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AN EXACT EXPRESSION FOR A FLAT CONNECTION ON THE COMPLEMENT OF A TORUS KNOT

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ABSTRACT

Simple physics ideas are used to derive an exact expression for a flat connection on the complement of a torus knot. The result is of some importance in the context of constructing representations of the knot group — a topological invariant of the knot. It is also a step forward in the direction of obtaining a generalization of the Aharonov–Bohm effect in which charged particles moving through force-free regions are scattered by impenetrable, knotted solenoids.

Keywords: Flat connection; torus knot complement.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

This paper deals with the problem of deriving a flat connection on the complement of a torus knot. The motivation for undertaking this exercise is two-fold. First, any exact expression is interesting in its own right. Besides, in the present case, the result is useful in constructing holonomies and hence the representations of the knot group — a well-known topological invariant of the knot [1]. Second, the result has a direct physical application. It is a first step toward generalizing the seminal work of Aharonov and Bohm, on the quantum mechanical scattering of charged particles moving through force-free fields [2, 5], to a situation in which the impenetrable solenoid is knotted [4, 8]. The key idea that is used in this paper to obtain the result relies on modeling a knot with wires and solenoids carrying steady currents. The fields associated with these objects can then be calculated in a straightforward manner by standard methods of classical electrodynamics.

As a prelude to the calculation, it is useful to briefly recapitulate some relevant, but well-known, mathematical facts about knots [10]. A knot K is a closed, oriented, loop of string in $\mathbf{R}^3$. It is defined by the map $K : S^1 \to \mathbf{R}^3$. Two knots $K_1, K_2 \in \mathbf{R}^3$ are equivalent if there exists an orientation-preserving homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(K_1) = K_2$.

Let X be the complement, $\mathbf{R}^3 - K$, of the knot K. This is a path-connected (noncompact) 3-manifold. The knot determines the complement. Clearly, equivalent knots have homeomorphic complements.

The knot group is, by definition, the fundamental group $\pi_1(X)$. Since complements of equivalent knots are homeomorphic, their fundamental groups are isomorphic.

A converse theorem [6] states that knots are determined by their complements. In other words, two knots having homeomorphic complements are equivalent. A second converse theorem [14] states that if two prime knots have isomorphic groups, their complements are homeomorphic which, by virtue of the first theorem, implies that the two knots are equivalent. Hence, the knot group determines the knot. It is a topological invariant associated with the knot. In general, knot groups are nonabelian. The knot group of a trefoil knot, for example, is the braid group on three strings, $\mathbf{B}_3$.

To put the above statements into perspective, let us consider a two-dimensional analogue. The space relevant to the standard Aharonov–Bohm effect is a plane with a hole. The fundamental group of this space is the (abelian) additive group of integers, $\mathbf{Z}$. As is well-known, an exact expression, in Cartesian coordinates, for a flat connection on this space is given by the formula $\vec{A}(x, y, z) = \frac{\Phi}{2\pi(x^2+y^2)}(-y, x, 0)$, where Φ is the flux through the hole. What is the corresponding expression for a flat connection on the complement of a knot? The present exercise answers this question.

2. A General Expression for a Flat Connection on a Knot Complement

Consider a small tubular neighborhood of the knot with cross-sectional radius ϵ. Imagine a densely packed winding around it, with a wire carrying a constant current i per winding. Let n be the number of windings per unit length, and dl' an infinitesimal line element at $\vec{r}'$ along the tube. This knotted solenoid is a simple, albeit nontrivial, generalization of the more familiar current distributions used in the study of the Aharonov–Bohm effect *viz.* the solenoidal and toroidal distributions in which the winding is done around a cylinder and a torus, respectively. From basic magnetostatics, the winding produces a magnetic field which has support only inside the knotted tube. In the complement of the knot, the vector potential (connection) is non-zero but the magnetic field vanishes imposing the flatness condition.

The brute-force method to calculate the vector potential at any point in the complement of the knotted solenoid follows the standard technique of adding contributions from the individual loops. This leads to an integral (over the length of the knot) of elliptic integrals (contribution from the individual windings). This answer is not very illuminating: A simplification by way of a reduction in the number of integrals is desirable.

Strictly speaking, the winding produces a component of the current along the knot which produces a non-zero magnetic field outside the knotted tube. The resulting field can be cancelled by passing an appropriate current through the axis of the knotted tube, in the opposite direction to the winding. This in turn produces an additional contribution to the vector potential which should be accounted for.

That is not all: The above current distribution also produces a field due to contact terms which we call the "knot moments" — analogous to the toroidal moment, also known as the anapole [15] in the case of a toroidal winding. Such fields typically have their support only in the source region; nevertheless, they produce a vector potential outside the sources. This produces a further contribution to the result.

All the three caveats mentioned above can be circumvented by the following simple expedient: We let the radius ϵ of the knot tube to be small. Each winding can then be approximated by a magnetic dipole of strength μ, at the position $\vec{r}\,'$, pointing in the direction of the tangent to the knot at $\vec{r}\,'$. In this limit, the knotted solenoid reduces to a collection of magnetic moments (magnets) which, for a trefoil knot, are aligned as shown in Fig. 1.

Each moment contributes a vector potential equal to $\frac{\vec{\mu}\times\vec{R}}{R^3}$ at the point $\vec{r}$, where $\vec{R} = \vec{r} - \vec{r}\,'$ and $R = |\vec{R}|$. The total vector potential is then obtained by integrating over $\vec{r}\,'$. Thus

$$\vec{A}(\vec{r}\,) = \int_K dl'\vec{m}(\vec{r}\,') \times \frac{\vec{R}}{R^3}, \tag{2.1}$$

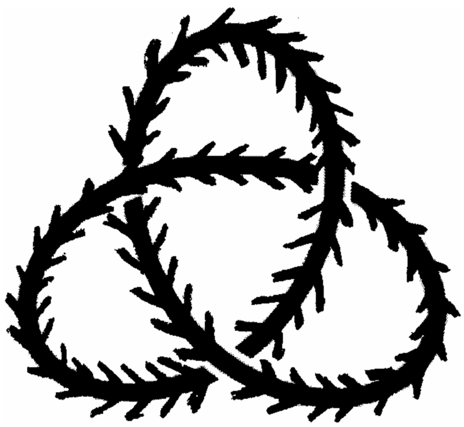

Fig. 1. Alignment of magnetic moments.

where $\vec{m}$ is the magnetic dipole moment density, i.e. magnetic dipole moment per unit length. This expression for the vector potential is identical to the expression for the magnetic field produced by a knotted wire carrying steady current, given by the Biot–Savart law, when $\vec{m}$ is replaced by current. The vector potential is given by an expression usually reserved for the magnetic field because the former (in the Coulomb gauge) satisfies the same equations as the latter in the magnetostatics limit in the region of interest *viz.* the source-free complement of the knot. Assuming that the knot tube is of uniform cross-sectional area, and defining the Hertz potential $\vec{H}$ by

$$\vec{H}(\vec{r}) = |\vec{m}| \int_K \frac{d\vec{l}'}{R}, \tag{2.2}$$

the expression for the flat connection (2.1) can be obtained by taking the curl of $\vec{H}$, $\vec{A} = \vec{\nabla} \times \vec{H}$. Equation (2.2) is deceptively simple since, although the integration is over a one-dimensional (filamentary) current, the nontrivial embedding of the knot makes the description manifestly three-dimensional. The key to evaluating this integral is to reduce the description to a one-dimensional integral. This simplification is easily effected for a class of knots called torus knots by using toroidal coordinates.

3. Flat Connections on Torus Knot Complements

A (p, q) torus knot can be obtained by considering a closed path that loops around one of the cycles of a putative torus p times, while looping around the other cycle q times, p, q being relatively prime integers. The toroidal coordinates are denoted by $0 \leq \eta < \infty, -\pi < \theta \leq \pi, 0 \leq \phi < 2\pi$. Given a toroidal surface of major radius R and minor radius d, we introduce a dimensional parameter a, defined by $a^2 = R^2 - d^2$, and a dimensionless parameter η_0, defined by $\eta_0 = \cosh^{-1}(R/d)$. The equation $\eta' =$ constant, say η_0, defines a toroidal surface. The combination R/d is called the aspect ratio. Clearly, larger η_0 corresponds to smaller thickness of the putative torus. Further, since we are interested in torus knots, we impose the constraint: $p\theta' + q\phi' = 0$, p and q being relatively prime integers. It follows that $\theta' \to \theta' + 2\pi q \Rightarrow \phi' \to \phi' - 2\pi p$, i.e. as we complete q cycles in the θ direction, we are forced to complete p cycles in the ϕ direction — as required. These constraints on the source coordinates effectively reduce the calculation to a one-dimensional problem.

It should be mentioned that in a different context, namely in the study of helical windings on a tokomak, this problem has been studied in great detail [3, 11]. For our purposes the results of [9] are more suitable, and we use them in what follows.

The toroidal coordinates are related to the usual Cartesian coordinates by the equations $x = \frac{a \sinh\eta \cos\phi}{(\cosh\eta - \cos\theta)}$, $y = \frac{a \sinh\eta \sin\phi}{(\cosh\eta - \cos\theta)}$, $z = \frac{a \sin\theta}{(\cosh\eta - \cos\theta)}$. The metric coefficients are $h_1 = h_2 = \frac{a}{(\cosh\eta - \cos\theta)}$, $h_3 = h_1 \sinh\eta$ and the volume element is $dV = \frac{a^3 \sinh\eta}{(\cosh\eta - \cos\theta)^3}$. These results are useful in expressing the Cartesian components of the Hertz potential in terms of the toroidal coordinates.

Likewise, the Green's functions, for $\eta' > \eta$ and $\eta' < \eta$ respectively, are readily expanded in toroidal harmonics as follows:

$$\begin{aligned}\frac{1}{R} &= \frac{1}{a\pi}\sqrt{(\cosh\eta' - \cos\theta')(\cosh\eta - \cos\theta)}\\ &\quad\times \sum_{m,n=0}^{\infty} \epsilon_m\epsilon_n(-1)^m \cos m(\phi - \phi')\cos n(\theta - \theta')\\ &\quad\times \begin{cases} P_{n-1/2}^{-m}(\cosh\eta)Q_{n-1/2}^{m}(\cosh\eta'),\\ P_{n-1/2}^{-m}(\cosh\eta')Q_{n-1/2}^{m}(\cosh\eta),\end{cases}\end{aligned} \tag{3.1}$$

where the Neumann factor ϵ_n is equal to 1 for $n = 0$ and 2 for $n \neq 0$, and $P_{n-1/2}^{-m}$ and $Q_{n-1/2}^{m}$ are generalized associated Legendre functions of the first and second kind with half-integral degree. Note that since $\eta' = \eta_0$ defines the putative torus on which the knot (of moments) winds, both $\eta > \eta'$ and $\eta < \eta'$ are coordinates in the complement of the knot. Hence both solutions are of interest. Substituting the above results in Eq. (2.2), the Cartesian components of the Hertz potential can be calculated by first expanding them in toroidal coordinates as follows: $H^i(\eta,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H_{nm}^i$ where

$$\begin{aligned}H_{nm}^i &= \sqrt{\cosh\eta - \cos\theta}\,D_{nm}Q_{n-1/2}^{m}(\cosh\eta)[\alpha_{nm}^i \cos m\phi\cos n\theta\\ &\quad + \beta_{nm}^i\cos m\phi\sin n\theta + \gamma_{nm}^i \sin m\phi\cos n\theta + \delta_{nm}^i \sin m\phi \sin n\theta],\end{aligned} \tag{3.2}$$

for $\eta > \eta'$ and $D_{nm} = \epsilon_n\epsilon_m(-1)^m/a\pi$. The coefficients α, β, γ, and δ are obtained by integrating over the source currents and hence contain the information about the knot. The expression for α_{nm}^i, for example, is given by

$$\alpha_{nm}^i = \int \frac{a^3 J_i}{(\cosh\eta' - \cos\theta')^{5/2}} P_{n-1/2}^{-m}(\cosh\eta')\sinh\eta'\cos m\phi'\cos n\theta' d\eta' d\theta' d\phi'. \tag{3.3}$$

The β_{nm}^i and γ_{nm}^i are obtained from α_{nm}^i by the changes $\cos n\theta' \to \sin n\theta'$ and $\cos m\phi' \to \sin m\phi'$, respectively. The δ_{nm}^i is obtained from α_{nm}^i by making both the changes mentioned above. The results for $\eta < \eta'$, are simply obtained by exchanging the roles of $P_{n-1/2}^{-m}$ and $Q_{n-1/2}^{m}$. The J_i stand for the Cartesian components of the current density and can be obtained from the corresponding toroidal components by a change of coordinates *viz.* $J_i = \sum_\alpha \gamma_{\alpha i}J_\alpha$. Here, $i = x, y, z; \alpha = \eta, \theta, \phi$ and $\gamma_{\alpha i} = \frac{1}{h_\alpha}\frac{\partial\xi_i}{\partial\xi_\alpha}$. The non-vanishing toroidal components of the current are given by

$$J_\phi \propto \cos\sigma\delta(\eta' - \eta_0)\delta(p\theta' + q\phi')(h_1h_2)^{-1} \tag{3.4}$$

and

$$J_\theta \propto \sin\sigma\delta(\eta' - \eta_0)\delta(p\theta' + q\phi')(h_1h_3)^{-1}. \tag{3.5}$$

In the above, σ is the pitch angle of the knot which winds around the torus, with respect to the azimuthal direction. The delta function in η' specifies the putative torus around which the knot (of moments) winds. The angular delta function enforces the knot constraint $p\theta' + q\phi' = 0$. The irksome denominator in (3.3) can be tamed by using the identity [7]

$$[\cosh\eta - \cos\theta]^{-\frac{1}{2}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\epsilon_n Q_{n-1/2}(\cosh\eta)\cos n\theta. \tag{3.6}$$

Substituting the above results in (3.3) and performing the integrals gives, for the x-component of α,

$$\begin{aligned}\alpha^x_{nm} = &\sqrt{\frac{2}{1+\Lambda_0^2}}\left(\frac{a}{\pi}\right)\sinh\eta_0 P^{-m}_{n-1/2}(\cosh\eta_0)\\ &\times\sum_{r=0}^{\infty}\epsilon_r[(-2\Lambda_0)Q'_{r-1/2}(\cosh\eta_0)\mathcal{I}^{(\alpha)x}_{rmn}(p,q) + Q_{r-1/2}(\cosh\eta_0)\mathcal{J}^{(\alpha)x}_{rmn}(p,q)],\end{aligned} \tag{3.7}$$

where $\Lambda_0 = \tan\sigma = -\frac{(q/p)}{\sinh\eta_0}$, and the prime on Q denotes a derivative of Q with respect to η_0.

The $\mathcal{I}^{(\alpha)x}$ and $\mathcal{J}^{(\alpha)x}$ are given by simple integrals over elementary trigonometric functions, and are easily evaluated. Equation (3.7) holds also for α^i_{nm} with corresponding integrals $\mathcal{I}^{(\alpha)i}$ and $\mathcal{J}^{(\alpha)i}$. For the z-component, α^z_{nm}, however, the $\mathcal{I}$ and $\mathcal{J}$ integrals pick up the following extra multiplicative factors, respectively.

$$\mathcal{I} \to \sinh\eta_0\mathcal{I} \quad \text{and} \quad \mathcal{J} \to \Lambda_0\coth\eta_0\mathcal{J}. \tag{3.8}$$

The expressions for the $\mathcal{I}$ and $\mathcal{J}$ integrals for the other cases *viz.* β, γ, δ can be worked out similarly.

In evaluating the above integrals, it is useful to let $\lambda = -q/p$, and define

$$\begin{aligned} a_1 &= \lambda(1-n+r), \quad & a_2 &= \lambda(1+n-r),\\ a_3 &= \lambda(1-n-r), \quad & a_4 &= \lambda(1+n+r)\end{aligned} \tag{3.9}$$

and

$$b_1 = \lambda(n-r), \quad b_2 = \lambda(n+r). \tag{3.10}$$

With these definitions, the values of the integrals for the various cases are given below.

$$\mathcal{I}^x_\alpha = \frac{1}{16}\sum_{i=1}^{4} f(a_i)(\cos 2\pi a_i - 1), \tag{3.11}$$

$$\mathcal{I}^y_\alpha = \frac{1}{16}\sum_{i=1}^{4} g(a_i)\sin 2\pi a_i, \tag{3.12}$$

$$\mathcal{I}_\alpha^z = -\frac{1}{4}\sum_{i=1}^{2} h(b_i)\sin 2\pi b_i, \tag{3.13}$$

$$\mathcal{J}_\alpha^x = \frac{1}{8}\sum_{i=1}^{2} g(b_i)(\cos 2\pi b_i - 1), \tag{3.14}$$

$$\mathcal{J}_\alpha^y = \frac{1}{8}\sum_{i=1}^{2} f(b_i)\sin 2\pi b_i, \tag{3.15}$$

$$\mathcal{J}_\alpha^z = \frac{1}{4}\sum_{i=1}^{2} h(b_i)\sin 2\pi b_i, \tag{3.16}$$

$$\mathcal{I}_\beta^x = \frac{1}{16}\sum_{i=1}^{4} f(a_i)\sin 2\pi a_i, \tag{3.17}$$

$$\mathcal{I}_\beta^y = -\frac{1}{16}\sum_{i=1}^{4} g(a_i)(\cos 2\pi a_i - 1), \tag{3.18}$$

$$\mathcal{I}_\beta^z = \frac{1}{4}\sum_{i=1}^{2} h(b_i)(\cos 2\pi b_i - 1), \tag{3.19}$$

$$\mathcal{J}_\beta^x = \frac{1}{8}\sum_{i=1}^{2} g(b_i)\sin 2\pi b_i, \tag{3.20}$$

$$\mathcal{J}_\beta^y = -\frac{1}{8}\sum_{i=1}^{2} f(b_i)(\cos 2\pi b_i - 1), \tag{3.21}$$

$$\mathcal{J}_\beta^z = -\frac{1}{4}\sum_{i=1}^{2} h(b_i)(\cos 2\pi b_i - 1), \tag{3.22}$$

$$\mathcal{I}_\gamma^x = \frac{1}{16}\sum_{i=1}^{4} k(a_i)\sin 2\pi a_i, \tag{3.23}$$

$$\mathcal{I}_\gamma^y = \frac{1}{16}\sum_{i=1}^{4} l(a_i)(\cos 2\pi a_i - 1), \tag{3.24}$$

$$\mathcal{I}_\gamma^z = \frac{1}{4}\sum_{i=1}^{2} \tilde{h}(b_i)(\cos 2\pi b_i - 1), \tag{3.25}$$

$$\mathcal{J}_\gamma^x = -\frac{1}{8}\sum_{i=1}^{2} l(b_i)\sin 2\pi b_i, \tag{3.26}$$

$$\mathcal{J}_\gamma^y = -\frac{1}{8}\sum_{i=1}^{2} k(b_i)(\cos 2\pi b_i - 1), \tag{3.27}$$

$$\mathcal{J}_\gamma^z = -\frac{1}{4}\sum_{i=1}^{2} \tilde{h}(b_i)(\cos 2\pi b_i - 1), \tag{3.28}$$

$$\mathcal{I}_\delta^x = -\frac{1}{16}\sum_{i=1}^{4} k(a_i)(\cos 2\pi a_i - 1), \tag{3.29}$$

$$\mathcal{I}_\delta^y = \frac{1}{16}\sum_{i=1}^{4} l(a_i)\sin 2\pi a_i, \tag{3.30}$$

$$\mathcal{I}_\delta^z = \frac{1}{4}\sum_{i=1}^{2} \tilde{h}(b_i)\sin 2\pi b_i, \tag{3.31}$$

$$\mathcal{J}_\delta^x = \frac{1}{8}\sum_{i=1}^{4} l(a_i)(\cos 2\pi b_i - 1), \tag{3.32}$$

$$\mathcal{J}_\delta^y = -\frac{1}{8}\sum_{i=1}^{2} k(b_i)\sin 2\pi b_i, \tag{3.33}$$

$$\mathcal{J}_\delta^z = -\frac{1}{4}\sum_{i=1}^{2} \tilde{h}(b_i)\sin 2\pi b_i. \tag{3.34}$$

In the above equations $f(x), h(x), l(x)$ are odd functions and $g(x), \tilde{h}(x)$ and $k(x)$ are even functions defined by the following combinations

$$f(x) = \frac{1}{x+m+1} + \frac{1}{x+m-1} + \frac{1}{x-m+1} + \frac{1}{x-m-1}, \tag{3.35}$$

$$g(x) = \frac{1}{1-x-m} + \frac{1}{1+x+m} + \frac{1}{1+x-m} + \frac{1}{1-x+m}, \tag{3.36}$$

$$h(x) = \frac{1}{x+m} + \frac{1}{x-m}, \quad \tilde{h}(x) = \frac{1}{x+m} - \frac{1}{x-m}, \tag{3.37}$$

$$k(x) = \frac{1}{m+x+1} + \frac{1}{m-x+1} + \frac{1}{m+x-1} + \frac{1}{m-x-1}, \tag{3.38}$$

$$l(x) = \frac{1}{m+1-x} + \frac{1}{m-1+x} + \frac{1}{-m+x+1} + \frac{1}{-m-x-1}. \tag{3.39}$$

This completes the derivation of the Hertz potential (2.2) for an arbitrary torus knot.

To develop some insight into the result, we now specialize to the case of a trefoil knot. As mentioned earlier, a trefoil is a (2, 3) torus knot. Substituting $\lambda = -p/q = -3/2$ in the expressions (3.9)–(3.34), the coefficients $\alpha, \beta, \gamma, \delta$ can be evaluated explicitly and we find that only two of them are non-zero for each component. Plugging the results into Eq. (2.2), the Hertz potential is found to have the following form:

$$\begin{aligned} H^x &= \sum_{m,n=0}^{\infty} H^x_{nm} \\ &= \sqrt{\cosh\eta - \cos\theta} \sum_{m,n=0}^{\infty} D_{nm} Q^m_{n-1/2}(\cosh\eta) \\ &\quad \times [X_{nm}(\eta_0)\cos m\phi \cos n\theta + \tilde{X}_{nm}(\eta_0)\sin m\phi \sin n\theta], \end{aligned} \tag{3.40}$$

$$\begin{aligned} H^y &= \sum_{m,n=0}^{\infty} H^y_{nm} \\ &= \sqrt{\cosh\eta - \cos\theta} \sum_{m,n=0}^{\infty} D_{nm} Q^m_{n-1/2}(\cosh\eta) \\ &\quad \times [Y_{nm}(\eta_0)\cos m\phi \sin n\theta + \tilde{Y}_{nm}(\eta_0)\sin m\phi \cos n\theta], \end{aligned} \tag{3.41}$$

$$\begin{aligned} H^z &= \sum_{m,n=0}^{\infty} H^z_{nm} \\ &= \sqrt{\cosh\eta - \cos\theta} \sum_{m,n=0}^{\infty} D_{nm} Q^m_{n-1/2}(\cosh\eta) \\ &\quad \times [Z_{nm}(\eta_0)\cos m\phi \sin n\theta + \tilde{Z}_{nm}(\eta_0)\sin m\phi \cos n\theta] \end{aligned} \tag{3.42}$$

for $\eta > \eta'$ and $D_{nm} = \epsilon_n \epsilon_m (-1)^m / a\pi$. The coveted expression for the flat connection can then be obtained by taking the curl of the Hertz potential given by the following generic expressions for the curl of a vector field in toroidal coordinates.

$$\begin{aligned} A_\eta &= \frac{(\cosh\eta - \cos\theta)}{a}\left[-\sin\phi \frac{\partial H_x}{\partial\theta} + \cos\phi \frac{\partial H_y}{\partial\theta}\right] \\ &\quad + \frac{(\sin\theta)}{a}\left[\cos\phi \frac{\partial H_x}{\partial\phi} + \sin\phi \frac{\partial H_y}{\partial\phi}\right] + \frac{(1 - \cosh\eta\cos\theta)}{a\sinh\eta}\frac{\partial H_z}{\partial\phi}, \end{aligned} \tag{3.43}$$

$$\begin{aligned} A_\theta &= \frac{(1 - \cosh\eta\cos\theta)}{a\sinh\eta}\left[\cos\phi \frac{\partial H_x}{\partial\phi} + \sin\phi \frac{\partial H_y}{\partial\phi}\right] - \frac{(\sin\theta)}{a}\frac{\partial H_z}{\partial\phi} \\ &\quad - \frac{(\cosh\eta - \cos\theta)}{a}\left[-\sin\phi \frac{\partial H_x}{\partial\eta} + \cos\phi \frac{\partial H_y}{\partial\eta}\right], \end{aligned} \tag{3.44}$$

$$\begin{aligned} A_\phi &= -\frac{(1 - \cosh\eta\cos\theta)}{a}\left[\cos\phi \frac{\partial H_x}{\partial\theta} + \sin\phi \frac{\partial H_y}{\partial\theta} + \frac{\partial H_z}{\partial\eta}\right] \\ &\quad - \frac{(\sinh\eta\sin\theta)}{a}\left[\cos\phi \frac{\partial H_x}{\partial\eta} + \sin\phi \frac{\partial H_y}{\partial\eta} - \frac{\partial H_z}{\partial\theta}\right]. \end{aligned} \tag{3.45}$$

4. Summary and Outlook

To summarize, we have derived an exact expression for a flat connection on the complement of a torus knot. The derivation relies on successfully mapping the mathematical problem into a simple physics problem in magnetostatics. We conclude by noting that the results for the Hertz potential and the flat connection can be taken over to represent the vector potential and the magnetic field respectively, produced by a knotted wire of the same size and shape carrying steady current.

A few other problems — some readily doable, and some harder — naturally come to mind. First, it may be of interest in engineering, for some special purposes, to design knotted antennae [13]. This would require going beyond the magnetostatic limit discussed in this paper to time-dependent situations. Second, it would be interesting to study multipole expansions of knot currents in general and, in particular, construct the generalization of the toroidal moment (anapole) for knotted solenoids. Third, it would be of considerable mathematical interest to work out analogous results on the complement of a figure-eight knot (which is not a torus knot). This is an example of a three-dimensional hyperbolic space and plays an important role in Thurston's geometrization programme [12]. Next, some effort needs to be devoted toward generalizing the ideas to the nonabelian case. Finally, it would be of considerable interest to study the diffraction and scattering effects of knotted solenoids on electrons, both theoretically and experimentally; thus generalizing the work initiated by Ehrenberg, Siday, Aharonov and Bohm. I hope to return to these issues elsewhere.

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