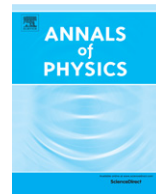




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# The classical and quantum mechanics of a particle on a knot

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## ABSTRACT

A free particle is constrained to move on a knot obtained by winding around a putative torus. The classical equations of motion for this system are solved in a closed form. The exact energy eigenspectrum, in the thin torus limit, is obtained by mapping the time-independent Schrödinger equation to the Mathieu equation. In the general case, the eigenvalue problem is described by the Hill equation. Finite-thickness corrections are incorporated perturbatively by truncating the Hill equation. Comparisons and contrasts between this problem and the well-studied problem of a particle on a circle (planar rigid rotor) are performed throughout.

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## 1. Introduction

The example of a particle constrained to move along a circle – the so-called planar rigid rotor – is one of the simplest problems that is discussed in text-books of quantum mechanics. The beguiling simplicity of this problem is at the heart of many non-trivial ideas that pervade modern physics. For understanding many issues like the existence of inequivalent quantisations of a given classical system [1], the role of topology in the definition of the vacuum state in gauge theories [2], band structure of solids [3], generalised spin and statistics of the anyonic type [4], and the study of mathematically interesting algebras of quantum observables on spaces with non-trivial topology [5], the problem of a particle on a circle serves as a toy model.

In this paper, we consider the problem of a particle constrained to move on a torus knot. Besides adding a new twist to the aforementioned problems, the present system can be thought of as a

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double-rotor (analogous to the double-pendulum, but without the gravitational field) which is a genuine non-planar generalisation of the planar rotor.

The paper is organised as follows: In the next section we introduce toroidal coordinates in terms of which the constraints which restrict the motion of the particle to the torus knot are most naturally incorporated. As a warm-up, we then analyse the particle on a circle in toroidal coordinates. This prelude allows us to compare and contrast the results of the subsequent sections with the well-known results for the particle on a circle. The following two sections deal with the classical and quantum mechanics of a particle on a torus knot. In the penultimate section we briefly touch upon the possibility of inequivalent quantisations of the particle on a knot. These will be labelled by two parameters, in contrast to the particle on a circle. The concluding section summarises and presents an outlook.

## 2. Toroidal coordinates

The toroidal coordinates [6] are denoted by  $0 \leq \eta < \infty$ ,  $-\pi < \theta \leq \pi$ ,  $0 \leq \phi < 2\pi$ . Given a toroidal surface of major radius  $R$  and minor radius  $d$ , we introduce a dimensional parameter  $a$ , defined by  $a^2 = R^2 - d^2$ , and a dimensionless parameter  $\eta_0$ , defined by  $\eta_0 = \cosh^{-1}(R/d)$ . The equation  $\eta = \text{constant}$ , say  $\eta_0$ , defines a toroidal surface. The combination  $R/d$  is called the aspect ratio. Clearly, larger  $\eta_0$  corresponds to smaller thickness of the torus. In the limit  $\eta_0 \rightarrow \infty$ , the torus degenerates into a limit circle.

The toroidal coordinates are related to the usual Cartesian coordinates by the equations

$$x = \frac{a \sinh \eta \cos \phi}{(\cosh \eta - \cos \theta)}, \quad y = \frac{a \sinh \eta \sin \phi}{(\cosh \eta - \cos \theta)}, \quad z = \frac{a \sin \theta}{(\cosh \eta - \cos \theta)}. \tag{1}$$

The metric coefficients are given by the equations

$$h_1 = h_2 = \frac{a}{(\cosh \eta - \cos \theta)}, \quad h_3 = h_1 \sinh \eta \tag{2}$$

and the volume element is

$$dV = \frac{a^3 \sinh \eta}{(\cosh \eta - \cos \theta)^3} d\eta d\theta d\phi. \tag{3}$$

With the help of these basic relations, it is straightforward to rewrite well-known Cartesian expressions in toroidal coordinates.

### 2.1. A particle constrained to move on a circle

The Lagrangian for a free particle of mass  $m$  in Cartesian coordinates  $(x, y, z)$  is

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{4}$$

In the above expression, and henceforth, an overdot refers to a time derivative. After some algebra, this expression can be rewritten in toroidal coordinates as

$$L = \frac{m}{2} a^2 \frac{(\dot{\eta}^2 + \dot{\theta}^2 + \sinh^2 \eta \dot{\phi}^2)}{(\cosh \eta - \cos \theta)^2}. \tag{5}$$

To restrict the motion of the particle to lie on a circle in the  $xy$  plane, we impose the constraints  $\eta = \eta_0$ , a constant, and  $\theta = \theta_0$ , another constant. The Lagrangian then takes the form

$$L = \frac{ma^2}{2} \frac{\sinh^2 \eta_0 \dot{\phi}^2}{(\cosh \eta_0 - \cos \theta_0)^2}. \tag{6}$$

The Euler–Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \tag{7}$$

then yields, as expected,

$$\ddot{\phi} = 0 \Rightarrow \phi(t) = \omega t + \phi_0 \tag{8}$$

where  $\omega$  is a constant and has the physical interpretation of frequency, and  $\phi_0$  is a constant of integration which specifies the position of the particle on the circle at time  $t = 0$ —similar to plane polar coordinates.

Defining a rescaled mass  $M = m \frac{\sinh^2 \eta_0}{(\cosh \eta_0 - \cos \theta_0)^2}$ , we get the Hamiltonian  $H = \frac{p_\phi^2}{2Ma^2}$  with the momentum canonically conjugate to  $\phi$  being given by  $p_\phi = Ma^2 \dot{\phi}$  as usual. Using this to set up the Schrödinger equation and solving it, we get, for the eigenvalues and the normalised eigenfunctions respectively,

$$E_n = \frac{n^2 \hbar^2}{2Ma^2}, \quad \psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{\pm in\phi} \quad n = 0, 1, 2, \dots \tag{9}$$

For large  $\eta_0$ , the thickness of the putative torus decreases and  $M \rightarrow m$ : we approach the well-known expressions in plane polar coordinates.

Interestingly, it is also possible to get a particle to move on a circle by imposing the constraints  $\eta = \eta_0$ , a constant, and  $\phi = \phi_0$ , another constant. This however results in a more complicated Lagrangian *viz.*

$$L = \frac{ma^2}{2} \frac{\dot{\theta}^2}{(\cosh \eta_0 - \cos \theta)^2}. \tag{10}$$

The resulting Euler–Lagrange equation is

$$\ddot{\theta}(\cosh \eta_0 - \cos \theta) = -\sin \theta \dot{\theta}^2 \tag{11}$$

which can be re-written as

$$\frac{d}{dt} [\dot{\theta}(\cosh \eta_0 - \cos \theta)] = 0$$

and readily integrated to yield

$$\dot{\theta}(\cosh \eta_0 - \cos \theta) = \kappa \tag{12}$$

$\kappa$  being a constant. Thus the solution is reduced to quadratures. Thanks to the presence of the factor  $(\cosh \eta - \cos \theta)$ , the solution is not as simple as the one in plane polar coordinates. The Hamiltonian can be obtained in a straightforward manner and is given by

$$H = \frac{p_\theta^2}{2ma^2} (\cosh \eta_0 - \cos \theta)^2. \tag{13}$$

The presence of the  $\theta$ -dependent multiplicative factor is portentous of additional complications that arise when we make a transition to quantum mechanics. In particular, the fact that the conjugate operators  $p_\theta$  and  $\theta$  do not commute requires us to perform an operator-ordering of the classical Hamiltonian.

The above analysis shows that while toroidal coordinates are ideally suited to consider the motion of a particle on a circle in the  $xy$ -plane, they are more cumbersome when it comes to handling paths which stray from the  $xy$ -plane. Since a knot is intrinsically non-planar, we should be prepared to confront the attendant complications. It should be mentioned, however, that these complications would also be present in other coordinate systems. We choose to work with toroidal coordinates because of their suitability in imposing the constraints that define a torus knot.

### 3. Classical mechanics of a particle on a knot

As already mentioned, the constraint  $\eta = \eta_0$  defines a toroidal surface. A  $(p, q)$  torus knot can be obtained by considering a closed path that loops  $p$  times around one of the cycles of a torus while looping around the other cycle  $q$  times,  $p, q$  being relatively prime integers. The desired property can be enforced by imposing the constraint:  $p\theta + q\phi = 0$ . It is easy to check that  $\theta \rightarrow \theta + 2\pi q \Rightarrow \phi \rightarrow \phi - 2\pi p$  i.e. as we complete  $q$  cycles in the  $\theta$  direction, we are forced to complete  $p$  cycles in the  $\phi$  direction—as required. Imposing the above two constraints on Eq. (5), we get the Lagrangian for a particle on a torus knot to be

$$L = \frac{M}{2} f(\phi) \dot{\phi}^2 \tag{14}$$

where

$$f(\phi) = \frac{a^2}{(\gamma - \cos \alpha \phi)^2} \quad \text{and} \quad M = m(\alpha^2 + \beta^2) \tag{15}$$

with

$$\alpha = -q/p, \quad \beta = \sinh \eta_0, \quad \gamma = \cosh \eta_0. \tag{16}$$

The main difference between the Lagrangian in (14) and the one for a particle on a circle viz. Eq. (6), lies in the appearance of the  $\phi$ -dependent factor  $f(\phi)$  in the Lagrangian which contains the information about the non-trivial embedding of the knot in three dimensions.

The Euler–Lagrange equation is given by

$$f(\phi) \ddot{\phi} + \frac{1}{2} f'(\phi) \dot{\phi}^2 = 0 \tag{17}$$

where the prime denotes a derivative of the function  $f$  with respect to its argument  $\phi$ . Now, using the above equation of motion, it is straightforward to show that

$$\frac{d}{dt} [\sqrt{f} \dot{\phi}] = 0 \Rightarrow \sqrt{f} \dot{\phi} = \mathcal{A} \Rightarrow \dot{\phi} = \frac{\mathcal{A}}{a} (\gamma - \cos \alpha \phi) \tag{18}$$

where  $\mathcal{A}$  is a constant. Noting that  $(1 - \gamma^2) < 0$ , the latter equation can be integrated to get

$$\phi(t) = \frac{1}{\alpha} \tan^{-1} \left[ \sqrt{\frac{\gamma - 1}{\gamma + 1}} \tan \left( \frac{\mathcal{A} \alpha \beta t}{2a} \right) \right]. \tag{19}$$

In the limit  $\eta_0 \rightarrow \infty$ , the right hand side is linear in  $t$ , as expected for a particle on a circle.

The momentum  $p_\phi$  canonically conjugate to  $\phi$  and the Hamiltonian  $H$  can be easily worked out and are given by the following expressions

$$p_\phi = Mf(\phi) \dot{\phi}, \quad H = \frac{p_\phi^2}{2Mf(\phi)}. \tag{20}$$

### 4. Quantum mechanics of a particle on a knot

In principle, once a Hamiltonian is given, it is a straightforward exercise to write down the Schrödinger equation. In the present case, the classical Hamiltonian involves a term which mixes the coordinate and the canonically conjugate momentum. Since these canonical pairs will be elevated to the level of operators in the quantum theory, we need to prescribe an ordering for the operator products. We choose the so-called Weyl ordering which symmetrises the product as follows:

$$H = \frac{1}{6M} \left[ \frac{1}{f} p_\phi^2 + p_\phi \frac{1}{f} p_\phi + p_\phi^2 \frac{1}{f} \right]. \tag{21}$$

In the above equation, and in what follows, we refrain from putting hats, but it should be remembered that both  $\phi$  and  $p_\phi$  are operators which obey the canonical commutation relations viz.  $[\phi, p_\phi]_- = i\hbar$ . Pulling all the momentum terms to the extreme right in preparation to make them act on a wavefunction, we get

$$H = \frac{1}{2M} \left[ \left( \frac{1}{f} \right) p^2 - (i\hbar) \left( \frac{1}{f} \right)' p - \frac{\hbar^2}{3} \left( \frac{1}{f} \right)'' \right]. \tag{22}$$

A few comments regarding the above Hamiltonian are in order here. First, as was first pointed out by Podolsky [7], a ‘natural’ candidate for the Hamiltonian of a free particle in arbitrary curvilinear coordinates is obtained by replacing the usual Laplace operator in Cartesian coordinates by the so-called Laplace–Beltrami operator viz.  $\frac{1}{\sqrt{g(x)}} \partial_a \sqrt{g(x)} g^{ab}(x) \partial_b$ . At the quantum level, this operator is self-adjoint in the weighted Hilbert space  $L_2(d\mu, \mathbf{R})$  with measure  $d\mu = \sqrt{g} dx$ . From (14), it is obvious that, in the present one-dimensional problem, the role of  $\sqrt{g}$  is played by  $[f(\phi)]^{-1}$ . A straightforward calculation for the Laplace–Beltrami operator using this metric yields the first two terms of the Hamiltonian in (22), but not the third term. This term has its origin in the choice of Weyl ordering, is higher order in  $\hbar$ , but more importantly, is independent of the momentum operator  $p$  and acts multiplicatively on coordinate wavefunctions. Hence, the full Hamiltonian (22) continues to be self-adjoint in the weighted Hilbert space  $L_2(d\mu, \phi)$  where  $d\mu = [f(\phi)]^{-1} d\phi$ . We make this explicit at the end of this section by presenting the inner product on the Hilbert space obtained by using the solutions of the Schrödinger equation.

Second, the correction term obtained by Weyl ordering above is very similar to the one obtained by Ogawa, Fujii, and Kobushukin [8] in a slightly different context. They showed that an additional term of order  $\hbar^2$  is needed to describe the Hamiltonian of a particle restricted to move on an  $(N - 1)$ -dimensional hyperspace of an  $N$ -dimensional Euclidean space and interpreted it as the energy associated with the fluctuations of the unit vector normal to the hypersurface.

It is possible to make the connection with [8] more concrete. For this we note that a knot is not just ‘topologically interesting’—it also has rich geometrical properties. These are encoded in the Frenet–Serret equations which describe how a triad of orthogonal unit vectors is transported along the curve in terms of the geometrical properties like the curvature and torsion of the curve. For given parametrisation of a knot, the unit tangent vector at a point is uniquely defined by taking the derivative of the coordinate with respect to the length of the curve. The principal normal at a point, obtained by taking the second derivative of the position vector, however, lies in a plane perpendicular to the tangent vector and is not uniquely defined. Once a principal normal is picked, the binormal—obtained by taking the cross product of the tangent and the principal normal is uniquely determined. One way of picking a principal normal at each point of a knot is to choose a knot different from the original one and construct a two-dimensional strip which is bounded by the two knots under consideration. This so-called ‘framed knot’ defines a two-dimensional hypersurface in three-dimensional Euclidean space for which the results of [8] can be applied. We hope to return to this interesting issue elsewhere.

Third, for the sake of completion, we stress that the corrections discussed above depend only on the embedding of the hypersurface in Euclidean space, and their origin is very different from the correction to the Podolsky result that was obtained by DeWitt [9] in formulating quantum mechanics over a general curved space. The latter correction is also of order  $\hbar^2$ , is proportional to the curvature of the space, and is identically zero in cases where the particle moves in Euclidean space.

With the foregoing comments, we can now return to the Hamiltonian (22) which is tailor-made for constructing the time-independent Schrödinger equation, with the usual prescription for replacing the momentum by the corresponding differential operator. The resulting Schrödinger equation is

$$\left[ -\frac{\hbar^2}{2M} \left( \frac{1}{f} \right) \frac{d^2}{d\phi^2} - \frac{\hbar^2}{2M} \left( \frac{1}{f} \right)' \frac{d}{d\phi} - \frac{\hbar^2}{6M} \left( \frac{1}{f} \right)'' \right] \psi = E\psi. \tag{23}$$

The first derivative in  $\phi$  can be eliminated by the well-known trick of substituting  $\psi = \chi \Sigma$  in the above equation and getting rid of terms proportional to  $d\Sigma/d\phi$  by choosing  $\chi$  appropriately. This yields for  $\chi$ ,

$$\chi \propto \sqrt{f}. \tag{24}$$

The inner product on the weighted Hilbert space  $L_2(d\mu, \phi)$ , with measure  $d\mu = [f(\phi)]^{-1}d\phi$  is

$$\langle \psi_1 | \psi_2 \rangle = \int [f(\phi)]^{-1}d\phi \psi_1^*(\phi)\psi_2(\phi) = \int \Sigma_1^*(\phi)\Sigma_2(\phi)d\phi. \tag{25}$$

The latter equality follows from the factorisation of  $\psi$  into  $\chi = \sqrt{f}$  and  $\Sigma$ . This implies that the solutions of the equation for  $\Sigma$  are orthogonal functions whose inner product is defined by the usual Lebesgue measure. Indeed, all the explicit solutions we obtain in the following subsections, have this property. In particular, it is straightforward to check that  $\langle \psi_1 | H\psi_2 \rangle = \langle H\psi_1 | \psi_2 \rangle$  for the Hamiltonian in (22), establishing its self-adjointness on  $L_2(d\mu, \phi)$ .

Substituting  $\chi \propto \sqrt{f}$  in (23) and collecting the remaining terms, the Schrödinger equation reduces to the following equation for  $\Sigma$

$$\left[ \frac{d^2}{d\phi^2} + V(\phi) \right] \Sigma = 0 \tag{26}$$

where the ‘potential’  $V$  is defined by

$$V(\phi) = \left[ \frac{2f''f - f'^2}{12f^2} + \frac{2MEf}{\hbar^2} \right]. \tag{27}$$

$V$  is an even function of  $\phi$ . Substituting for  $f$  from (15), we get after some algebra,

$$V = \frac{2MEa^2/\hbar^2 + \alpha^2/2 - \alpha^2\gamma \cos \alpha\phi/3 - \alpha^2 \cos 2\alpha\phi/6}{(\gamma - \cos \alpha\phi)^2}. \tag{28}$$

Since  $V(\phi)$  is a periodic function, the above potential can be expanded in a Fourier series and Eq. (26) gets identified with the Hill differential equation [10].

#### 4.1. The thin-torus approximation

As already mentioned, large values of  $\eta_0$  and hence large values of  $\gamma$ , correspond to a thin torus around which the particle’s trajectory winds. In this limit, we can restrict to terms of the order of  $1/\gamma$ . Then  $\beta^2 \sim \gamma^2$ , hence  $M \rightarrow m$ , and Eq. (28) simplifies to

$$V = \frac{\alpha^2}{4}\lambda - \frac{\alpha^2}{3\gamma} \cos \alpha\phi \tag{29}$$

where

$$\lambda = \frac{8mEa^2}{\hbar^2 \alpha^2}. \tag{30}$$

Eq. (26) now takes the form

$$\left[ \frac{d^2}{d\phi^2} + \frac{\alpha^2}{4}\lambda - \frac{\alpha^2}{3\gamma} \cos \alpha\phi \right] \Sigma = 0. \tag{31}$$

Changing variables such that  $\alpha\phi = 2z$ , the above equation becomes

$$\left[ \frac{d^2}{dz^2} + \lambda - \frac{4}{3\gamma} \cos 2z \right] \Sigma = 0 \tag{32}$$

which is immediately recognised to be the well-known Mathieu equation [11].

The solutions  $\Sigma$  of the Mathieu equation with the required periodicity are given by the Mathieu functions of fractional order  $\nu$  viz.

$$ce_\nu(z, \gamma) = \cos \nu z - \frac{1}{6\gamma} \left[ \frac{\cos(\nu + 2)z}{(\nu + 1)} - \frac{\cos(\nu - 2)z}{(\nu - 1)} \right] \dots \tag{33}$$

$$se_\nu(z, \gamma) = \sin \nu z - \frac{1}{6\gamma} \left[ \frac{\sin(\nu + 2)z}{(\nu + 1)} - \frac{\sin(\nu - 2)z}{(\nu - 1)} \right] \dots \tag{34}$$

which satisfy the usual inner product rules with the standard Lebesgue measure [11]. The complete solution with two arbitrary coefficients  $A$  and  $B$  is given by

$$\Sigma = Ase_\nu(z, \gamma) + Bce_\nu(z, \gamma). \tag{35}$$

Setting  $\nu = \frac{2n}{q}$  where  $n$  is an integer, we see that the above functions have a periodicity  $q\pi$  in  $z$ , which translates into the required periodicity  $2p\pi$  in  $\phi$ .

The condition relating  $\lambda$  to  $\nu$  is given by

$$\lambda = \nu^2 + \boxed{\frac{2}{9\gamma^2(\nu^2 - 1)}} \dots \tag{36}$$

Since we are restricting our attention to  $1/\gamma$  order, the boxed terms can be neglected. The allowed values of  $\lambda$  follow by setting  $\nu = \frac{2n}{q}$ . These values of  $\lambda$ , in conjunction with Eq. (30), determine the energy eigenvalues to be

$$E_n = \frac{n^2 \hbar^2 \alpha^2}{2ma^2q^2}. \tag{37}$$

Before proceeding further, it is worth recalling that the complete solution of Eq. (23) that we are trying to solve is given by  $\psi = \chi \Sigma$  with  $\chi \propto \sqrt{f}$ . The complete solutions for the (un-normalised) eigenfunctions  $\psi$  with the correct boundary conditions are therefore given by

$$\psi_+^{(n)}(\phi) = \frac{a}{(\gamma - \boxed{\cos\alpha\phi})} \tag{38}$$

$$\times \left\{ \cos(n\alpha\phi/q) \boxed{-\frac{1}{6\gamma} \left[ \frac{\cos((n+q)\alpha\phi/q)}{(2n/q+1)} - \frac{\cos((n-q)\alpha\phi/q)}{(2n/q-1)} \right] \dots} \right\} \tag{38}$$

$$\psi_-^{(n)}(\phi) = \frac{a}{(\gamma - \boxed{\sin\alpha\phi})} \tag{39}$$

$$\times \left\{ \sin(n\alpha\phi/q) \boxed{-\frac{1}{6\gamma} \left[ \frac{\sin((n+q)\alpha\phi/q)}{(2n/q+1)} - \frac{\sin((n-q)\alpha\phi/q)}{(2n/q-1)} \right] \dots} \right\} \tag{39}$$

where we have used  $2z = \alpha\phi$ . Further, since we retain only terms of order  $1/\gamma$ , the boxed terms in Eqs. (38) and (39) can be neglected.

In passing, we mention that the two independent solutions (33) and (34) can be combined into a single equation given by [11,12]

$$\Sigma = e^{i\nu z} u \tag{40}$$

where

$$u = \sin(z - \sigma) + a_3 \cos(3z - \sigma) + b_3 \sin(3z - \sigma) + a_5 \cos(5z - \sigma) + b_5 \sin(5z - \sigma) + \dots \tag{41}$$

where  $\sigma$  is a new parameter such that  $\sigma = \pi/2$  yields the solution (33) and  $\sigma = 0$  yields the solution (34). In the above, the coefficients  $a, b$  are determined in terms of  $\gamma$  and  $\sigma$ . To order  $1/\gamma$  that we are

interested in, only  $b_3 = -\frac{1}{12\gamma}$ , is non-zero. This succinct way of writing the general solution will be particularly useful in incorporating finite-thickness corrections.

It may be noted that for  $q = 1$ ,  $p = -1$ , and hence  $\alpha = -1$ , the above results (36)–(39) reduce to the well-known results for a particle on a circle. For a (2,3) torus knot, namely, the trefoil,  $p = 2$ , and the eigenfunctions have a period  $4\pi$ . The general solutions of Mathieu equations with a period  $4\pi$  were first worked out by Lars Onsager in 1935 in his dissertation for a Ph.D. at Yale [13]. While it is gratifying to note this, it is also a little disappointing. The energy levels and energy eigenfunctions are the same as that of a particle on a circle, except for the factor of  $\alpha$ . This is a consequence of the fact that, in the weak coupling limit (large  $\gamma$ ), the putative torus degenerates into a limit circle, with the attendant vagueness associated with the winding in the  $\theta$  direction. Correspondingly, the Mathieu functions degenerate into trigonometric functions. It may be tempting to think that the general solution (for arbitrary  $\gamma$ ) will be given by Mathieu functions, with the boxed expressions in Eqs. (36) and (38)–(39) being the next order corrections. The story, however, is slightly more complicated. Our penchant for boxing negligible pieces relates to this fact.

#### 4.2. The slightly-thick-torus correction

To the next order in  $1/\gamma$ , the correct expression for the potential is obtained by starting with Eq. (28), making a binomial expansion of the denominator, and collecting terms up to order  $1/\gamma^2$ . This straightforward exercise, followed by the steps that lead up to Eq. (32), yields the so-called Hill–Whittaker equation [12]

$$\left[ \frac{d^2}{dz^2} + \Theta_0 + 2\Theta_1 \cos 2z + 2\Theta_2 \cos 4z \right] \Sigma = 0 \tag{42}$$

with

$$\Theta_0 = \lambda + \frac{2}{3\gamma^2}, \quad \Theta_1 = -\frac{2}{3\gamma}, \quad \Theta_2 = -\frac{1}{\gamma^2} \tag{43}$$

where now

$$\lambda = \frac{8MEa^2}{\hbar^2 \alpha^2 \gamma^2}. \tag{44}$$

Following Ince [12], the most general solution of the Hill–Whittaker equation can be obtained along the same lines adopted for solving Mathieu’s equation and yields the following energy eigenvalues

$$E_n = \frac{\hbar^2 \alpha^2 \gamma^2}{8Ma^2} \left\{ -16n^2/q^2 + \left( 1 - \frac{4}{3\gamma} \cos 2\sigma - \frac{2}{3\gamma} \sin 2\sigma - \frac{7}{9\gamma^2} \right) \right\}. \tag{45}$$

The corresponding solutions are

$$\begin{aligned} \Sigma^{(n)} = e^{2inz/q} & \left\{ \sin(z - \sigma) + \frac{2\gamma}{3} \sin(3z - \sigma) + \left( \frac{1}{108} - \frac{4\gamma^2}{9} \right) \sin(5z - \sigma) \right. \\ & \left. + \frac{2\gamma}{3} \cos(3z - \sigma) - \frac{4\gamma^2}{9} \cos(5z - \sigma) \right\} \end{aligned} \tag{46}$$

where  $\sigma$  is the parameter introduced earlier. Once again multiplying by the factor  $\chi = N\sqrt{f}$ , expanding the denominator, retaining terms up to order  $1/\gamma^2$  and, rewriting everything in terms of  $\phi$  using  $\alpha\phi = 2z$ , gives the final expression for the eigenstate to be

$$\begin{aligned} \psi^{(n)}(\phi) = Ne^{in\alpha\phi/q} & \times \left\{ -\frac{4\gamma}{9} \left[ \sin\left(\frac{5\alpha\phi}{2} - \sigma\right) + \cos\left(\frac{5\alpha\phi}{2} - \sigma\right) \right] \right. \\ & \left. + \frac{2}{3} \left[ \sin\left(\frac{3\alpha\phi}{2} - \sigma\right) + \cos\left(\frac{3\alpha\phi}{2} - \sigma\right) - \frac{2}{3} \cos \alpha\phi \left\{ \sin\left(\frac{5\alpha\phi}{2} - \sigma\right) \right\} \right] \right\} \end{aligned}$$



$$\begin{aligned}
 & - \cos\left(\frac{5\alpha\phi}{2} - \sigma\right)\Bigg\} + \frac{1}{\gamma} \left[ \sin\left(\frac{\alpha\phi}{2} - \sigma\right) + \frac{1}{108} \sin\left(\frac{5\alpha\phi}{2} - \sigma\right) \right. \\
 & + \frac{2 \cos \alpha\phi}{3} \left\{ \sin\left(\frac{3\alpha\phi}{2} - \sigma\right) + \cos\left(\frac{3\alpha\phi}{2} - \sigma\right) \right\} \\
 & - \frac{4 \cos^2 \alpha\phi}{9} \left\{ \sin\left(\frac{5\alpha\phi}{2} - \sigma\right) + \cos\left(\frac{5\alpha\phi}{2} - \sigma\right) \right\} \Bigg] \\
 & + \frac{1}{\gamma^2} \left[ \cos \alpha\phi \left\{ \sin\left(\frac{\alpha\phi}{2} - \sigma\right) + \frac{1}{108} \sin\left(\frac{5\alpha\phi}{2} - \sigma\right) \right\} \right. \\
 & + \frac{2 \cos^2 \alpha\phi}{3} \left\{ \sin\left(\frac{3\alpha\phi}{2} - \sigma\right) + \cos\left(\frac{3\alpha\phi}{2} - \sigma\right) \right\} \\
 & \left. \left. - \frac{4 \cos^3 \alpha\phi}{9} \left\{ \sin\left(\frac{5\alpha\phi}{2} - \sigma\right) + \cos\left(\frac{5\alpha\phi}{2} - \sigma\right) \right\} \right] \right\} \Bigg\} \tag{47}
 \end{aligned}$$

where  $N$  is a normalisation constant.

### 4.3. The result for an arbitrarily thick torus

For the sake of completion, we mention that this method can be systematically continued to arbitrary orders in  $1/\gamma$ . The corresponding equation satisfied by  $\Sigma$  is the Hill equation given by

$$\left[ \frac{d^2}{dz^2} + \Theta_0 + 2 \sum_{r=1}^{\infty} \Theta_{2r} \cos 2rz \right] \Sigma = 0. \tag{48}$$

As in the earlier section, we follow Ince [12], and try a general solution of the form

$$\Sigma = e^{ivz} u \tag{49}$$

where

$$\begin{aligned}
 v = & p_1(\sigma)\Theta_1 + p_2(\sigma)\Theta_2 + \dots + q_1(\sigma)\Theta_1^2 + q_2(\sigma)\Theta_2^2 + \dots + q_{12}\Theta_1\Theta_2 \\
 & + q_{13}\Theta_1\Theta_3 + q_{23}\Theta_2\Theta_3 + \dots + r_1(\sigma)\Theta_1^3 + \dots \tag{50}
 \end{aligned}$$

and

$$\begin{aligned}
 u = & \sin(z - \sigma) + A_1(z, \sigma)\Theta_1 + A_2(z, \sigma)\Theta_2 \dots + B_1(z, \sigma)\Theta_1^2 + B_2(z, \sigma)\Theta_2^2 + \dots \\
 & + B_{12}(z, \sigma)\Theta_1\Theta_2 + \dots \tag{51}
 \end{aligned}$$

with  $\sigma$  being determined by the relation

$$\begin{aligned}
 \Theta_0 = & 1 + \lambda_1(\sigma)\Theta_1 + \lambda_2(\sigma)\Theta_2 \dots + \mu_1(\sigma)\Theta_1^2 + \mu_2(\sigma)\Theta_2^2 + \dots \\
 & + \mu_{12}(\sigma)\Theta_1\Theta_2 \dots + \nu_1(\sigma)\Theta_1^3 \dots \tag{52}
 \end{aligned}$$

Substituting these expressions in Eq. (48) we can solve for the coefficients to any desired order, and hence obtain the corresponding eigenvalues and eigenvectors. We do not pursue this exercise since it does not shed any further light on the solution to the problem.

## 5. Inequivalent quantisations

Let us briefly recapitulate the interesting consequences that arise if the particle which is constrained to move on a circle is charged, and if the circle encloses an infinitely long, infinitesimally thin, and impenetrable solenoid carrying a uniform current. As is well-known, the wavefunction of the particle picks up a nontrivial phase factor which depends on the net flux enclosed by the trajectory of the particle as it goes around the circle. Thus the wavefunction is multi-valued, which is a manifestation of the nontrivial topology of the circle which, in turn, is a consequence of the fact

that the path cannot be shrunk to a point in the presence of the impenetrable solenoid. Redefining the wavefunction such that it is single-valued modifies the Hamiltonian in such a way that the energy spectrum depends on the enclosed flux. Given that the corresponding Lagrangians, with and without the flux, differ only by a total derivative term, the classical theory is unaltered; although different values of the flux yield different energy spectra, and hence inequivalent quantum theories. It is reasonable to expect similar features in the case of a charged particle constrained to move along a knot.

For the torus knot of interest, two independent magnetic fluxes can be introduced. The first is the usual magnetic field obtained by placing a uniform current carrying, long, thin solenoid parallel to the  $z$ -axis and passing through the centre of the putative torus around which the knot winds. Let us denote the corresponding flux by  $\Phi_S$ . The second flux is obtained by a uniform poloidal current winding around the torus which produces a magnetic field which has support only inside the torus, the so-called toroidal magnetic field. Let us denote this flux by  $\Phi_T$ .

A particle constrained to move on a  $(p, q)$  torus knot, starts at a point on the surface of the putative torus and returns to the initial point after completing one circuit of the knot; in the process winding around the solenoidal flux  $p$  times and the toroidal flux  $q$  times. The total flux enclosed is therefore:  $\Phi = p\Phi_S + q\Phi_T$ . Thus we have the equation which highlights the multi-valued nature of the wavefunction *viz.*

$$\psi(\eta_0, \theta + 2\pi q, \phi - 2\pi p) = \exp(i\Phi)\psi(\eta_0, \theta, \phi). \quad (53)$$

Defining the single-valued wavefunction

$$\tilde{\psi}(\eta_0, \theta, \phi) = \exp\left(-i\frac{\Phi}{2p\pi}\phi\right)\psi(\eta_0, \theta, \phi) \quad (54)$$

and the corresponding Hamiltonian obtained by the transformation

$$\tilde{H} = \exp\left(-i\frac{\Phi}{2p\pi}\phi\right)H\exp\left(i\frac{\Phi}{2p\pi}\phi\right) \quad (55)$$

we see that the momentum operator in the Hamiltonian is shifted by  $\frac{i\Phi}{2p\pi}$ , which leads to a corresponding shift in  $\nu$  and hence the energy spectrum defined in Eqs. (36) and (44). It is noteworthy that the phase picked up by the wavefunction of the particle, for a given  $(p, q)$  knot, is a sum of two independent fluxes. Thus the inequivalent quantisations are labelled by two parameters.

## 6. Conclusions and outlook

The classical and quantum mechanics of a particle constrained to move on a torus knot were studied. The results were compared and contrasted with the well-known results for a particle constrained to move on a circle. Defining the knot as a trajectory which winds around a putative torus in a well-defined fashion, and using toroidal coordinates to parametrise the knot, makes it possible to rewrite the time-independent Schrödinger equation as a Hill equation which can then be studied perturbatively in the thickness of the putative torus.

Attributing a charge to the particle and introducing two independent magnetic fields having supports in physically disconnected, but topologically linked, regions, leads to a two-parameter family of inequivalent quantisations of the particle moving on a knot.

The model discussed in this paper has several features which are worth discussing further. First, it would be natural to study the model non-perturbatively *i.e.* using instanton methods made popular in [2,3]. Second, it would be interesting to generalise the treatment to more than one particle moving on the knot. The non-trivial phase factor can then be related to exotic quantum statistics of the anyonic type. It would also be interesting to construct coherent states and study algebras of quantum observables associated with a particle on a knot. All these problems have natural analogues for the corresponding, but much simpler, example of a particle constrained to move on a circle.

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