Perturbed Chern–Simons theory, fractional statistics, and Yang–Baxter algebra

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Topological Chern–Simons theory coupled to matter fields is analysed in the frame work of Dirac's method of quantising constrained systems in a general class of linear, non-local gauges. We show that in the weak coupling limit gauge invariant operators in the theory transform under an exchange according to a higher dimensional representation of the braid group which is built out of the fundamental representation matrices of the gauge group and thus behave like anyons. We also discover new solutions of the Yang–Baxter equation which emerges as a consistency condition on the structure functions of the operator algebra of the matter fields.

Quantum field theories defined by lagrangians which are given purely by the Chern–Simons (CS) term or along with other terms, have emerged at the heart of some profound connections between certain seemingly disparate areas of mathematics and physics in the last couple of years. Of central importance with respect to physics is their connection with anyons – particles and fields with fractional spin and statistics [1]. CS gauge fields in quantum field theories become relevant because they naturally induce these exotic properties in the sources they couple to [2]. It is our endeavour in this letter to generalise the results of ref. [2] to the non-abelian case and it is in this context that a mention of one of the most important properties of the CS term becomes pertinent.

This has to do with the intimate relationship it shares with conformal field theories in (1+1)-dimensional spacetime, first noticed by Witten [3] and elaborated subsequently by several others [4,5]. It is worth mentioning at this juncture that since it has been known for some time now that adding certain special perturbations to conformally invariant systems leads to integrable or exactly solvable systems [6], the question whether such systems can be obtained straight away from CS theory by introducing

perturbations which explicitly break the metric-independence of the theory is, at once, both interesting and important.

It is compelling to believe, in the light of these points, that further surprises are in store for us in the study of topological Chern–Simons field theory coupled to matter. In what follows we offer the reader glimpses into the beautiful results that lie dotted along this hitherto uncharted terrain.

Let us therefore begin by considering the following three dimensional model:

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi)$$

$$+ \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \mathrm{Tr}(A_\mu \partial_\nu A_\lambda + i \cdot \tfrac{2}{3} A_\mu A_\nu A_\lambda) . \tag{1}$$

Here Φ is an N-component scalar field which transforms according to the fundamental representation of the SU(N) group, $D_\mu = \partial_\mu + iA_\mu$, $A_\mu = A_\mu^a T^a$ where the fundamental representation matrices T^a are traceless and hermitian and

$$[T^a, T^b]_- = if^{abc}T^c, \quad [T^a, T^b]_+ = d^{abc}T^c + \frac{1}{N}\delta^{ab},$$

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab} .$$

The components of the gauge fields A_μ^a transform under the adjoint representation of the gauge group and the trace in the Chern–Simons term is over the fun-

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damental representation. We adopt the conventions $\epsilon^{012}=\epsilon_{012}=1$, $g_{\mu\nu}=(1, -1, -1)$ and sum over repeated indices without comment. The minimal coupling to the matter fields through D_μ makes (1) metric dependent and therefore spoils the topological nature of the CS theory. Note however, that under a trivial scaling, $A \to k^{-1/2}A$, k appears in the covariant derivative and it is therefore a measure of the strength of the minimal interaction. This is the sense in which we call (1) a perturbed Chern–Simons field theory.

The canonically conjugate momenta are defined and given by

$$\Pi_\alpha \equiv \frac{\delta\mathscr{L}}{\delta(\partial_0\Phi_\alpha)} = \left(\partial_0\Phi_\alpha^\dagger - i\Phi_\beta^\dagger A_0^a T_{\beta\alpha}^a\right), \qquad (2a)$$

$$\Pi_\alpha^\dagger \equiv \frac{\delta\mathscr{L}}{\delta(\partial_0\Phi_\alpha^\dagger)} = \left(\partial_0\Phi_\alpha + iA_0^a T_{\alpha\gamma}^a\Phi_\gamma\right), \qquad (2b)$$

$$\Pi_0^a \equiv \frac{\delta\mathscr{L}}{\delta(\partial_0 A_0^a)} = 0, \qquad (2c)$$

$$\Pi_i^a \equiv \frac{\delta\mathscr{L}}{\delta(\partial_0 A_i^a)} = \frac{k}{8\pi}\epsilon_{ij}A^{aj}. \qquad (2d)$$

Eqs. (2c) and (2d) do not involve any velocities and are merely some relations between coordinates and momenta. Systems with such complications are called constrained systems and they can be consistently quantised by using Dirac's procedure [7]. Thus, to begin with we have the primary constraints

$$P_0^a \equiv \Pi_0^a \approx 0, \qquad (3a)$$

$$P_i^a \equiv \Pi_i^a - \frac{k}{8\pi}\epsilon_{ij}A^{aj} \approx 0. \qquad (3b)$$

The canonical hamiltonian density is given by

$$\mathscr{H}_c = \Pi_\alpha^\dagger\Pi_\alpha + (\partial_i\Phi_\alpha^\dagger)(\partial_i\Phi_\alpha) + A_0^a j_0^a + A_i^a j_i^a$$

$$+ \Phi_\beta^\dagger A_i^a T_{\beta\alpha}^a A_i^b T_{\alpha\gamma}^b\Phi_\gamma$$

$$- \frac{k}{4\pi}\text{Tr }\epsilon^{ij}(A_0\,\partial_i A_j + A_i\,\partial_j A_0 + 2iA_0 A_i A_j), \qquad (4)$$

where

$$j_0^a = i(\Pi_\alpha^\dagger\Phi_\beta^\dagger - \Phi_\alpha\Pi_\beta)T_{\beta\alpha}^a, \qquad (5a)$$

$$j_i^a = i[\Phi_\alpha(\partial_i\Phi_\beta^\dagger) - (\partial_i\Phi_\alpha)\Phi_\beta^\dagger]T_{\beta\alpha}^a \qquad (5b)$$

are the matter charge–current densities. The canonical hamiltonian is $H_c = \int d^2x\,\mathscr{H}_c(x)$. We also define

$$H = \int d^2x\,(\mathscr{H}_c + u_a P_0^a + v_a^i P_i^a), \qquad (6)$$

where u_a and v_a^i are some arbitrary functions of coordinates and momenta. In order that the primary constraints are preserved in time, we require that they have at least weakly vanishing Poisson brackets with H. Such a requirement on P_0^a yields the following secondary constraint:

$$S_0^a(x) \equiv -j_0^a(x)$$

$$+ \frac{k}{4\pi}\epsilon^{ij}[\partial_i^x A_j^a(x) - \tfrac{1}{2}f^{bca}A_i^b(x)A_j^c(x)]$$

$$\approx 0, \qquad (7)$$

upon imposing the boundary conditions on the gauge fields,

$$A_i^a(x) \to 0 \quad \text{as } x_j \to \pm\infty, \qquad i,j = 1, 2. \qquad (8)$$

It is easy to verify that there are no further secondary constraints in the theory and therefore eq. (7) along with eqs. (3a) and (3b) gives the full set of constraints. A preliminary classification of these constraints is readily done by evaluating the PBs between them. P_0^a has a vanishing PB with each one of the other constraints and it is therefore a first class constraint. The rest are second class. We called this a preliminary classification because we might have naively concluded from here that the theory has one first class and three second class constraints. There is, however, more to it than meets the eye. In the present case it is possible to form a combination of the second class constraints which has a vanishing PB with all the constraints in the theory and is therefore first class. It is this new constraint which contains the germ of all that is to unfold. Let us denote this constraint by $\mathscr{P}^a(x)$. It has the following form:

$$\mathscr{P}^a(x) \equiv \int d^2y\,[F_1^{ab}(x,y)P_1^b(y)$$

$$+ F_2^{ab}(x,y)P_2^b(y) + F_0^{ab}(x,y)S_0^b(y)] \approx 0, \qquad (9)$$

where $F_1^{ab}(x,y)$ and $F_2^{ab}(x,y)$ have the functional forms in terms of $F_0^{ab}(x,y)$ given by

$$F_1^{ab}(x,y) = \partial_1^y F_0^{ab}(x,y) + f^{cbd}F_0^{ad}(x,y)A_1^c(y), \qquad (10a)$$

$$F_2^{ab}(x,y) = \partial_2^y F_0^{ab}(x,y) + f^{cbd}F_0^{ad}(x,y)A_2^c(y), \qquad (10b)$$

where

$$F_0^{ab}(x, y) = f_{01}^a(x, y) f_{02}^b(y) ,\qquad (10c)$$

with $f_{01}^a(x, y)$ and $f_{02}^b(y)$ being some arbitrary functions, such that $F_0^{ab}(x, y) \to 0$ as $y_i \to \pm\infty$.

Notice that since we have been able to construct a new (first class) constraint from the three second class constraints we were having before, we have to throw away one of them. We decide to throw away $S_0^a(x)$. We are then left with two first class constraints in eqs. (3a) and (9) and two second class constraints in eq. (3b). Since the theory exhibits gauge symmetries generated by the first class constraints, we need to fix the gauge before we proceed further. We choose the following general class of linear but non-local gauge fixing conditions corresponding to the two first class constraints we have, namely P_0^a and $\mathscr{P}^a$:

$$\chi_0^a(x) \equiv \int d^2y\, K_0^{ab}(x, y) A_0^b(x_0, y) \approx 0 ,\qquad (11a)$$

$$\chi^a(x) \equiv \int d^2y\, K_i^{ab}(x, y) A_i^b(x_0, y) \approx 0 ,\qquad (11b)$$

where the kernels $K_0^{ab}(x, y)$ and $K_i^{ab}(x, y)$ are, until further commented upon, arbitrary. This is all the infrastructure we need in order to compute the commutation relations of the basic fields in the theory.

Using the definition of the Dirac bracket (DB) [7], we arrive at

$$\{\Phi_\alpha(u), \Phi_\beta(v)\}^*$$
$$= \frac{4\pi}{k} \Delta^{ab}(u, v) T_{\alpha\mu}^a T_{\beta\nu}^b \Phi_\mu(u) \Phi_\nu(v) ,\qquad (12)$$

if we set, for convenience,

$$\mathscr{D}_i^y K_i^{ab}(x, y) \equiv \partial_i^y K_i^{ab}(x, y) + f^{cbd} A_i^c(y) K_i^{ad}(x, y)$$
$$= \delta^{ab} \delta^2(x - y) .\qquad (13)$$

In the above we have

$$\Delta^{ab}(u, v) = \int d^2z\, \epsilon^{ij} K_i^{ac}(u, z) K_j^{bc}(v, z) .\qquad (14)$$

Obviously, $\Delta^{ab}(u, v) = -\Delta^{ba}(v, u)$. The commutator of the matter fields can now be obtained in the standard way and it reads

$$[\Phi_\alpha(u), \Phi_\beta(v)]_-$$
$$= i\frac{2\pi}{k} \Delta^{ab}(u, v) T_{\alpha\mu}^a T_{\beta\nu}^b [\Phi_\mu(u), \Phi_\nu(v)]_+ .\qquad (15)$$

In abstracting the above commutator from the classical DB relation we needed to prescribe an ordering for the bilinear in Φ's on the RHS of eq. (15). We adopted Weyl ordering as it ensures the antisymmetry of the commutator under the simultaneous exchanges $\alpha \leftrightarrow \beta$ and $u \leftrightarrow v$, without any further restrictions on $\Delta^{ab}(u, v)$. Eq. (15) embodies the central result of our analysis and will serve as the springboard for the rest of the exercises we perform. As a warm-up let us note that eq. (15) can be rewritten in the matrix form

$$\Phi(u)\Phi(v) = R(u, v)\Phi(v)\Phi(u) ,\qquad (16)$$

where the structure functions of the operator algebra of the Φ's are given by

$$R(u, v) = \frac{1 + i\Delta(u, v)}{1 - i\Delta(u, v)} ,\qquad (17)$$

with the obvious notational simplification

$$\Delta(u, v) = \frac{2\pi}{k} \Delta^{ab}(u, v) T^a \otimes T^b .\qquad (18)$$

The hermiticity of $\Delta(u, v)$ in eq. (18) ensures the unitarity of $R(u, v)$ in eq. (17),

$$R(u, v) R^\dagger(u, v) = R^\dagger(u, v) R(u, v) = 1 ,\qquad (19)$$

and therefore eq. (16) can be recast as

$$\Phi(u)\Phi(v) = \exp[i\Theta(u, v)]\, \Phi(v)\Phi(u) .\qquad (20)$$

Here $\Theta(u, v)$ is an $N^2 \times N^2$ hermitian matrix constructed out of the fundamental representation matrices T^a. Thus, under a permutation, the product of two basic fields in our theory picks up a phase. As it stands, however, there are several worrisome aspects of this phase which need to be clarified before we can conclude that we have anyons in the theory. As is well known from quantum mechanics, when two anyons are exchanged, for non-trivial results the phase that is picked up must be multi-valued. It is natural to expect a similar feature to be reflected at the field theoretic level we are working. A related point in the comparison with the quantum mechanical result concerns the two possible phases that can occur in

quantum mechanics depending on the way one particle is rotated around the other, namely clockwise or anticlockwise. The above result does not appear to accommodate such a possibility. Moreover, eq. (20) implies that the statistics of the fields under consideration depends upon their spatial coordinates as well as the choice of the gauge fixing kernels. This is physically absurd. It is, however, not very difficult to realise that all these problems are rooted in the fact that the basic fields we are considering do not correspond to physical particles because they are gauge non-invariant. Scaling the gauge fields $A \rightarrow k^{-1/2} A$ for convenience and choosing the gauge transformation parameter in such a way that the element of the gauge group goes to the identity at spatial infinity, gauge invariant operators in the theory can be constructed according to the standard line integral prescription of Schwinger and are given by

$$\hat{\Phi}_\alpha(u) = \left[P \exp\left(ik^{-1/2} \int_\infty^u dx^i A_i(x) \right) \right]_{\alpha\mu} \Phi_\mu(u) ,$$

$$(21)$$

where the P in front of the exponential refers to path ordering. It is now straightforward to compute the algebra of these gauge invariant operators, in doing so we need the expressions for two more commutators involving the basic fields. They are recorded here for the convenience of the reader:

$$[A_i^a(u), A_j^b(v)]_-$$

$$= \frac{-2\pi i}{k} \int d^2z\, \epsilon^{kl} [\mathscr{D}_i^u K_k^{ac}(u, z), \mathscr{D}_j^v K_l^{bc}(v, z)]_+ ,$$

$$(22)$$

$$[\Phi_\alpha(u), A_i^a(v)]_-$$

$$= \frac{4\pi}{k} [\mathscr{D}_i^v \Delta^{ba}(u, v) - \epsilon_{ij} K_j^{ba}(u, v)] T_{\alpha\beta}^b \Phi_\beta(u) .$$

$$(23)$$

As in eq. (13), $\mathscr{D}_i$ is the covariant derivative in the adjoint representation. Notice that the RHS of eq. (22) has been Weyl ordered just like in eq. (15) in order to ensure the proper antisymmetry of the commutator. With the help of these results the algebra of the gauge invariant operators in eq. (21) can be worked out. For large k, i.e. in the weak coupling limit, the following result holds to first order in k^{-1}:

$$\hat{\Phi}(u)\hat{\Phi}(v) = \exp[(2i/k)(\pi \bmod 2\pi) T^a \otimes T^a]$$

$$\times \hat{\Phi}(v)\hat{\Phi}(u) .$$

$$(24)$$

An exactly similar expression can be worked out for the $\hat{\Phi}^\dagger$ operators. Now, an N-particle quantum mechanical state can be produced by conventional Fock space methods by acting a string of N $\hat{\Phi}^\dagger$'s on the vacuum state. The multivaluedness of the operators can then be transferred to the definition of the physical states. The resulting multivalued state is a non-abelian generalization of Laughlin's state for the fractional quantum Hall effect [8]. In terms of this state the particle interpretation as well as the two possible phases that are associated with the clockwise or anti clockwise rotation by 2π of one particle around the other are rendered immediately obvious. Thus the multivalued gauge invariant operators in the theory are anyon field operators. What is significant is the fact that this phase is not merely a number as in ref. [2] but a matrix and hence it furnishes a higher dimensional representation of the braid group. What we have obtained is in fact known to be the monodromy representation of the braid group [5]. It is most instructive to work through the details of the procedure outlined so far in the simpler case of the abelian theory where we get the one-dimensional representation of the braid group as an exact answer. The details in this regard will be presented in a forthcoming publication [9].

Let us now consider the product of three fields $\Phi(u)$ $\Phi(v)$ $\Phi(w)$ and permute it in two ways, viz.

(1) $\Phi(u)$ with $\Phi(v)$, then $\Phi(u)$ with $\Phi(w)$ and then $\Phi(v)$ with $\Phi(w)$;

(2) $\Phi(v)$ with $\Phi(w)$, then $\Phi(u)$ with $\Phi(w)$ and then $\Phi(u)$ with $\Phi(v)$.

Demanding associativity produces, upon using the result in eq. (16),

$$R_{\alpha\beta\mu\nu}(u, v) R_{\mu\gamma\rho\kappa}(u, w) R_{\nu\kappa\lambda\omega}(v, w)$$

$$= R_{\beta\gamma\mu\nu}(v, w) R_{\alpha\nu\kappa\omega}(u, w) R_{\kappa\mu\rho\lambda}(u, v) .$$

$$(25)$$

In order to write this as a matrix equation, let us consider the direct product space $V_1 \otimes V_2 \otimes V_3$, where V_i ($i = 1, 2, 3$) are isomorphic vector spaces (in this case the representation spaces furnished by the Φ's). Then define

$$R_{12} \equiv R|_{V_1 \otimes V_2} \otimes \mathbb{1}|_{V_3} , \quad R_{23} \equiv \mathbb{1}|_{V_1} \otimes R|_{V_2 \otimes V_3} , \quad (26)$$

and similarly R_{13}. Then, eq. (25) can be recast as

$$R_{12}(u, v) \cdot R_{13}(u, w) \cdot R_{23}(v, w)$$
$$= R_{23}(v, w) \cdot R_{13}(u, w) \cdot R_{12}(u, v) . \qquad (27)$$

This classic equation lies at the heart of integrability of certain $(1+1)$-dimensional field theories and two-dimensional classical statistical mechanical models and is called the Yang–Baxter equation (YBE) [10]. Thus we have the important new result that the structure functions of the operator algebra of the matter fields coupled to a Chern–Simons gauge field furnish solutions of the YBE.

The fact that the arguments of $R(u, v)$ in eq. (27) are not scalars and that the functional dependence is not through $u - v$, is a crucial difference between these solutions and those which fall under the purview of the Belavin–Drinfeld classification scheme [11]. Solutions of the star–triangle equation of the chiral Potts model involving functions on algebraic curves of genus $\geqslant 1$ are also known to exhibit a similar qualitative feature [12].

We conclude by making a few remarks. Firstly, it would appear that if normal gauge fixing conditions, viz. $A_0^a \approx 0$ and $\partial_i A_i^a \approx 0$, were chosen, than all the fields in the theory would have normal statistics. It is, however, important to note that the two gauge fixing conditions in this set are actually incompatible with each other in any gauge theory coupled to matter fields [6]. This can be most easily seen by taking the time derivative of the second condition which leads to a new condition which clashes with the first. This hurdle can be overcome by smearing out the gauge fixing conditions using some non-local distribution functions $K_0^{ab}(x, y)$ and $K_i^{ab}(x, y)$. The incompatibility can now be traded for some conditions on these distribution functions. These conditions can then be shown to be precisely the ones which completely fix the local gauge invariance in the theory. It is interesting to note that the condition on $K_0^{ab}(x, y)$ that arises this way and eq. (13) simply does not admit solutions which will reduce our gauge fixing conditions to the conventional ones which are simply forbidden on the grounds of algebraic consistency [9]. We therefore conclude that there exists no special choice of the gauge fixing kernels for which the basic fields in the theory obey a set of normal commutation relations. In this sense our results are not

gauge artefacts. Our set of gauge fixing conditions is merely being used as a tool to unearth the non-trivial physics in the model. This fact finds its most eloquent expression in eq. (24), were gauge invariant operators are shown to obey statistics which knows nothing about the gauge fixing conditions used. This is a remarkable departure from the results in ref. [2]. Another interesting feature to note is that if the initial lagrangian density (1) had a polynomial potential term (including possibly a mass term) for the scalar field the results would still not change because the constraint structure of the theory remains unaltered. The results therefore are independent of the details of the bosonic interaction.

We plan to discuss the details of all the results announced in this letter, the construction of exactly solvable two-dimensional classical statistical mechanical models and the hamiltonians of the underlying quantum spin chains, using our new solutions of the YBE along with some related aspects like the quantum group structure, in a longer publication elsewhere. It will be also interesting to know whether the statistics phase in eq. (24) receives any higher order corrections in powers of k^{-1}.

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