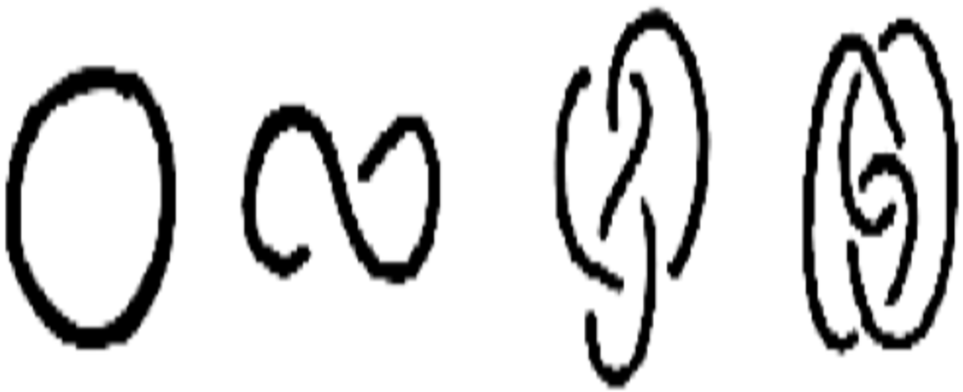
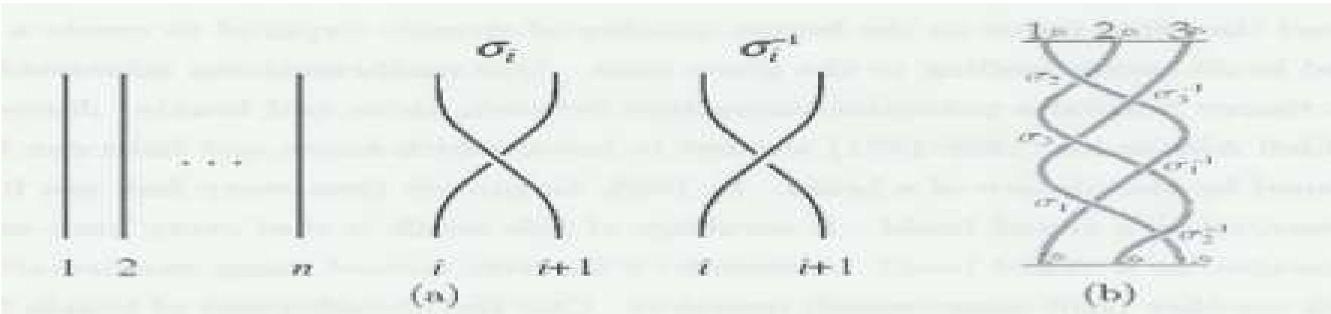


- *The fundamental problem of knot theory is to be able to distinguish between inequivalent knots*.

Artin and His Braids



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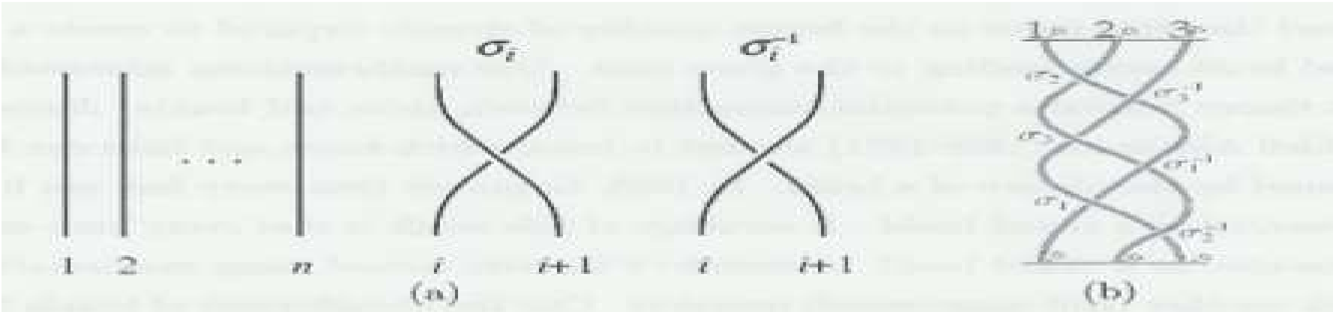
- 

- $\sigma_i \sigma_j = \sigma_j \sigma_i, \quad | \, i - j \, | \geq 2; \qquad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \;\; \forall \;\; i$

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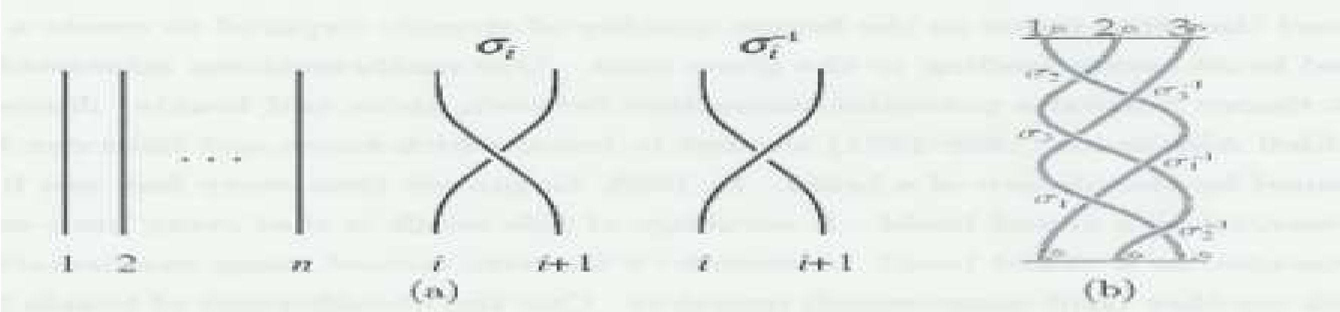
- 

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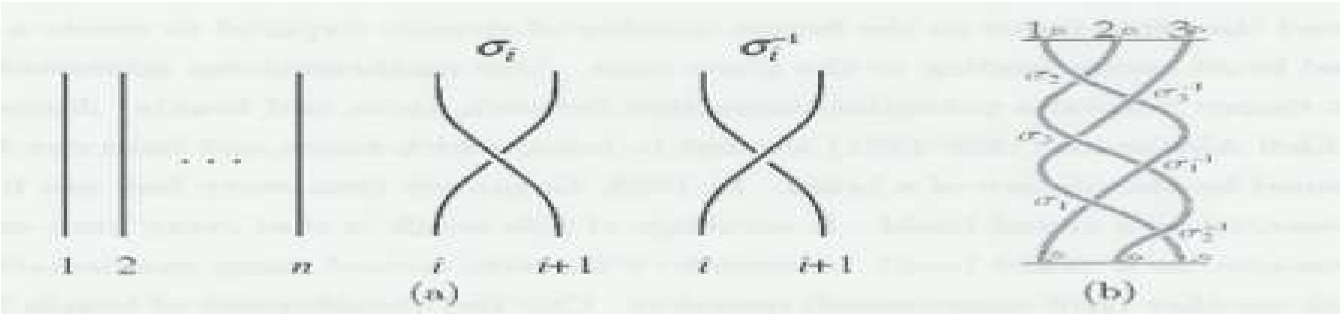
- 

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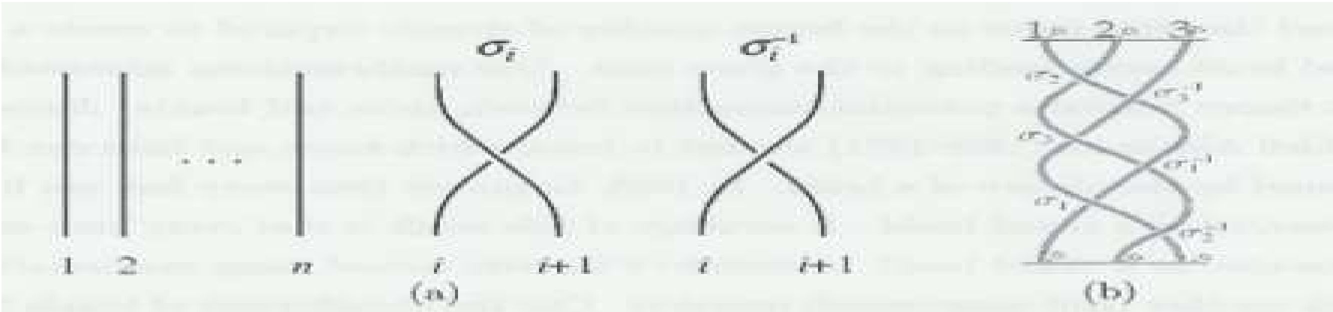
- 

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- *Alexander's Theorem (1928): Braids related to knots and links through closure.*

Reidemeister and His Moves

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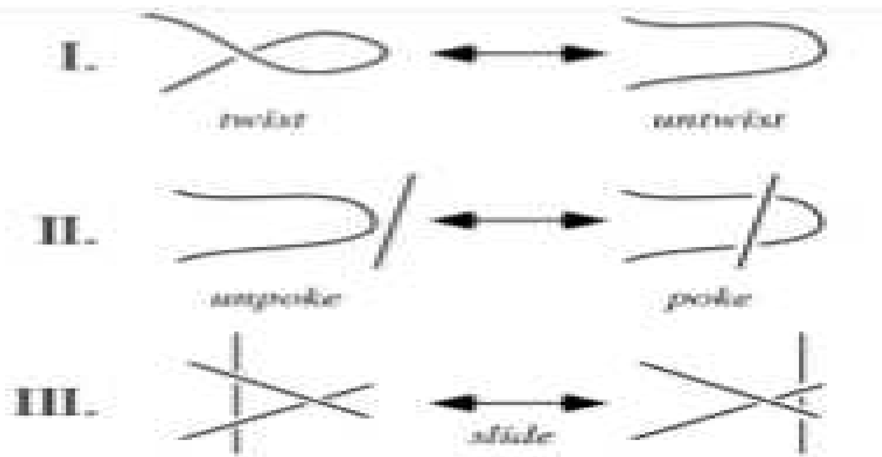

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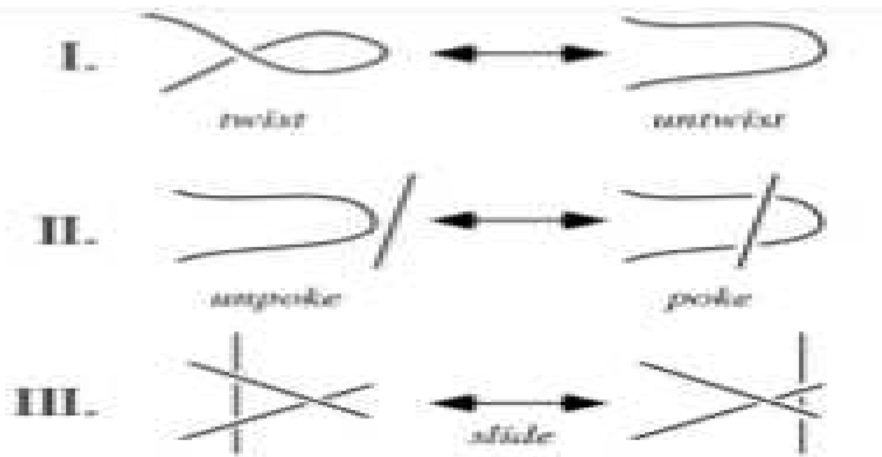
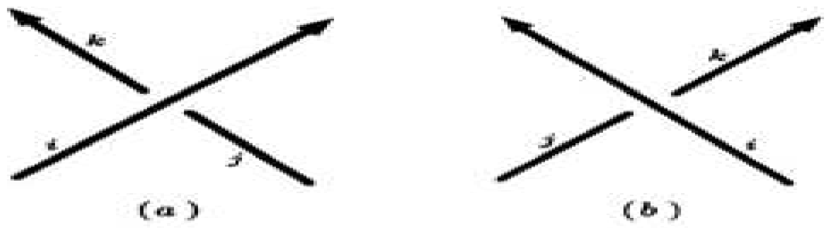


- *J. C. Maxwell had discovered this result earlier, but refused to publish the results despite his friend Tait's prodding*.

Alexander and His Polynomial

- 1. Pick an oriented diagram for the knot K; separately number the n arcs and crossings of the diagram.



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(a) (b)

- *2. Define an $n \times n$ matrix with the row index representing a crossing and the column index an arc. For a right-handed crossing, l, the entries are: $(1-t),\ (-1),\ (t)$ for the li, lj and lk elements, and for a left-handed crossing the entries are $(1-t),\ t,\ (-1)$ respectively.*

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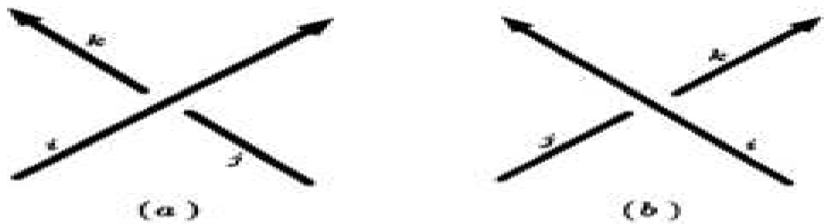

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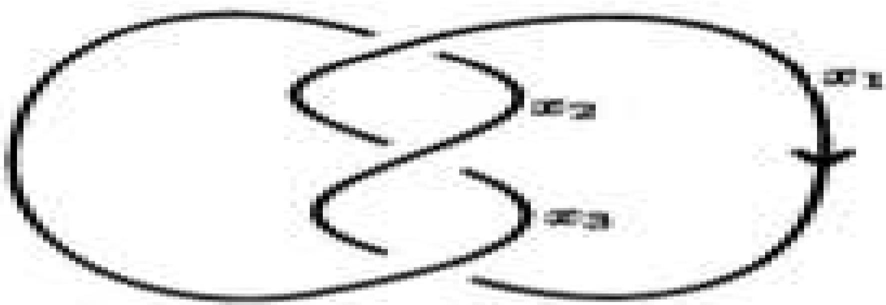
- *3. The $(n-1) \times (n-1)$ matrix obtained by removing the last row and column is called the Alexander matrix of K and its determinant is the Alexander polynomial $A_K(t)$.*

Alexander Polynomial of a Trefoil



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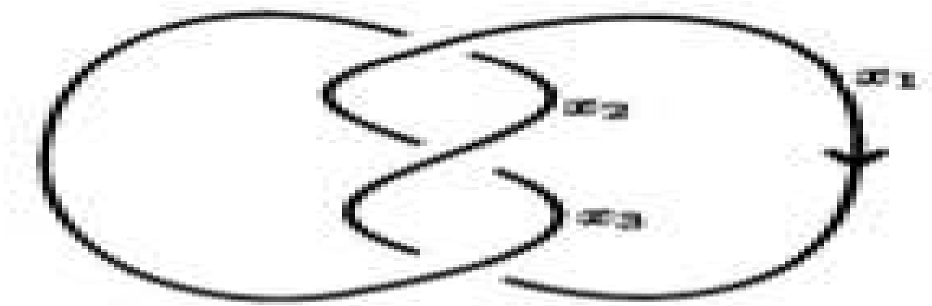

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$$\begin{pmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{pmatrix}$$

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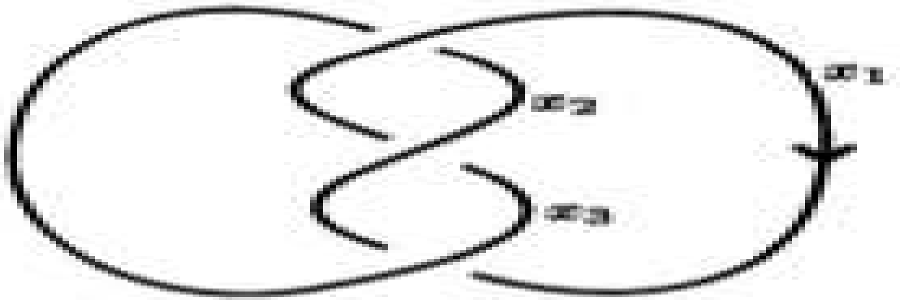
- 

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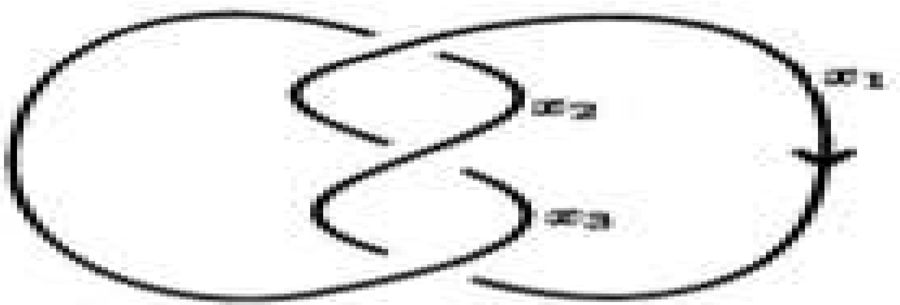

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- K *and* K' *are inequivalent if* $A_K(t) \neq A'_K(t)$. *Does not distinguish between mirror images.*

Conway and His Polynomial (1960)

- *Axioms*:
  1. *Invariance:* $K \sim K^{'} \Rightarrow \nabla(K) = \nabla(K^{'})$
  2. *Normalization:* $\nabla(O) = 1$
  3. *Skein Relation:* $\nabla(K_{+}) - \nabla(K_{-}) = x\nabla(K_{0})$

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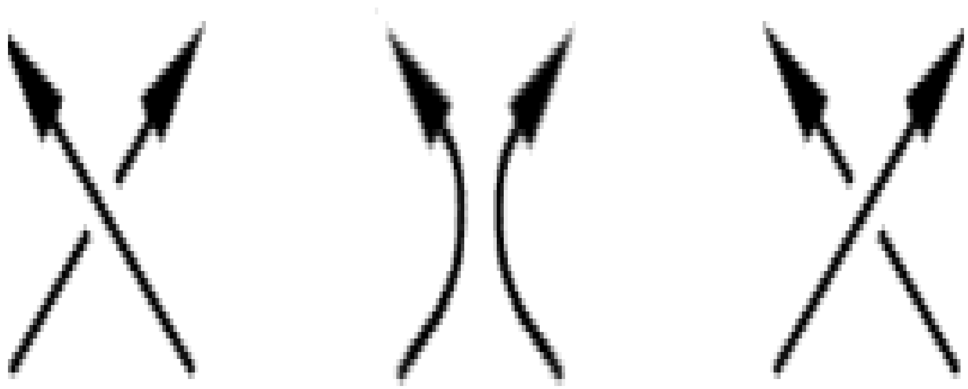

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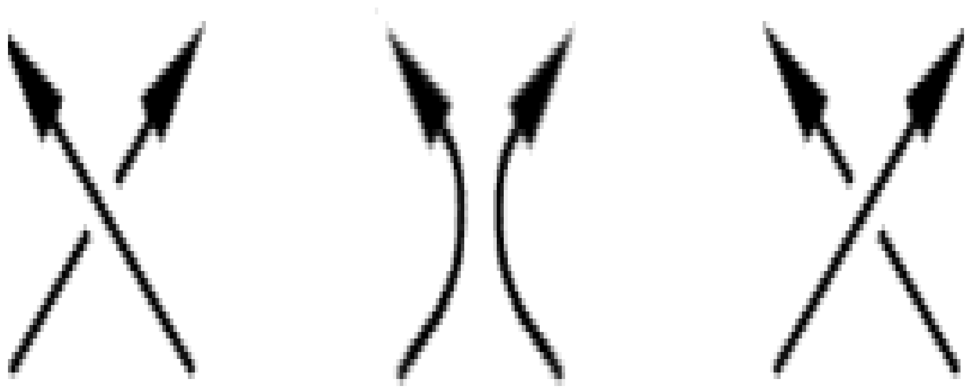

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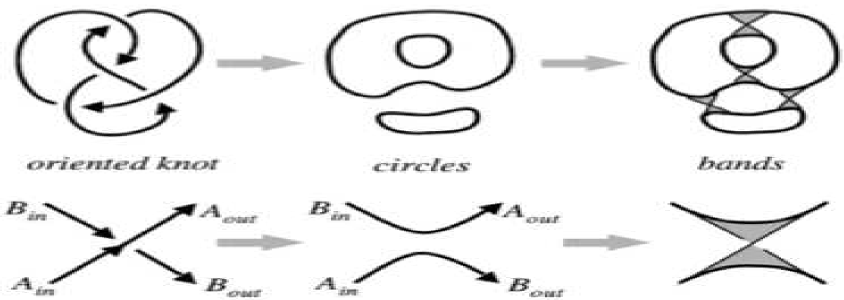

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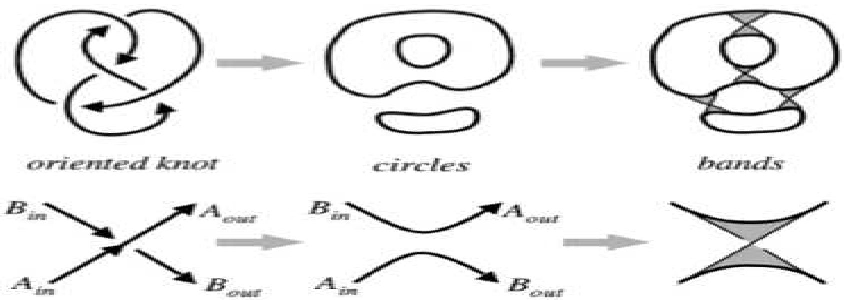

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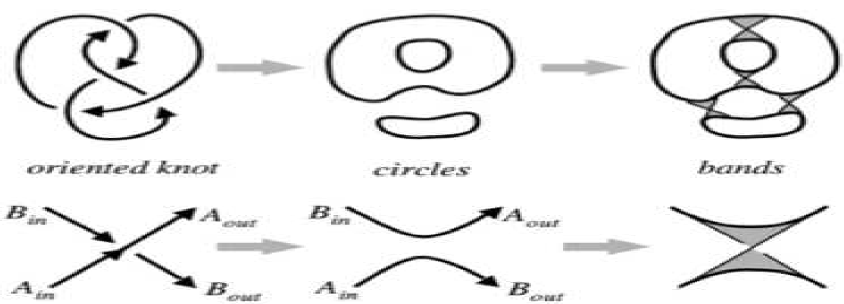

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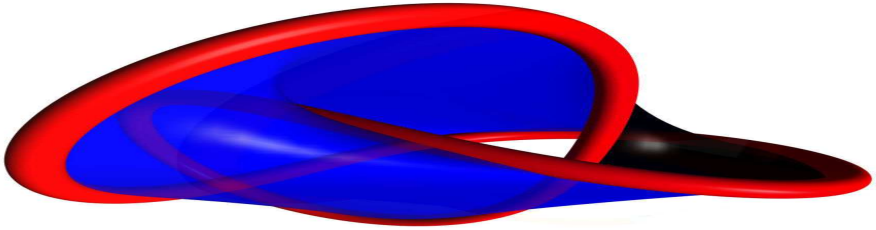

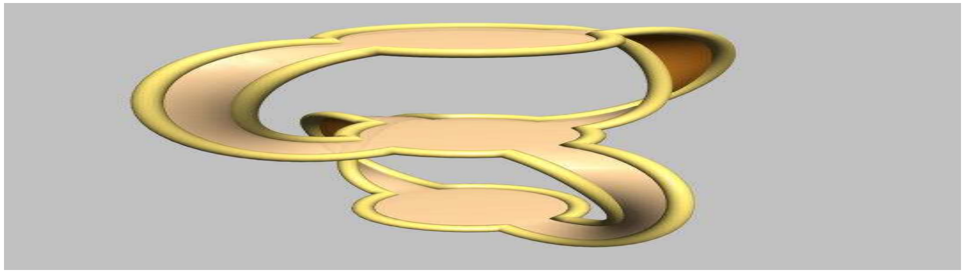

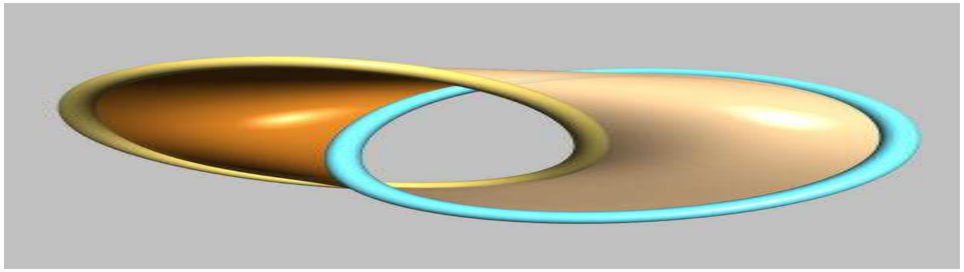

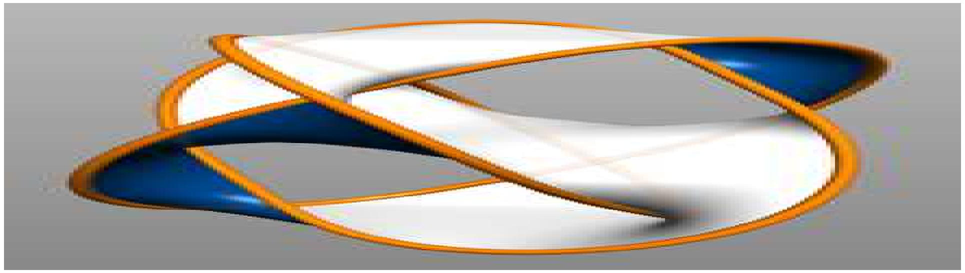

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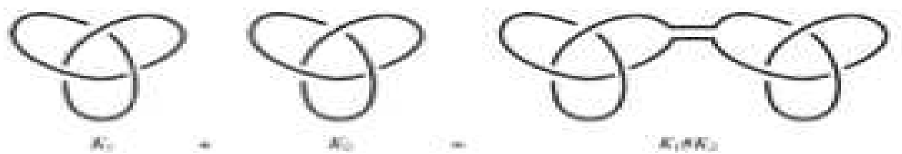
- 

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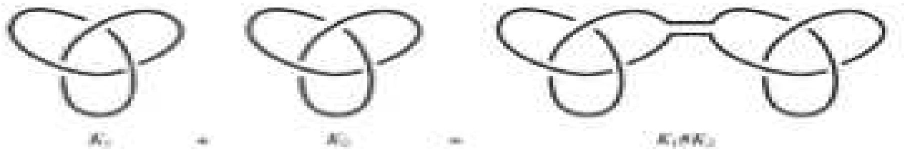
- 
- *A nontrivial knot is called a prime knot, if it cannot be decomposed into a nontrivial connected sum.*
- *The fundamental theorem of arithmetic says that every integer greater than one, is either a prime itself, or is the product of prime numbers.*
- *Schubert's theorem: Any nontrivial knot has a finite decomposition into prime knots, and this decomposition is unique up to order.*

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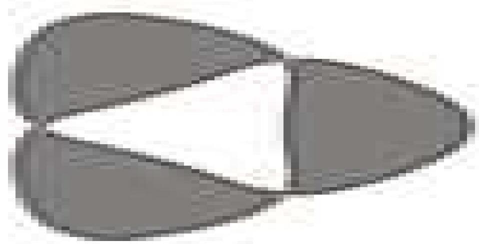

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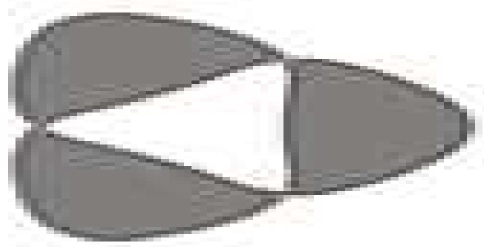
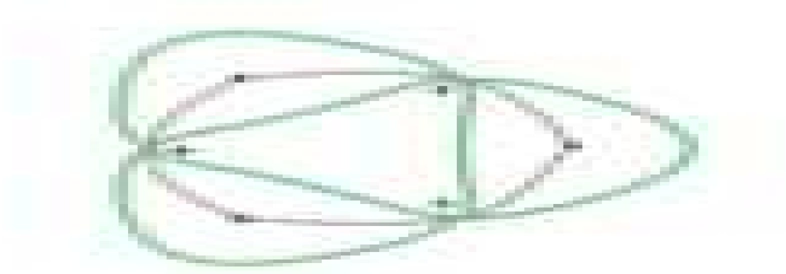


- 

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- *The knot group is an invariant of the knots. Tietze showed that the knot group can be used to distinguish between the unknot and a trefoil knot. The knot group of the trefoil is the braid group on three strings.*

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- *For $p = 2,\ q = 3$, for example, the curve C is a trefoil knot.*

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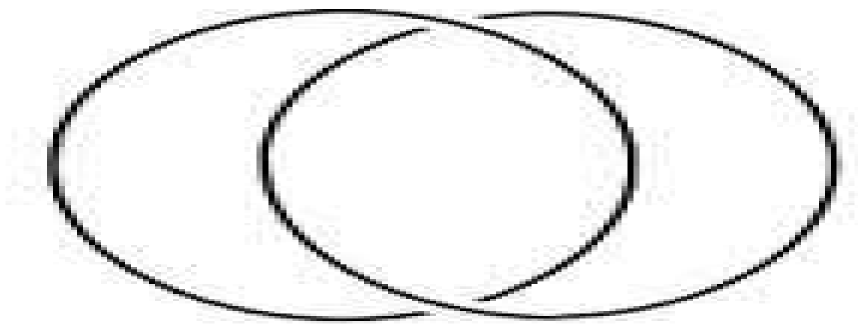

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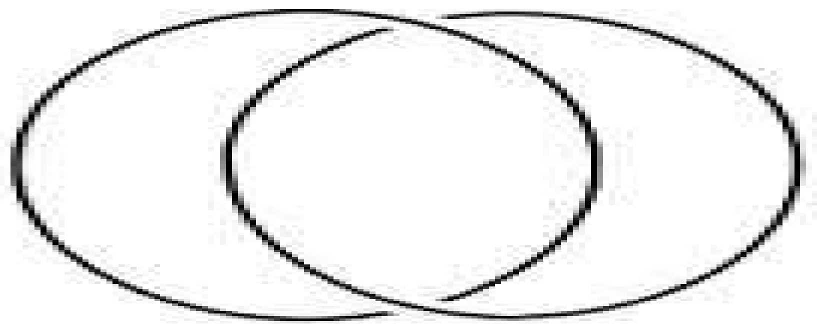

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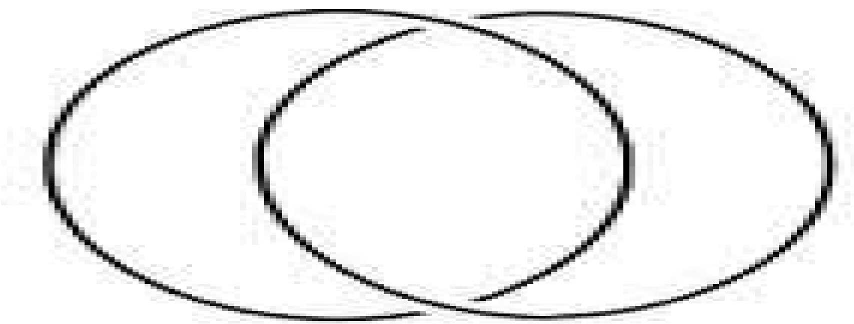

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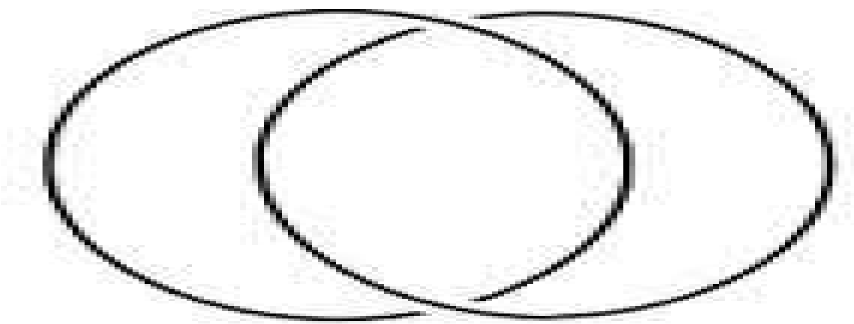

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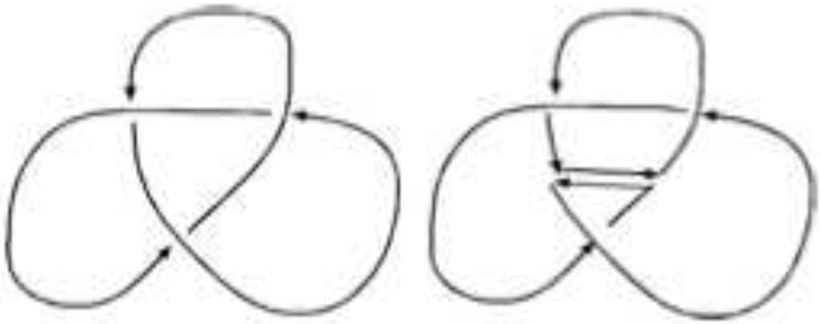

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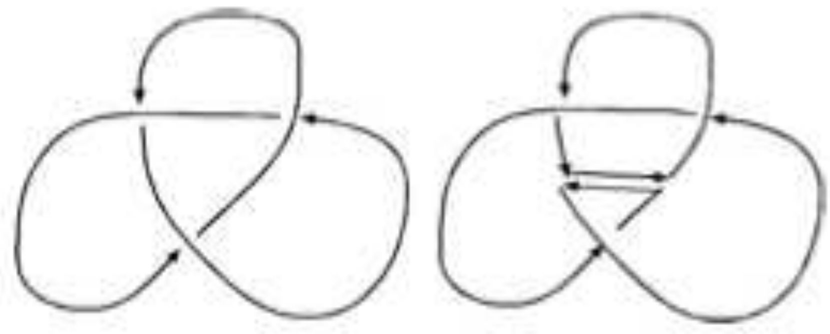

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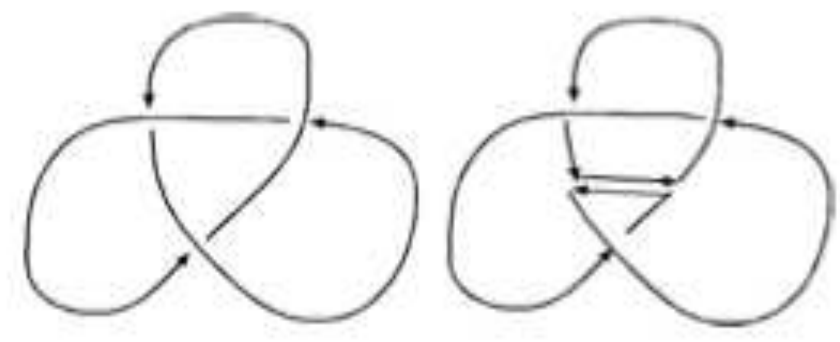

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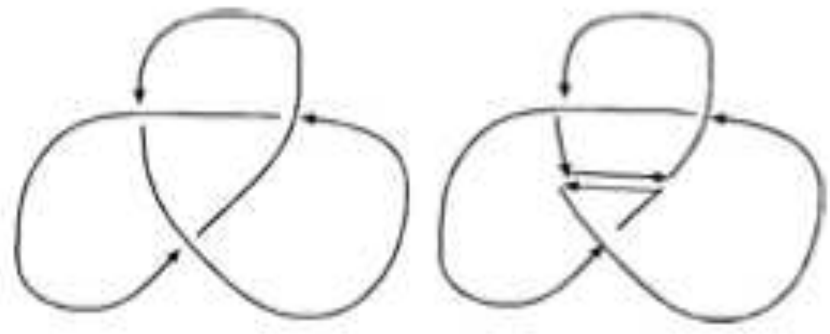

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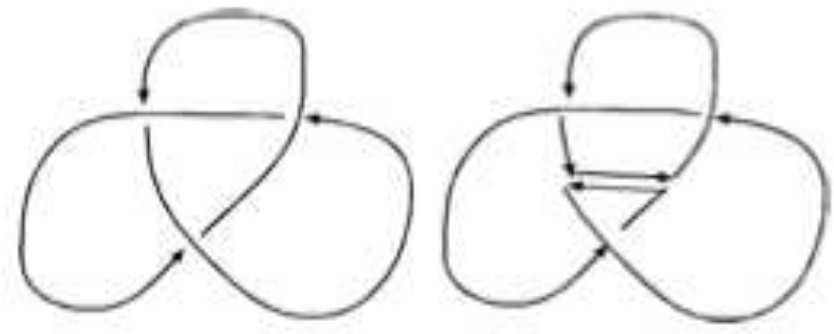

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- *This is not to say that* $W = 0$. *For* $t \neq s$, *the integral has a finite, non-zero, answer.*

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- *The Frenet-Serret Equations*

$$\frac{d\hat{t}}{ds} = \kappa\hat{n}, \quad \frac{d\hat{n}}{ds} = -\kappa\hat{t} + \tau\hat{b}, \quad \frac{d\hat{b}}{ds} = -\tau\hat{n}$$

κ: *Curvature*, τ: *Torsion*: *Illustration*

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- *Framing: Given a space curve, we can define another small ribbon based on it, with width ϵ by the following prescription:*
- $\vec{R}(s) = \vec{r}(s) + \epsilon\hat{n}, \quad \epsilon > 0$
- *The linking number between the two edges of the ribbon can be easily obtained using Gauss's formula. It will be an integer. We can also actually evaluate the integral in two small steps.*
- *Substitute for the two edges using the above equations, and evaluate the integral for $s - \delta \leq t \leq s + \delta$ in the limit $\epsilon \to 0$, to get the following expression*:

$$\frac{1}{2\pi}\int_0^1 ds\hat{t} \cdot \hat{n} \times \hat{b}$$

This is the torsion of the curve.

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- *Suppose a molecule has $n = 4,\ T = 2.5,\ W = 1.5$. Suppose that $\beta >> \alpha$, the molecule will minimise the bending energy (curvature) by forming a flat circle for which $W = 0$. But this will be very twisted i.e. $T = 4$.*

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- *This is an example of a twist-to-writhe transformation.*

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- THANKS!