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Canonical Quantization and Gauge Invariant Anyon Operators in Chern–Simons Scalar Electrodynamics

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Complex scalar fields minimally coupled to an abelian Chern–Simons gauge field are systematically quantized using Dirac's method for constrained systems. Manifestly gauge invariant anyon operators are constructed following the line integral prescription of Schwinger. Stringent consistency conditions like the spin–statistics connection, Poincaré invariance, and tree level vector current Ward identity are scrutinised. The effect of the addition of the Maxwell term to the lagrangian density on all the above results is examined. The gauge invariant anyon operators constructed in this case are shown not to exhibit any spin or statistics transmutation as one moves from shorter length scales to longer ones, where the residual coulomb interaction is effectively screened.

1. INTRODUCTION

The classification scheme based on spin and statistics, which divides the particle world in three-dimensional space into bosons and fermions, is one of the pillars on which the edifice of modern physics rests. This scheme has its mathematical genesis in the twin facts that the rotation group allows at most double-valued representations and the permutation group, relevant for statistics, has only two one-dimensional representations, viz., the identity and the alternating. While the former implies that the only allowed values of spin are those which are even or odd integer multiples of one-half, the latter restricts the change in the wavefunction of identical particles under an arbitrary exchange of their positions, barring parastatistics, to a mere sign at best.

It is well known, however, that in two space dimensions, unlike in the above case, spin need not quantized and the wavefunction of identical particles can in general change by an arbitrary multiplicative phase factor under an exchange of their positions. The first of these properties is a consequence of the fact that in two spatial dimensions there is no non-trivial angular momentum algebra, as there is

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only one axis to rotate about. The second result follows from the fact that the fundamental group of the configuration space of identical particles in two dimensions, which is obtained by deleting the so-called diagonal points—positions which can be occupied by two or more particles—from the tensor product of one-particle configuration spaces and modding out the resultant space by the relevant permutation group, is not merely the permutation group, but a more complicated non-abelian group called the braid group whose one-dimensional representations correspond to arbitrary phase factors $e^{i\theta}$. As a result, spin and statistics in two spatial dimensions can in general take any arbitrary values, the bosonic and fermionic incarnations merely corresponding to some special cases of this more generic mathematical possibility [1]. That the above ad-hoc deletion of the diagonal points is an inessential assumption was shown by Goldin, Menikoff, and Sharp who independently discovered these possibilities through their approach based on diffeomorphism groups and current algebras [2]. Objects which physically realise these mathematical possibilities are called anyons and they have attracted a lot of interest in the last few years because of their exotic properties [3].

Motivation for further studies in anyons, however, transcends the esoteric interests alluded to above. The most striking physical effect for whose theoretical explanation the existence of anyons is a sine qua non, is the fractional quantum Hall effect. The importance of anyons in this context was first elucidated by Laughlin [4]. It is widely believed that anyons also have an important role to play in the theoretical explanation of high-T_c superconductivity observed in thin CuO layers. This point of view has received a particularly eloquent advocacy in the work of Chen *et al.* [5]. In all such physical applications it is profitable for us to realise these exotic possibilities in spin and statistics through some dynamical mechanism, instead of restricting ourselves to quantum kinematics. A simple model which fulfills this requirement was first introduced by Wilczek [6]. A brief description of this model is in order at this juncture as it is germane to what follows.

Let us consider a point particle which carries in addition to a charge q a magnetic flux ϕ in the two-dimensional plane. The associated magnetic field in the plane has its support only at the position of the particle. There is, however, a non-trivial vector potential felt by an identical particle at some other position in the plane. Consequently, when the latter moves round the former, assumed to be held fixed, the amplitude picks up a Bohm–Aharonov phase [7] with the fixed flux-carrying particle acting like an impenetrable point solenoid. Now, since the exchange of two identical particles can always be interpreted as a rotation of one of them around the other through π, what we have obtained above is essentially the statistics phase masquerading in the form of a Bohm–Aharonov phase. The physics of fractional statistics, like that of the Bohm–Aharonov effect, is, therefore, rooted in the profound principle of non-locality in quantum theory. Any attempts to construct theories of anyons must, therefore, necessarily prescribe a way of incorporating this non-locality in a consistent way.

The charge–flux composite that was advanced as a prototype anyon above can be fabricated in a particularly simple way. This is done by coupling the conserved current j^μ of the point-charged particle to a gauge field A^μ whose dynamics is governed by the Chern–Simons (CS) term which is special to $2+1$ dimensions [8]. In other words, we add to the lagrangian of the point particle the following terms

$$\Delta L = qj^\mu A_\mu + \theta \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{1.1}$$

the latter being the CS term. Note that it is gauge invariant up to a boundary term. More interestingly, it is linear in the derivative of the gauge field. As a result, the gauge field has no independent dynamics. This can be easily seen by working out the equation of motion from (1.1),

$$qj^\mu = -\theta \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}. \tag{1.2}$$

Thus, the gauge invariant dynamics of A_μ is completely determined by j^μ. For a point particle of charge q, j^μ has its support only on the world line of the particle and hence $F_{\nu\lambda}$ vanishes away from it. Hence, if we integrate the zeroth component of the above equation over a small spatial disc intersected by the world line of the particle, we obtain

$$q = -\theta\phi. \tag{1.3}$$

This tells us that the charged particle actually behaves like a magnetic flux point.

For most physical applications and to gain a deeper understanding of the concepts of fractional spin and statistics, however, a straightforward relativistic quantum field theoretic generalization of the very successful quantum mechanical picture of representing an anyon as a charged particle, to which a flux line is attached through the CS mechanism, is of inestimable importance. Unfortunately, such holistic approaches in the past have been only partially successful. The first such attempt was made by Semenoff [9] who canonically quantized a variant of the scalar electrodynamics model in three dimensions in which the CS term replaces the usual Maxwell term. The operators which carry fractional spin and statistics in this model are given by multivalued, non-local, composites of the basic fields in the theory. This model was later extended to include a Maxwell term also [10]. However, there seems to be an internal algebraic inconsistency in this calculation because of the choice of gauge-fixing conditions. This can be easily seen following [11]. The gauge-fixing conditions chosen in Refs. [9, 10] are $\partial_i A_i \approx 0$ and $A_0 \approx 0$. Is is easy to show that taking the time derivative of the first constraint above leads to a condition on A_0 which can be solved to obtain a non-trivial value for A_0. This immediately clashes with the second gauge-fixing condition above. This is a non-trivial aspect of gauge theories coupled to external sources which clearly distinguishes them from free theories. In addition one needs to employ some formal manipulations involving the exchange of a derivative and an integral of a multivalued function which are mathematically wrong and lead to physically indefensible conclusions [12]. While restoring compatibility of the two gauge-fixing conditions may not, in itself, be an insurmountable difficulty, the above-mentioned unjustifiable manipulations and the fact that the operators advanced to represent anyons in Refs. [9, 10] are not manifestly gauge invariant, makes it desirable for us to look for more elegant constructions.

The work of Foerster and Girotti [13], on the other hand, begins by making a polar decomposition of the fields and using non-local gauge fixing conditions involving the new fields. In term of the original fields these conditons are not only non-local but are also highly non-linear. Even if this highly unaesthetic choice of gauge-fixing conditions is accepted, the change of variables itself is questionable at least on two grounds. As is only too well known, this decomposition is singular and creates problems when the modulus field goes to zero. Furthermore, under such change of variables the effective action defined properly through a functional integral would pick up contributions from the change in the measure which are ignored in their analysis. Given this obscure relationship between the model in terms of the old and new variables, the validity of the results they obtain is certainly debatable because all their calculations are done in the model in terms of new variables and only in the end are the results obtained used to reconstruct the commutators of the old fields. There is yet another problem with this approach. It is easy to check that their model exhibits completely different constraint structures when expressed in terms of the original and transformed sets of variables. Such a difference in terms of the second-class constraints would have been completely innocuous, but, while their model in terms of the old variables contains two first-class constraints, in terms of the new variables it contains just one. Such redefinitions of fields that change the number of first-class constraints of the theory are clearly inadmissible as they destroy part of the gauge invariance of the theory.

To summarise, therefore, attempts to construct anyon operators in model field theories have either revolved around construction of complicated, non-local, multivalued composites of the basic fields or thinking of the basic matter fields themselves to be composed of some new fields of another theory. All such attempts have been, however, fraught with inconsistencies and most importantly they have failed to shed light on a connection with the simple quantum mechanical model of the anyon based on Wilczek's holistic principle. This is not very surprising because the operators purportedly representing anyons in the above theories are not gauge invariant. One has no option, therefore, except to look askance at the physical significance of such constructions. The question whether one can give a consistent particle interpretation to such operators in intimately related to the question whether anyons are real physical excitations or mere gauge artefacts. This paper is devoted towards finding answers to these questions.

In the rest of this section we discuss the organization of this paper. In Section 2 we study the canonical quantization of Chern–Simons scalar electrodynamics defined through complex scalar fields minimally coupled to abelian Chern–Simons gauge fields. The constraint structure of the theory is analysed in detail and quantization is carried out using Dirac's procedure after choosing linear, but, non-local gauge-fixing conditions. The commutation relations between all the basic fields of the theory and their canonically conjugate momenta are worked out. In

Section 3 we construct gauge invariant operators by using Schwinger's line integral prescription and show that they pick up a multivalued phase factor under a permutation. We also work out the momenta canonically conjugate to these operators and all the commutation relations involving them. We show that the Fock space constructed using conventional quantum field theoretic techniques yields a multi-particle state reminiscent of the Laughlin state for quantum Hall systems. The spin of a one-particle state so obtained is also calculated and a generalised spin-statistics connection is established. Section 4 examines the effect of adding a Maxwell term to the lagrangian density, which yields the so-called topologically massive scalar electrodynamics. It is seen that gauge invariant variables constructed according to the standard line integral prescription of Schwinger do not carry fractional spin and statistics as in the pure Chern–Simons case. Nevertheless, one can construct other manifestly gauge invariant operators which have all the properties of the gauge invariant operators of the earlier section and hence behave like anyons. Moreover, these operators reduce to the corresponding ones in the pure Chern–Simons case as the coupling of the Maxwell term goes to zero although the two theories are completely unrelated at the level of the algebraic structure of the basic fields in this limit. In Section 4 we prove that the quantization scheme developed by us, despite being very different from the conventional ones, does not disturb the Poincaré invariance of the two theories considered by us. Similarly, the vector current Ward identity for two anyons, vector current three-point function, is established at the tree level. It is also shown that the gauge fixing conditions completely fix the local gauge invariance in the theory. In the last section we summarise all the results obtained by us and conclude presenting the outlook for the future.

2. Canonical Quantization

The $U(1)$ gauge theory we are considering is defined by the following lagrangian density in three-dimensional space time:

$$\mathscr{L} = (D_\mu \phi)^* (D^\mu \phi) + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \tag{2.1}$$

Here ϕ is a one-component complex scalar field, A_μ is a $U(1)$ gauge field, $D_\mu = \partial_\mu + iA_\mu$, $\varepsilon^{\mu\nu\lambda}$ is the completely antisymmetric Levi–Civita tensor, and the covariant current $j_\mu = i(\phi^* D_\mu \phi - \phi(D_\mu \phi)^*)$ is conserved, i.e., $\partial^\mu j_\mu = 0$. We adopt the conventions $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$ and $\varepsilon^{012} = \varepsilon_{012} = 1$ and sum over repeated indices without comment.

The canonically conjugate momenta are defined and given by

$$\pi \equiv \frac{\delta \mathscr{L}}{\delta(\partial_0 \phi)} = (D_0 \phi)^* \tag{2.2a}$$

$$\pi^* \equiv \frac{\delta \mathscr{L}}{\delta(\partial_0 \phi^*)} = (D_0 \phi) \tag{2.2b}$$

$$\pi_0 \equiv \frac{\delta \mathscr{L}}{\delta(\partial_0 A_0)} = 0 \tag{2.2c}$$

$$\pi_i \equiv \frac{\delta \mathscr{L}}{\delta(\partial_0 A^i)} = \frac{\theta}{4\pi^2} \varepsilon_{ij} A^j. \tag{2.2d}$$

Unlike Eqs. (2.2a) and (2.2b), Eqs. (2.2c) and (2.2d) do not involve any velocities and are merely some relations between coordinates and momenta. Systems with such complications are called constrained systems and they can be consistently quantized by using Dirac's procedure [14]. Thus, to begin with, we have the following primary constraints:

$$P_0 \equiv \pi_0 \approx 0 \tag{2.3a}$$

$$P_i \equiv \pi_i - \frac{\theta}{4\pi^2} \varepsilon_{ij} A^j \approx 0. \tag{2.3b}$$

The canonical hamiltonian density is given by

$$\mathscr{H}_c = \pi^* \pi + (D_i \phi)^* (D_i \phi) + A_0 j_0 - \frac{\theta}{4\pi^2} \varepsilon^{ij} (A_0 \, \partial_i A_j + A_i \, \partial_j A_0), \tag{2.4}$$

where

$$j_0(\mathbf{x}) = i(\pi^*(\mathbf{x}) \, \phi^*(\mathbf{x}) - \pi(\mathbf{x}) \, \phi(\mathbf{x})) \tag{2.5}$$

is the matter charge density. The canonical hamiltonian is $H_c = \int d^2x \mathscr{H}_c(\mathbf{x})$. We also define the hamiltonian that generates translations in time

$$H = \int d^2\mathbf{x} [\mathscr{H}_c(\mathbf{x}) + u^0 P_0(\mathbf{x}) + v^i P_i(\mathbf{x})], \tag{2.6}$$

where u^0 and v^i are some arbitrary functions of the coordinates ϕ, ϕ^*, A_μ and their canonically conjugate momenta. In order that the primary constraints are preserved in time, we require that they have at least weakly vanishing Poisson brackets (PB) with H. Such a requirement on P_0 yields the secondary constraint

$$S_0(\mathbf{x}) = -j_0(\mathbf{x}) + \frac{\theta}{2\pi^2} \varepsilon^{ij} \, \partial_i A_j(\mathbf{x}) \approx 0, \tag{2.7}$$

upon using the natural boundary conditions on the gauge fields,

$$A_i(\mathbf{x}) \to 0 \quad \text{as} \quad x_j \to \pm\infty \; (i, j = 1, 2). \tag{2.8}$$

It is easy to verify that there are no further secondary constraints in the theory and, therefore, Eq. (2.7), along with Eqs. (2.3a), (2.3b) gives the full set of constraints. A preliminary classification of these constraints is readily done by evaluating the PBs between them. P_0 has a vanishing PB with each one of the other constraints and it is therefore a first-class constraint. The rest are second-class. We called this a preliminary classification because we might have naively concluded from here that the theory has one first-class and three second-class constraints. There is, however, more to it than meets the eye. In the present case it is possible to form a combination of the second-class constraints which has a vanishing PB with all the other constraints in the theory and is therefore first-class. It is necessary to embark on such an exercise because the theory exhibits gauge symmetries generated by the first-class constraints. Dirac's procedure relies on eliminating such redundant degrees of freedom at the classical level by imposing gauge-fixing conditions, which effectively convert the first-class constraints into second-class ones, before a transition to the quantum theory is made. It is, therefore, imperative to extract the maximal set of first-class constraints in the theory. Towards such an end let us define the following most general linear combination of the second-class constraints in the theory,

$$\mathscr{P} \equiv \int d^2\mathbf{y} [F_1(\mathbf{x}, \mathbf{y})\, P_1(\mathbf{y}) + F_2(\mathbf{x}, \mathbf{y})\, P_2(\mathbf{y}) + F_0(\mathbf{x}, \mathbf{y})\, S_0(\mathbf{y})] \approx 0. \tag{2.9}$$

Requiring $\mathscr{P}$ to have at least weakly vanishing PBs with P_1 and P_2 yields the following solutions for $F_1(\mathbf{x}, \mathbf{y})$ and $F_2(\mathbf{x}, \mathbf{y})$ in terms of $F_0(\mathbf{x}, \mathbf{y})$ which is subject to the boundary condition $F_0(\mathbf{x}, \mathbf{y}) \to 0$ as $y_i \to \pm\infty$ but is otherwise arbitrary:

$$F_1(\mathbf{x}, \mathbf{y}) = \partial_1^y F_0(\mathbf{x}, \mathbf{y}) \tag{2.10a}$$

$$F_2(\mathbf{x}, \mathbf{y}) = \partial_2^y F_0(\mathbf{x}, \mathbf{y}). \tag{2.10b}$$

Note that since we have been able to construct a new (first-class) constraint from the three second-class constraints we were having before, we have to throw away one of them. We decide to throw away $S_0(\mathbf{x})$. We are then left with two first-class constraints in Eqs. (2.3a) and (2.9) and two second-class constraints in Eq. (2.3b). As already mentioned, the theory exhibits gauge symmetries generated by the first-class constraints and we need to fix them before we proceed further. We choose the following general class of linear but non-local gauge-fixing conditions corresponding to the two first-class constraints namely P_0 and $\mathscr{P}$,

$$\chi_0(\mathbf{x}) \equiv \int d^2\mathbf{y}\, K_0(\mathbf{x}, \mathbf{y})\, A_0(x_0, \mathbf{y}) \approx 0 \tag{2.11a}$$

$$\chi(\mathbf{x}) \equiv \int d^2\mathbf{y}\, K_i(\mathbf{x}, \mathbf{y})\, A_i(x_0, \mathbf{y}) \approx 0, \tag{2.11b}$$

where kernels $K_0(\mathbf{x}, \mathbf{y})$ and $K_i(\mathbf{x}, \mathbf{y})$ are, until further commented upon, arbitrary.

It is also necessary for us to demand that these gauge fixing conditions are preserved in time:

$$\dot{\chi}_0(x) = i\left[\int d^2y K_0(\mathbf{x}, \mathbf{y})\, A_0(x_0, \mathbf{y}), H\right] = 0 \tag{2.12a}$$

$$\dot{\chi}(x) = i\left[\int d^2y K_i(\mathbf{x}, \mathbf{y})\, A_i(x_0, \mathbf{y}), H\right] = 0 \tag{2.12b}$$

Such a requirement can be easily verified to produce two conditions on the arbitrary velocities appearing in the hamiltonian (2.6) which can be, in principle, solved to eliminate two of the three velocities. At first sight, therefore, it might appear that Eq. (2.11) fix the gauge invariance in the theory only partially. That this is misleading is easily understood by realising that only two of the three velocities are really independent. The reason for this can be traced to the fact that requiring the constraint (2.3b) to be preserved in time also leads to a condition on the velocities which already fixes one of them in terms of the others. The fact that there are only two independent arbitrary velocities which need to be fixed by imposing conditions from outside is a direct reflection of the fact that there are only two gauge invariances in the theory which are generated by the two first-class constraints, (2.3a) and (2.9). The stage is now set for the computation of the commutation relations for the basic fields in the theory.

The DB between two variables $A(\mathbf{x})$ and $B(\mathbf{y})$ is defined as

$$\{A(\mathbf{x}), B(\mathbf{y})\}^* = \{A(\mathbf{x}, B(\mathbf{y})\} - \int d\mathbf{z}_1 \int d\mathbf{z}_2 \times \{A(\mathbf{x}), \theta_\alpha(\mathbf{z}_1)\}\, C_{\alpha\beta}(\mathbf{z}_1, \mathbf{z}_2)\{\theta_\beta(\mathbf{z}_2), B(\mathbf{y})\}.$$

In the above $\theta_\alpha(\mathbf{x})$ represents the set of all constraints and $C_{\alpha\beta}(\mathbf{x}, \mathbf{y})$ is an element of the inverse of the matrix of Poisson brackets of the constraints.

We focus our attention on the matter sector first. Using the above definition of the DB, we obtain

$$\{\phi(\mathbf{x}), \phi(\mathbf{y})\}^* = \frac{2\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{x})\, \phi(\mathbf{y}), \tag{2.13}$$

where we have used the freedom in choosing the kernels $K_0(\mathbf{x}, \mathbf{y})$ and $K_i(\mathbf{x}, \mathbf{y})$ to set

$$\partial_i^y K_i(\mathbf{x}, \mathbf{y}) = \delta^2(\mathbf{x} - \mathbf{y}). \tag{2.14}$$

In the above,

$$\Delta(\mathbf{x}, \mathbf{y}) = \int d^2\mathbf{z}\, \varepsilon^{ij} K_i(\mathbf{x}, \mathbf{z})\, K_j(\mathbf{y}, \mathbf{z}). \tag{2.15}$$

Obviously, $\Delta(\mathbf{x}, \mathbf{y}) = -\Delta(\mathbf{y}, \mathbf{x})$.

The commutator of the matter fields can now be obtained in the standard way and it reads

$$[\phi(\mathbf{x}), \phi(\mathbf{y})]_- = \frac{i\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})[\phi(\mathbf{x}), \phi(\mathbf{y})]_+. \tag{2.16a}$$

In abstracting the above commutator from the classical DB relation we needed to prescribe an ordering for the bilinear in ϕ's on the r.h.s. of Eq. (2.16a). We adopted Weyl ordering as it ensures the antisymmetry of the commutator under exchange $\mathbf{x} \leftrightarrow \mathbf{y}$, without any further restrictions on $\Delta(\mathbf{x}, \mathbf{y})$. In exactly the same manner all the other commutators in the matter sector can be extracted from their corresponding DBs. A generic feature of this set of DBs is the fact that the r.h.s. is a bilinear in the fields and/or their momenta. Consistent commutators, however, can be obtained as in the above example, by symmetrizing the bilinear. Some of the commutators are recorded below for the convenience of the reader. The others can be obtained by hermitian conjugation and/or antisymmetry:

$$[\phi(\mathbf{x}), \phi^*(\mathbf{y})]_- = -\frac{i\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})[\phi(\mathbf{x}), \phi^*(\mathbf{y})]_+ \tag{2.16b}$$

$$[\phi(\mathbf{x}), \pi(\mathbf{y})]_- = i\delta^2(\mathbf{x}-\mathbf{y}) - \frac{i\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})[\phi(\mathbf{x}), \pi(\mathbf{y})]_+ \tag{2.16c}$$

$$[\phi(\mathbf{x}), \pi^*(\mathbf{y})]_- = \frac{i\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})[\phi(\mathbf{x}), \pi^*(\mathbf{y})]_+ \tag{2.16d}$$

$$[\pi(\mathbf{x}), \pi(\mathbf{y})]_- = \frac{i\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})[\pi(\mathbf{x}), \pi(\mathbf{y})]_+ \tag{2.16e}$$

$$[\pi(\mathbf{x}), \pi^*(\mathbf{y})]_- = -\frac{i\pi^2}{\theta} \Delta(\mathbf{x}, \mathbf{y})[\pi(\mathbf{x}), \pi^*(\mathbf{y})]_+. \tag{2.16f}$$

We now concentrate on the gauge sector. Once again by using the definition of the DB we can construct the commutators of the gauge fields and their canonically conjugate momenta. They read as follows:

$$[A_i(\mathbf{x}), A_j(\mathbf{y})]_- = -\frac{2\pi^2 i}{\theta} \partial_i^x \partial_j^y \Delta(\mathbf{x}, \mathbf{y}) \tag{2.17a}$$

$$[A_i(\mathbf{x}), \pi_j(\mathbf{y})]_- = \frac{i}{2} \varepsilon_{jk} \partial_i^x \partial_j^y \Delta(\mathbf{x}, \mathbf{y}) \tag{2.17b}$$

$$[\pi_i(\mathbf{x}), \pi_j(\mathbf{y})]_- = -\frac{i\theta}{8\pi^2} \varepsilon_{il}\varepsilon_{jm} \partial_l^x \partial_m^y \Delta(\mathbf{x}, \mathbf{y}). \tag{2.17c}$$

Note that all the commutators in this sector are equal to c-numbers. As a result there are no operator ordering problems.

Finally, we present the details of the commutators in the mixed sector:

$$[\phi(\mathbf{x}), A_i(\mathbf{y})]_- = \frac{2\pi^2}{\theta} [\partial_i^y \Delta(\mathbf{x}, \mathbf{y}) - \varepsilon_{ij} K_j(\mathbf{x}, \mathbf{y})] \phi(\mathbf{x}) \tag{2.18a}$$

$$[\phi(\mathbf{x}), \pi_i(\mathbf{y})]_- = -\frac{1}{2} [\varepsilon_{ij} \partial_j^y \Delta(\mathbf{x}, \mathbf{y}) - K_i(\mathbf{x}, \mathbf{y})] \phi(\mathbf{x}) \tag{2.18b}$$

$$[\pi(\mathbf{x}), A_i(\mathbf{y})]_- = -\frac{2\pi^2}{\theta} [\partial_i^y \Delta(\mathbf{x}, \mathbf{y}) - \varepsilon_{ij} K_j(\mathbf{x}, \mathbf{y})] \pi(\mathbf{x}) \tag{2.18c}$$

$$[\pi(\mathbf{x}), \pi_i(\mathbf{y})]_- = \frac{1}{2} [\varepsilon_{ij} \partial_j^y \Delta(\mathbf{x}, \mathbf{y}) - K_i(\mathbf{x}, \mathbf{y})] \pi(\mathbf{x}). \tag{2.18d}$$

As in the matter sector the other commutators in this sector can be easily worked out by using hermiticity and/or antisymmetry.

Before we conclude this section a few general comments are in order. The structure of the commutators in the matter sector tells us that one can in fact continuously interpolate between bosonic and fermionic limits by smoothly varying θ. This is the first signal of the possibility of the matter fields behaving like anyons. We will return to a detailed discussion of this question in Section 3. It is interesting to check the internal consistency of the algebra of the fields in the gauge sector using the definition of the momenta π_i. Such an exercise simply demonstrates that there is only one independent commutator in the gauge sector—a feature of the symplectic structure of Chern–Simons theory by virtue of which the two spatial components of the gauge fields are conjugate to each other. Last, note that if in the lagrangian defining the model, Eq. (2.1), we make the replacement $A_\mu \to \theta^{-1/2} A_\mu$ then θ appears in the covariant derivative and is therefore a measure of the strength of the minimal coupling between the matter and gauge fields. Under such a scaling note that, while θ disappears from the r.h.s. of commutators in the gauge sector, it remains unaltered in the matter sector and becomes $\theta^{-1/2}$ in the mixed sector. If we now look at the limit $\theta \to \infty$, i.e., the limit in which the matter fields decouple from the gauge fields, we find that the matter fields commute amongst themselves and also with the gauge fields, as indeed they should. The gauge fields themselves do not commute amongst themselves which is, again, a consequence of the symplectic structure of the CS term.

3. Spin, Statistics, and Particle Interpretation

As mentioned earlier the set of commutation relations presented in the previous section suggests the possibility of having anyons in our model. In order to explore this possibility fully, let us recast Eq. (2.16a) in a more suggestive form,

$$\phi(\mathbf{x}) \phi(\mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \phi(\mathbf{x}), \tag{3.1}$$

where

$$\lambda(\mathbf{x}, \mathbf{y}) = \frac{(1 + (i\pi^2/\theta)\, \Delta(\mathbf{x}, \mathbf{y}))}{(1 - (i\pi^2/\theta)\, \Delta(\mathbf{x}, \mathbf{y}))}. \tag{3.2}$$

Note that, since $\lambda(\mathbf{x}, \mathbf{y})\, \lambda^*(\mathbf{x}, \mathbf{y}) = 1$, the above equations imply

$$\phi(\mathbf{x})\, \phi(\mathbf{y}) = [\exp i\Theta(\mathbf{x}, \mathbf{y})]\, \phi(\mathbf{y})\, \phi(\mathbf{x}), \tag{3.3}$$

where $\Theta(\mathbf{x}, \mathbf{y})$ is real. Thus, under a permutation, the product of two basic fields in the theory picks up a phase factor. As it stands, however, there are several worrisome aspects of this phase which need to be clarified before we can conclude that we have anyons in the theory. As is well known from quantum mechanics, when two anyons are exchanged, for non-trivial results, the phase picked up must be multivalued. It is natural to expect a similar feature to be reflected at the field theoretic level we are working. A related point in the comparison with quantum mechanical result concerns the two possible phases that can occur in quantum mechanics, depending on the way one particle is rotated around the other, namely, clockwise or anticlockwise. The above result does not appear to accommodate such a possibility. Most objectionably, Eq. (3.3) implies that the statistics of the fields under consideration depends upon their spatial coordinates as well as the choice of gauge fixing kernels.[1] This is physically absurd. It is, however, not very difficult to realise that all these problems are rooted in the fact that the basic fields we are considering do not correspond to physical particles because they are gauge non-invariant. Choosing the gauge transformation parameter in such a way that the element of the gauge group goes to the identity at the spatial infinity, gauge invariant operators in the theory can be constructed according to the standard line integral prescription of Schwinger and are given by

$$\hat{\phi}(\mathbf{x}) = P\left[\exp i \int_{\infty}^{\mathbf{x}} dz^i A_i(\mathbf{z})\right] \phi(\mathbf{x}). \tag{3.4}$$

The line integral appearing in the above expression is along a spacelike path from the point at infinity to $\mathbf{x}$, on a fixed time slice. The gauge invariant field, therefore, depends not only on the point $\mathbf{x}$ but also on the whole path. It is completely defined once the path is prescribed. Hence, to be precise we need to denote the path dependence of the operator explicitly. The $\mathbf{x}$ dependence itself can be spared explicit mention because it is redundant to specify the end-point of a path when the path itself is specified completely. Nevertheless, since we are interested in examining the algebra of the gauge invariant fields, it is sufficient to display the $\mathbf{x}$-dependence explicit and suppress the explicit display of the path dependence for the sake

[1] Strictly speaking one has to solve Eq. (2.14) before making such an assertion. As we will see presently such an exercise reveals that the exponential in Eq. (3.3) actually collapses to unity.

of brevity. Note that it is necessary to path order the exponential in the above expression, even in the abelian theory being considered here, because of the non-commutativity of the gauge fields $A_i(\mathbf{z})$, obvious from Eq. (2.17a). It is now straightforward to compute the algebra of these gauge-invariant operators. Using the result of the previous section this works out to be

$$\hat{\phi}(\mathbf{x})\,\hat{\phi}(\mathbf{y})=\Bigg[\exp i\Theta(\mathbf{x},\mathbf{y})\times\exp\frac{2\pi^2 i}{\theta}\int_\infty^{\mathbf{x}} dz_1^i\int_\infty^{\mathbf{y}} dz_2^j\,\partial_i^{z_1}\partial_j^{z_2}\Delta(\mathbf{z}_1,\mathbf{z}_2)$$
$$\times\exp\frac{2\pi^2 i}{\theta}\Bigg(\int_\infty^{\mathbf{x}} dz^i\varepsilon_{ij}K_j(\mathbf{y},\mathbf{z})$$
$$-\int_\infty^{\mathbf{y}} dz^i\varepsilon_{ij}K_j(\mathbf{x},\mathbf{z})\Bigg)\Bigg]\hat{\phi}(\mathbf{y})\,\hat{\phi}(\mathbf{x}). \tag{3.5}$$

Although the above algebra looks more complicated than that in Eq. (3.3), the appearance of the exponentials of the line integrals of the gauge fixings kernels in the right-hand side augurs well for our cherished goal of obtaining multivaluedness of the phase factor. All the other problems listed in the context of Eq. (3.3), however, persist. In order to make any headway, therefore, we now have to solve Eq. (2.14) for the kernels $K_i(\mathbf{x},\mathbf{y})$. Towards such an end let us express them through the relation

$$K_i(\mathbf{x},\mathbf{y})\equiv-\partial_i^{y}\Phi(\mathbf{x},\mathbf{y}). \tag{3.6}$$

Plugging Eq. (3.6) into Eq. (2.14), we find that $\Phi(\mathbf{x},\mathbf{y})$ is merely the two-dimensional massless propagator and is given by

$$\Phi(\mathbf{x},\mathbf{y})=\frac{1}{2\pi}\ln\mu|\mathbf{x}-\mathbf{y}|, \tag{3.7}$$

where μ is an infrared cutoff. As is well known, the above expression for the propagator is not unique. In particular, one can always add to the right-hand side of Eq. (3.7) any arbitrary solution $f(\mathbf{x},\mathbf{y})$ of the two-dimensional Laplace equation. These merely correspond to the zero modes of the two-dimensional Laplace operator. However, if we require such solutions to be regular over the entire two-dimensional space, then the only possible solutions for $f(\mathbf{x},\mathbf{y})$ are constants and hence they do not alter Eq. (3.6). Let us further define

$$G_i(\mathbf{x},\mathbf{y})\equiv\varepsilon_{ij}K_j(\mathbf{x},\mathbf{y})\equiv-\partial_i^{y}\Psi(\mathbf{x},\mathbf{y}) \tag{3.8}$$

and concentrate upon the line integrals in the last exponential on the r.h.s. of Eq. (3.5). They work out to be

$$\int_\infty^{\mathbf{x}} d\mathbf{z}^i\varepsilon_{ij}K_i(\mathbf{y},\mathbf{z})-\int_\infty^{\mathbf{y}} d\mathbf{z}^i\varepsilon_{ij}K_j(\mathbf{x},\mathbf{z})$$
$$=-\Psi(\mathbf{y},\mathbf{x})+\Psi(\mathbf{x},\mathbf{y})+\Psi(\mathbf{y},\infty)-\Psi(\mathbf{x},\infty). \tag{3.9}$$

It also follows from Eqs. (3.6) and (3.8) that

$$\partial_i^y \Phi(\mathbf{x}, \mathbf{y}) = \varepsilon_{ij} \partial_j^y \Psi(\mathbf{x}, \mathbf{y}). \tag{3.10}$$

Note that this equation is in conflict with Eq. (2.14) if $\mathbf{x} = \mathbf{y}$. Hence in what follows we will always work on the punctured two-dimensional plane which is obtained by assuming $\mathbf{x}$ to be fixed and deleting the above troublesome point from $\mathbf{R}^2$. This is an example of the diagonal point alluded to in Section 1. We immediately recognise the above to be the Cauchy–Riemann equations if we treat y_1 and y_2 as the real and imaginary parts of a complex variable. $\Phi(\mathbf{x}, \mathbf{y})$ and $\Psi(\mathbf{x}, \mathbf{y})$ are then the real and imaginary parts of an analytic function of such a complex variable. It follows from the fact that the real part of this analytic function is given by Eq. (3.7) that

$$\Psi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \arctan\left(\frac{x_2 - y_2}{x_1 - y_1}\right) \tag{3.11}$$

which is just the angle that the $\mathbf{x} - \mathbf{y}$ vector makes with the $x_1 - y_1$ axis. The first two terms on the r.h.s. of Eq. (3.9), therefore, merely give the relative angle between the $\mathbf{x} - \mathbf{y}$ and $\mathbf{y} - \mathbf{x}$ vectors, apart from the overall factor of $1/2\pi$. Hence,

$$\Psi(\mathbf{x}, \mathbf{y}) - \Psi(\mathbf{y}, \mathbf{x}) = \frac{1}{2\pi} (\pi \bmod 2\pi). \tag{3.12}$$

The remaining terms cancel each other because all lines pointing towards infinity are parallel and hence the relative angle between them is zero. This argument can be formalised as follows: In polar coordinates $\Psi(\mathbf{x}, \infty)$ can be written as

$$\Psi(\mathbf{x}, \infty) = \frac{1}{2\pi} \lim_{R \to \infty} \arctan\left(\frac{x_2 - R \sin\alpha}{x_1 - R\cos\alpha}\right), \tag{3.13}$$

where we have introduced the two-dimensional radial vector $\mathbf{R} = (R \sin\alpha, R\cos\alpha)$, α being the polar angle. For $R \to \infty$, x_2 and x_1 can be neglected and Eq. (3.13) reduces to

$$\Psi(\mathbf{x}, \infty) = \frac{1}{2\pi} \lim_{R \to \infty} \arctan\left(\frac{R\sin\alpha}{R\cos\alpha}\right) = \frac{\alpha}{2\pi}. \tag{3.14}$$

This term being a constant, precisely cancels with $\Psi(\mathbf{y}, \infty)$. Now that the structure of the gauge-fixing kernels is completely fixed, it is worthwhile computing $\Delta(\mathbf{x}, \mathbf{y})$. Recall from Eq. (2.15) that

$$\Delta(\mathbf{x}, \mathbf{y}) = \int d^2\mathbf{z}\, \varepsilon^{ij} K_i(\mathbf{x}, \mathbf{z})\, K_j(\mathbf{y}, \mathbf{z}).$$

Substituting for $K_i(\mathbf{x}, \mathbf{z})$ from Eqs. (3.6) and (3.7), we get obtain

$$\Delta(\mathbf{x}, \mathbf{y}) = \int d^2\mathbf{z}\, \varepsilon^{ij} \frac{(z_i - x_i)(z_j - y_j)}{|\mathbf{z}-\mathbf{x}|^2\, |\mathbf{z}-\mathbf{y}|^2}. \tag{3.15}$$

Although the above integral is superficially logarithmically divergent, the fact that the higest power of z in the numerator, viz., the $z_i z_j$ term is zero because it is multiplied by ε^{ij}, makes it convergent. It is then easy to show, following the sequence of steps $\mathbf{z} \to \mathbf{z}' = \mathbf{z} + (\mathbf{x}+\mathbf{y})/2$, $i \leftrightarrow j$, and $\mathbf{z} \to -\mathbf{z}$, that

$$\Delta(\mathbf{x}, \mathbf{y}) = -\Delta(\mathbf{x}, \mathbf{y}) = 0. \tag{3.16}$$

The upshot of this result is that in the r.h.s. if Eq. (3.5) the second exponential and the irksome first exponential both collapse to unity. We therefore arrive at the neat result

$$\hat{\phi}(\mathbf{x})\, \hat{\phi}(\mathbf{y}) = \left[\exp i \left(\frac{\pi}{\theta}\right) (\pi \bmod 2\pi) \right] \hat{\phi}(\mathbf{y})\, \hat{\phi}(\mathbf{x}). \tag{3.17}$$

The multivalued exponential factor in the above equation is then recognised to be the one-dimensional representation of the braid group [15]. Thus gauge-invariant operators in the theory defined in Eq. (3.4) are anyon field operators. Note that the path ordering in the above definition can now be dropped without ado because the gauge fields commute. The values $\theta = \pi/(2n+1)$, where n is an integer are particularly interesting because for these values of θ the gauge invariant operators anticommute and hence behave like fermions. Similarly, for $\theta = \pi/2n$, the gauge invariant operators commute and behave like bosons. These results may be compared with the ones obtained in Ref. [16], where the statistics of point particle sources coupled to an abelian Chern–Simons field was worked out by successfully computing the ratio between the scattering amplitudes of the exchange and direct processes within the path integral framework. To return to the present problem we find that exactly similar results hold for $\hat{\phi}^*$s. Now we can use the conventional Fock space methods to construct an N particle state by acting a string of $N\hat{\phi}^*$ operators on the vacuum. This yields

$$\hat{\phi}^*(\mathbf{x}_1)\, \hat{\phi}^*(\mathbf{x}_2) \cdots \hat{\phi}^*(\mathbf{x}_N)\, |0\rangle = \left[\exp \frac{\pi^2 i}{\theta} \sum_{i,j} \Psi(\mathbf{x}_i - \mathbf{x}_j) \right] |\ \rangle, \tag{3.18}$$

where $|\ \rangle$ is a single-valued state. Switching over to complex coordinates $\mathbf{z}_i = \mathbf{x}_{i1} + i\mathbf{x}_{i2}$, $\bar{\mathbf{z}}_i = \mathbf{x}_{i1} - i\mathbf{x}_{i2}$, where i and j run from 1 to N, the above equation can be recast as

$$\hat{\phi}^*(\mathbf{z}_1, \bar{\mathbf{z}}_1)\, \hat{\phi}^*(\mathbf{z}_2, \bar{\mathbf{z}}_2) \cdots \hat{\phi}^*(\mathbf{z}_N, \bar{\mathbf{z}}_N)\, |0\rangle = \prod_{ij} (\mathbf{z}_i - \mathbf{z}_j)^{\pi/2\theta}\, |\ \rangle, \tag{3.19}$$

where an unimportant factor of $\prod_i |\mathbf{z}_i - \mathbf{z}_j|^{-\pi/\theta}$ has been absorbed in the definition of the single-valued state. The N particle state we have thus obtained is an analogue of Laughlin's ansatz for quantum Hall effect [17]. In terms of this state it is easy to see that when one particle is taken around another through an angle $2n\pi$, $n \in Z$, the multivalued state picks up a factor of $\exp(\pm i(\pi/\theta)\, n\pi)$, depending upon the rotation being clockwise or anticlockwise. This completes the description of the conventional Fock space of the theory.

The one-particle state obtained by adopting the above procedure is an eigenstate of the charge operator with an eigenvalue equal to one as can be easily checked by using the commutation relations between the basic fields.

$$Q\hat{\phi}^*(\mathbf{x})\,|0\rangle \equiv \int d^2\mathbf{x}'\, j_0(\mathbf{x}')\, \hat{\phi}^*(\mathbf{x})\,|0\rangle$$
$$= \hat{\phi}^*(\mathbf{x})\,|0\rangle. \tag{3.20}$$

The stage is now set for discussing the spin of the one anyon state. The angular momentum operator is defined by

$$J = \int d^2\mathbf{x}\, \varepsilon^{ij} x_i T_{0j}(\mathbf{x}), \tag{3.21}$$

where T_{0j} is an element of the completely covariant and symmetric energy momentum tensor $T_{\mu\nu}$ which is obtained by coupling the theory to gravity in the usual way and considering the variation of the action around the flat metric. By virtue of the fact that the Chern–Simons term is metric-independent,this definition of the energy momentum tensor is insensitive to its presence and is independent of θ. It reads

$$T_{\mu\nu} = (D_\mu \phi)^*\,(D_\nu \phi) + (D_\nu \phi)^*\,(D_\mu \phi) - g_{\mu\nu}(D_\lambda \phi)^*\,(D^\lambda \phi). \tag{3.22}$$

Hence

$$T_{0j}(\mathbf{x}) = (D_0\phi)^*\,(D_j\phi) + (D_j\phi)^*\,(D_0\phi)$$
$$= \pi(\mathbf{x})(\partial_j \phi(\mathbf{x})) + (\partial_j \phi^*(\mathbf{x}))\,\pi^*(\mathbf{x})$$
$$+ iA_j(\mathbf{x})(\pi(\mathbf{x})\,\phi(\mathbf{x}) - \phi^*(\mathbf{x})\,\pi^*(\mathbf{x})). \tag{3.23}$$

The first two terms are the normal canonical terms that appear for complex scalar fields. The third term, however, requires more care. Recall that because the time component of the momentum conjugate to the gauge field vanishes (Eq. 2.2c), the Heisenberg equation of motion

$$\dot{\pi}_0(\mathbf{x}) = i[H, A_0(\mathbf{x})] = 0$$

leads to the constraint in Eq. (2.7) which just corresponds to the zeroth component of Eq. (1.2) in this theory. This equation can be solved formally for A_i and it yields

$$A_i(\mathbf{x}) = -\frac{2\pi}{\theta}\,\varepsilon_{ij}\,\frac{\partial^j_x}{\partial^2}\, j_0(\mathbf{x}) + \partial_i f(\mathbf{x}), \tag{3.24}$$

where $f(\mathbf{x})$ is an arbitrary function. Requiring the above solution to satisfy the gauge fixing condition in Eq. (2.11b) we obtain

$$0 = \int d^2\mathbf{x}\, K_i(\mathbf{x}, \mathbf{y})\, A_i(\mathbf{x})$$

$$= -\frac{\pi}{\theta} \varepsilon_{ij} \int d^2\mathbf{x}\, K_i(\mathbf{x}, \mathbf{y})\, \partial_x^j \int d^2\mathbf{z} \ln \mu\, |\mathbf{x}-\mathbf{z}|\, j_0(\mathbf{z})$$

$$+ \int d^2\mathbf{x}\, K_i(\mathbf{y}, \mathbf{x})\, \partial_i^x f(\mathbf{x}). \tag{3.25}$$

This yields, upon doing integration by parts in the last two terms and using the property (2.14),

$$f(\mathbf{x}) = 0.$$

Hence Eq (3.24) reduces to

$$A_i(\mathbf{x}) = -\frac{\pi}{\theta} \varepsilon_{ij} \partial_x^j \int d^2\mathbf{y} \ln \mu\, |\mathbf{x}-\mathbf{y}|\, j_0(\mathbf{y}). \tag{3.26}$$

Therefore, we once again find that the zero modes of the two-dimensional operator $\varepsilon^{ij} \partial_i$ do not play any role. Substituting for $A_i(\mathbf{x})$ from above in the expression for the angular momentum operator and concentrating only on the anomalous third term which we denote by J_s, we obtain

$$J_s = \frac{2\pi^2}{\theta} \int d^2\mathbf{x}\, \varepsilon_{ij} x_i j_0(\mathbf{x}) \left(\varepsilon_{jk} \frac{\partial^k}{\partial^2} j_0(\mathbf{x}) \right)$$

$$= -\frac{\pi}{\theta} \int d^2\mathbf{x} \int d^2\mathbf{y}\, j_0(\mathbf{x})\, x_i \partial_x^i \ln \mu\, |\mathbf{x}-\mathbf{y}|\, j_0(\mathbf{y})$$

which reduces, after some simple algebra [18], to

$$J_s = -\frac{\pi}{2\theta} Q^2. \tag{3.27}$$

The rotation operator acting on the one-particle state is now easily seen to rotate the state by a phase:

$$[\exp iJ_s \omega]\, \hat{\phi}^* \,|0\rangle = \left[\exp - \frac{i\omega\pi}{2\theta} \right] \hat{\phi}^* \,|0\rangle. \tag{3.28}$$

If the parameter of rotation ω takes a value 2π, then it is easy to check that for $\theta = \pi/(2n+1)$ the state picks up a minus sign which is a signal of the fact that the state is fermionic, while for $\theta = \pi/2n$, it does not change implying that it is bosonic.

For other values of θ the state does not return to itself, except for a possible sign change, after a 2π rotation and is therefore anyonic—in agreement with the conclusions that were drawn from the algebra of the gauge-invariant operators (3.17). This, therefore, establishes a generalised connection between spin and statistics within the framework of a relativistic quantum field theory under consideration here.

We conclude this section presenting the details of the algebra of the remaining gauge invariant operators. As a first step towards the construction of these operators it is useful to rewrite the Lagrangian density (2.1) in terms of gauge invariant operators, as was first done by Mandelstam [19]. Thus, define

$$\hat{\phi}(x, P) = \left[\exp i \int_{\infty}^{x} dz^{\mu} A_{\mu}(z)\right] \phi(x) \tag{3.29a}$$

$$\hat{\phi}^*(x, P) = \phi^*(x) \left[\exp -i \int_{\infty}^{x} dz^{\mu} A_{\mu}(z)\right] \tag{3.29b}$$

The above operators share all the properties of the ones defined in Eq. (3.4). We have therefore adopted the same notation for both. These is, however, a slight difference between the two because the sum over the repeated index inside the integral in Eq. (3.29) is over (2+1)-dimensional spacetime, unlike the one in Eq. (3.4) which in over two spatial dimensions only. The abuse of notation, however, is justified since in what follows we will always take the paths to lie on equal-time planes. In this case the difference referred to above disappears. The gauge covariant derivatives of $\hat{\phi}$ can also be defined, following Mandelstam, as

$$\partial_{\mu}\hat{\phi}(x, P) = \lim_{dx_{\mu} \to 0} \frac{\hat{\phi}(x + dx_{\mu}, P') - \hat{\phi}(x, P)}{dx_{\mu}}, \tag{3.30}$$

where the path P' is obtained from P simply by extending it by dx_{μ} in the μ direction. Thus the new path P' passes through the point x. Moreover, since it does not enclose any area with the path P, there are no additional contributions coming from the flux that is enclosed by the area between the two paths P and P' when the derivative acts on $\hat{\phi}$. In the special case of taking the time derivative of $\hat{\phi}$, we need to, in principle, allow the paths to have infinitesimal timelike portions. As it turns out, however, at least in so far as our purposes are concerned, it will not be necessary for us to ever explicitly compute this. It follows easily then that

$$\partial_{\mu}\hat{\phi}(x, P) = D_{\mu}\phi(x) \tag{3.31}$$

and Eq. (2.1) takes the form

$$\mathcal{L} = (\partial_{\mu}\hat{\phi})^*(\partial^{\mu}\hat{\phi}) + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_{\mu} \partial_{\nu} A_{\lambda}. \tag{3.32}$$

The striking thing about this lagrangian density is that the minimal interaction between the matter fields and the gauge fields is now eliminated. The price one has to pay is that the matter fields now become path-dependent, consequently multi-valued, and have non-trivial statistics. This trade-off between interaction and statistics is in total conformity with the quantum mechanical example mentioned in Section 1.

The momentum canonically conjugate to $\hat{\phi}$ can now be defined in the usual way by varying the above lagrangian density with respect to $\partial_0\hat{\phi}$. Nevertheless, it is more convenient to invert the relations in Eq (3.29) as shown below:

$$\phi(x) = \left[\exp -i\int_{\infty}^{x} dz^{\mu}A_{\mu}(z)\right]\hat{\phi}(x) \tag{3.33a}$$

$$\phi^{*}(x) = \hat{\phi}^{*}(x)\left[\exp i\int_{\infty}^{x} dz^{\mu}A_{\mu}(z)\right], \tag{3.33b}$$

where we have suppressed the explicit path dependence once again by appealing to brevity. Plugging the above equations into Eq. (2.1) yields

$$\begin{aligned}\mathscr{L} = &\left(D_{\mu}^{*}\left(\hat{\phi}^{*}\left[\exp i\int_{\infty}^{x} dz_1^{\mu}A_{\mu}(z_1)\right]\right)\right)\\ &\times\left(D^{\mu}\left(\left[\exp -i\int_{\infty}^{x} dz_2^{\nu}A_{\nu}(z_2)\right]\hat{\phi}(x)\right)\right)\\ &+\frac{\theta}{4\pi^2}\varepsilon^{\mu\nu\lambda}A_{\mu}\,\partial_{\nu}A_{\lambda}.\end{aligned} \tag{3.34}$$

The momentum canonically conjugate to $\hat{\phi}$ can now be calculated as

$$\begin{aligned}\hat{\pi}(x) &\equiv \frac{\delta\mathscr{L}}{\delta(\partial_0\hat{\phi}(x))}\\ &= \left(D_0\left(\hat{\phi}^{*}(x)\left[\exp i\int_{\infty}^{x} dz_1^{\mu}A_{\mu}(z_1)\right]\right)\right)\times\left[\exp -i\int_{\infty}^{x} dz_2^{\nu}A_{\nu}(z_2)\right]\\ &= \hat{\phi}^{*}(x)\left(\partial_0\left[\exp i\int_{\infty}^{x} dz_1^{\mu}A^{\mu}(z_1)\right]\right)\times\left[\exp -i\int_{\infty}^{x} dz_2^{\nu}A_{\nu}(z_2)\right]+(D_0\hat{\phi}(x))^{*}\\ &= (D_0\hat{\phi}(x))^{*}-\phi^{*}(x)\left(\partial_0\left[\exp -i\int_{\infty}^{x} dz_2^{\nu}A_{\nu}(z_2)\right]\right)\\ &= \pi(x)\left[\exp -i\int_{\infty}^{x} dz^{\mu}A_{\mu}(z)\right],\end{aligned} \tag{3.35a}$$

where we have used, in addition to Eq. (3.33b), the following non-trivial result which itself is a consequence of an essentially trivial resolution of the identity

$$0=\partial_0(\mathbf{1})=\partial_0\left(\left[\exp i\int_{\infty}^{x} dz_1^{\mu} A_{\mu}(z_1)\right]\left[\exp -i\int_{\infty}^{x} dz_2^{\nu} A_{\nu}(z_2)\right]\right)$$

$$=\left(\partial_0\left[\exp i\int_{\infty}^{x} dz_1^{\mu} A_{\mu}(z_1)\right]\right)\left[\exp -i\int_{\infty}^{x} dz_2 A_{\nu}(z_2)\right]$$

$$+\left[\exp i\int_{\infty}^{x} dz_1^{\mu} A_{\mu}(z_1)\right]\left(\partial_0\left[\exp -i\int_{\infty}^{x} dz_2^{\nu} A_{\nu}(z_2)\right]\right).$$

An exactly similar calculation or just hermitian conjugation, also gives the expression

$$\hat{\pi}^*(x)\equiv\frac{\delta\mathscr{L}}{\delta(\partial_0\hat{\phi}(x))^*}\doteq\left[\exp i\int_{\infty}^{x} dz^{\mu} A_{\mu}(z)\right]\pi^*(x). \tag{3.35b}$$

It is now straightforward to work out the algebra of the gauge-invariant operators by using the known commutation relations between the basic fields in the theory. On an equal-time plane, it works out to be the elegant structure displayed below:

$$\hat{\phi}(\mathbf{x})\,\hat{\phi}(\mathbf{y})=\left[\exp i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\phi}(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{3.36a}$$

$$\hat{\phi}(\mathbf{x})\,\hat{\phi}^*(\mathbf{y})=\left[\exp -i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\phi}^*(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{3.36b}$$

$$\hat{\phi}(\mathbf{x})\,\hat{\pi}(\mathbf{y})=\delta^2(\mathbf{x}-\mathbf{y})+\left[\exp -i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\pi}(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{3.36c}$$

$$\hat{\phi}(\mathbf{x})\,\hat{\pi}^*(\mathbf{y})=\left[\exp i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\pi}^*(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{3.36d}$$

$$\hat{\pi}(\mathbf{x})\,\hat{\pi}(\mathbf{y})=\left[\exp i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\pi}(\mathbf{y})\,\hat{\pi}(\mathbf{x}) \tag{3.36e}$$

$$\hat{\pi}(\mathbf{x})\,\hat{\pi}^*(\mathbf{y})=\left[\exp -i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\pi}^*(\mathbf{y})\,\hat{\pi}(\mathbf{x}). \tag{3.36f}$$

The rest of the algebra can be obtained by hermitian conjugation.

In this section we have comprehensively demonstrated that the systematic quantization scheme developed in Section 2 can be used to show that the manifestly gauge-invariant operators in the theory carry fractional spin and statistics and obey a generalised connection between them. When expressed in terms of these variables the matter and gauge sectors, hitherto minimally interacting, become decoupled. The new gauge-invariant matter field operators are free except for the complicated statisical interactions they carry by virtue of their multi-valuedness. The particle interpretation that can be given by using conventional Fock space methods leads to a multi-particle state which is analogous to the Laughlin state. It should be borne in mind, however, that all the results in this section hinge crucially on the constraint structure of the theory discussed elaborately in the last section. Since the constraint structure of the model is known to become drastically modified with the addition of a Maxwell term to the lagrangian density, it is worthwhile to ask at this juncture, if the same, or similar, constructions still work to produce manifestly gauge-invariant anyon operators. The next section analyses this issue.

4. Addition of the Maxwell Term

The addition of the Maxwell term to the lagrangian density (2.1) yields what may be called topologically massive scalar electrodynamics, adapting the terminology of Deser *et al.* [20]. The interest in this model emerges because pure Chern–Simons scalar electrodynamics, dealt with in the previous two sections, can be considered as a low energy limit of this model. Moreover, it has been argued that, even if the Maxwell term is absent at the tree level, there is no symmetry which prevents its generation by quantum corrections. Thus, it is of considerable interest for us to know if the results obtained in the last section hold even in the presence of the Maxwell term or whether they are merely low energy artefacts.

The model is defined by

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) + \frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}. \tag{4.1}$$

We adopt the same conventions as in Section 2 and once again the covariant current is conserved. The canonically conjugate momenta are defined and given by

$$\pi \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)} = (D_0 \phi)^* \tag{4.2a}$$

$$\pi^* = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi^*)} = (D_0 \phi) \tag{4.2b}$$

$$\pi_0 = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_0)} = 0 \tag{4.2c}$$

$$\pi_i \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 A^i)} = \frac{\theta}{4\pi^2} \varepsilon_{ij} A^j - \frac{1}{e^2} F_{0i}. \tag{4.2d}$$

It is immediately obvious that, unlike the pure Chern–Simons scalar electrodynamics model, in the present model, Eqs. (4.2a), (4.2b), (4.2d) define momenta in terms of velocities and are, therefore, not constraints. Equation (4.2c), however, leads to the primary constraint

$$\pi_0 \approx 0. \tag{4.3}$$

The fact that the other two primary constraints, present in the pure Chern–Simons case, are absent here is because the Maxwell term, unlike the Chern–Simons term, is bilinear in derivatives with respect to the gauge fields. The Hamiltonian of the model can be worked out as usual and is given by the expression, $H = \int d^2\mathbf{x}\, \mathscr{H}(x)$, where

$$\begin{aligned}\mathscr{H} &= \pi^*\pi + (D_i\phi)^*\,(D_i\phi) - \frac{\theta}{4\pi^2}\,\varepsilon^{ij}(A_0\,\partial_i A_j)\\ &+ \frac{1}{4e^2}\,F_{ij}F^{ij} + j_0 A^0\\ &- \frac{e^2}{2}\left(\pi_i - \frac{\theta}{4\pi^2}\,\varepsilon_{ij}\pi^j\right)\left(\pi^i - \frac{\theta}{4\pi^2}\,\varepsilon^{ik}A_k\right) + \pi_i\,\partial^i A_0.\end{aligned} \tag{4.4}$$

Requiring the primary constraint (4.2c) to be preserved in time yields the following secondary constraint:

$$S_0 \equiv -j_0 + \frac{\theta}{4\pi^2}\,\varepsilon^{ij}\,\partial_i A_j + \partial^i\pi_i \approx 0. \tag{4.5}$$

It is straightforward to check that there are no more constraints in the theory and that both (4.2c) and (4.5) are first-class constraints. Thus, while the addition of the Maxwell term drastically alters the constraint structure of the theory in so far as the second-class constraints are concerned, the number of first-class constraints and, consequently, the gauge invariances in the two theories are the same. Since our main motivation is to study the effect of adding the Maxwell term, we will adopt the same gauge-fixing conditions that were used in Section 2, viz.,

$$\int d^2\mathbf{y}\; K_0(\mathbf{x}, \mathbf{y})\, A_0(x_0, \mathbf{y}) \approx 0$$

and

$$\int d^2\mathbf{y}\; K_i(\mathbf{x}, \mathbf{y})\, A_i(x_0, \mathbf{y}) \approx 0,$$

where the kernels $K_i(\mathbf{x}, \mathbf{y})$ in the latter equation are again constrained by

$$\partial_i^y K_i(\mathbf{x}, \mathbf{y}) = \delta^2(\mathbf{x}, \mathbf{y}).$$

A rerun of the Dirac bracket machinery yields the following algebra for the basic fields in the theory:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\,\delta^2(\mathbf{x}-\mathbf{y}) \tag{4.6a}$$

$$[\phi(\mathbf{x}), \pi_i(\mathbf{y})] = -K_i(\mathbf{x}, \mathbf{y})\,\phi(\mathbf{x}) \tag{4.6b}$$

$$[\pi(\mathbf{x}), \pi_i(\mathbf{y})] = K_i(\mathbf{x}, \mathbf{y})\,\pi(\mathbf{x}) \tag{4.6c}$$

$$[A_i(\mathbf{x}), \pi_j(\mathbf{y})] = i\varepsilon_{ij}\,\delta^2(\mathbf{x}-\mathbf{y}) + i\,\partial_i^x K_j(\mathbf{x}, \mathbf{y}) \tag{4.6d}$$

$$[\pi_i(\mathbf{x}), \pi_j(\mathbf{y})] = \frac{i\theta}{4\pi^2}\,\varepsilon_{ij}\,\delta^2(\mathbf{x}-\mathbf{y}). \tag{4.6e}$$

The rest of the commutators can be obtained by hermitian conjugation and/or antisymmetry. The other commutators involving the basic fields which do not follow from above, e.g., $[A_i(\mathbf{x}), A_j(\mathbf{y})]$ are all trivial.

At this juncture it is important to note that, while the lagrangian density (4.1) and the two first class constraints (4.3) and (4.5) smoothly reduce to the corresponding expressions (2.1), (2.2c), and (2.7) when we take the limit $e^2 \to \infty$ and use Eq. (4.2d), the algebraic structure in Eq. (4.6) does not reduce to the corresponding structure in Section 2. This stems from the fact that the algebra of the gauge non-invariant basic fields of the theory is determined by the entire constraint structure of the theory. In the present case the constraint structure of the theory becomes drastically altered as we take the limit $e^2 \to \infty$ because the number of second-class constraints jumps from zero to two. This is a discontinuous change. It is, therefore, clear why taking this limit is not smooth at the level of the algebra of the basic fields of the theory. In view of this subtle complication, irrespective of the value of e^2, we have to use the commutation relations in Eq. (4.6) to work out the algebra of any composite operators we may construct in this theory. Such a construction is what we turn to now.

Gauge invariant operators as in Section 3 can once again be constructed by appealing to Schwinger's line integral prescription. However, since the matter fields and the gauge fields all commute amongst themselves, it is easy to check that the operators defined in Section 3 actually have trivial commutation relations. Can one, therefore, conclude that the Maxwell term incorporates a mechanism for suppression of fractional statistics in the theory? A little bit of thinking tells us that this is not quite the end of the story. In fact there exists a plethora of gauge-invariant operators in the theory, which may still have non-trivial statistics because of the unconventional algebra in Eq. (4.6). We choose to pick one which has the following merit: it reduces to the operator presented in Eq. (3.4) in limit $e^2 \to \infty$. It is given by

$$\begin{aligned}\hat{\phi}(\mathbf{x}) &= P\left[\exp -\frac{2\pi^2 i}{\theta}\int_\infty^{\mathbf{x}} dz^i \left(\varepsilon_{ij}\pi^j(\mathbf{z}) - \frac{\theta}{4\pi^2}A_i(\mathbf{z})\right)\right]\phi(\mathbf{x}) \\ &= P\left[\exp i\int_\infty^{\mathbf{x}} dz^\mu \left(A_\mu(z) - \frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z)\right)\right]\phi(x).\end{aligned} \tag{4.7}$$

The paths that appear in the definition of the above operator share all the properties of the ones in Eq. (3.4). Using the commutation relations of the basic fields and all the properties of the kernels discussed in the last section, it is now straightforward, although tedious, to show that

$$\hat{\phi}(\mathbf{x})\,\hat{\phi}(\mathbf{y})=\left[\exp\left(\frac{i\pi^2}{\theta}\right)\int_{\infty}^{\mathbf{x}} dz_1^i \int_{\infty}^{\mathbf{y}} dz_2^j\,\varepsilon_{ij}\delta^2(\mathbf{z}_1-\mathbf{z}_2)\right]$$
$$\times\left[\exp\left(i\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\phi}(\mathbf{y})\,\hat{\phi}(\mathbf{x}). \qquad (4.8)$$

The additional phase in the first exponential on the right-hand side of Eq. (4.8) is a result of the non-commutativity of the exponents in the definition of the $\hat{\phi}$s. It measures the difference between the right intersections and the left intersections of the path from ∞ to $\mathbf{x}$ and from ∞ to $\mathbf{y}$ [21]. For paths which do not intersect, the above algebra reduces to

$$\hat{\phi}(\mathbf{x})\,\hat{\phi}(\mathbf{y})=\left[\exp i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\phi}(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \qquad (4.9)$$

which is identical to the algebra in Eq. (3.36a). Similary, we can define

$$\hat{\phi}^*(\mathbf{x})=\phi^*(\mathbf{x})\bar{P}\left[\exp\frac{2\pi^2 i}{\theta}\int_{\infty}^{\mathbf{x}} dz^i\left(\varepsilon_{ij}\pi^j(\mathbf{z})-\frac{\theta}{4\pi^2}A_i(\mathbf{z})\right)\right]$$
$$=\phi^*(\mathbf{x})\,\bar{P}\left[\exp -i\int_{\infty}^{\mathbf{x}} dz^\mu\left(A_\mu(z)-\frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z)\right)\right], \qquad (4.10)$$

where $\bar{P}$—the anti-path-ordering operator—orders products of operators in the exponential along the path in exactly the opposite way a normal path-ordering operator P does. Equations (4.7) and (4.9) can once again be inverted to yield

$$\phi(\mathbf{x})=\bar{P}\left[\exp\frac{2\pi^2 i}{\theta}\int_{\infty}^{\mathbf{x}} dz^i\left(\varepsilon_{ij}\pi^j(\mathbf{z})-\frac{\theta}{4\pi^2}A_i(\mathbf{z})\right)\right]\hat{\phi}(\mathbf{x})$$
$$=\bar{P}\left[\exp -i\int_{\infty}^{\mathbf{x}} dz^\mu\left(A_\mu(z)-\frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z)\right)\right]\hat{\phi}(\mathbf{x}) \qquad (4.11a)$$

$$\phi^*(\mathbf{x})=\hat{\phi}^*(\mathbf{x})\,P\left[\exp -\frac{2\pi^2 i}{\theta}\int_{\infty}^{\mathbf{x}} dz^i\left(\varepsilon_{ij}\pi^j(\mathbf{z})-\frac{\theta}{4\pi^2}A_i(\mathbf{z})\right)\right]$$
$$=\hat{\phi}^*(x)\,P\left[\exp i\int_{\infty}^{\mathbf{x}} dz^\mu\left(A_\mu(z)-\frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z)\right)\right]. \qquad (4.11b)$$

Substuting the above expressions in the lagrangian density (4.1) produces

$$\mathscr{L}=\left\{D_\alpha^*\left(\hat{\phi}^*(x)\,P\left[\exp i\int_\infty^x dz_1^\mu\left(A_\mu(z_1)-\frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z_1)\right)\right]\right)\right\}$$

$$\times\left\{D^\alpha\left(\bar{P}\left[\exp -i\int_\infty^x dz_2^\rho\left(A_\rho(z_2)-\frac{\pi^2}{e^2\theta}\varepsilon_{\rho\eta\sigma}F^{\eta\sigma}(z_2)\right)\right]\hat{\phi}(x)\right)\right\}$$

$$+\frac{\theta}{4\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu\,\partial_\nu A_\lambda-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}. \tag{4.12}$$

The resolution of the identity used in deriving Eq. (3.35b) generalises to

$$0=\partial_0(1)=\partial_0\left\{\left[P\exp i\int_\infty^x dz_1^\mu\left(A_\mu(z_1)-\frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z_1)\right)\right]\right.$$

$$\left.\times\left[\bar{P}\exp -i\int_\infty^x dz_2^\rho\left(A_\rho(z_2)-\frac{\pi^2}{e^2\theta}\varepsilon_{\rho\eta\sigma}F^{\eta\sigma}(z_2)\right)\right]\right\}$$

and a repetition of the steps leading to (3.35b) yields

$$\begin{aligned}\hat{\pi}(x)&=\frac{\delta\mathscr{L}}{\delta(\partial_0\hat{\phi}(x))}\\&=\pi(x)\,\bar{P}\left[\exp -i\int_\infty^x dz^\mu\left(A_\mu(z)-\frac{\pi^2}{e^2\theta}\varepsilon_{\mu\nu\lambda}F^{\nu\lambda}(z)\right)\right]\\&=\pi(x)\,\bar{P}\left[\exp\frac{2\pi^2 i}{\theta}\int_\infty^x dz^i\left(\varepsilon_{ij}\pi^j-\frac{\theta}{4\pi^2}A_i(z)\right)\right].\end{aligned} \tag{4.13}$$

It is now once again straightforward to work out the commutation relations of the gauge-invariant operators. They are

$$\hat{\phi}(\mathbf{x})\,\hat{\phi}^*(\mathbf{y})=\left[\exp -i\frac{\pi^2}{\theta}\int_\infty^x dz_1^i\int_\infty^y dz_2^j\varepsilon_{ij}\,\delta(\mathbf{z}_1-\mathbf{z}_2)\right]$$

$$\times\left[\exp -i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\hat{\phi}^*(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{4.14a}$$

$$\hat{\phi}(\mathbf{x})\,\hat{\pi}(\mathbf{y})=\delta^2(\mathbf{x}-\mathbf{y})$$

$$+\left\{\left[\exp -i\frac{\pi^2}{\theta}\int_\infty^x dz_1^i\int_\infty^y dz_2^j\varepsilon_{ij}\,\delta(\mathbf{z}_1-\mathbf{z}_2)\right]\right.$$

$$\left.\times\left[\exp -i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\right\}\hat{\pi}(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{4.14b}$$

$$\hat{\phi}(\mathbf{x})\,\hat{\pi}^*(\mathbf{y}) = \left\{\left[\exp i\frac{\pi^2}{\theta}\int_{\infty}^{x} dz_1^i \int_{\infty}^{y} dz_2^j \varepsilon_{ij}\,\delta(\mathbf{z}_1 - \mathbf{z}_2)\right] \times \left[\exp i\frac{\pi}{\theta}(\pi \bmod 2\pi)\right]\right\} \hat{\pi}^*(\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{4.14c}$$

$$\hat{\pi}(\mathbf{x})\,\hat{\pi}(\mathbf{y}) = \left\{\left[\exp i\frac{\pi^2}{\theta}\int_{\infty}^{x} dz_1^i \int_{\infty}^{y} dz^j \varepsilon_{ij}\,\delta(\mathbf{z}_1 - \mathbf{z}_2)\right] \times \left[\exp i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\right\} \hat{\pi}(\mathbf{y})\,\hat{\pi}(\mathbf{x}) \tag{4.14d}$$

$$\hat{\pi}(\mathbf{x})\,\hat{\pi}^*(\mathbf{y}) = \left\{\left[\exp -i\frac{\pi^2}{\theta}\int_{\infty}^{x} dz_1^i \int_{\infty}^{y} dz_2^j \varepsilon_{ij}\,\delta(\mathbf{z}_1 - \mathbf{z}_2)\right] \times \left[\exp -i\left(\frac{\pi}{\theta}\right)(\pi \bmod 2\pi)\right]\right\} \hat{\pi}^*(\mathbf{y})\,\hat{\pi}(\mathbf{x}). \tag{4.14e}$$

If the paths are chosen to be non-intersecting, it is interesting to note that the algebra in Eqs. (4.8) and (4.14) is exactly identical to that in Eq. (3.36). Moreover, the gauge-invariant operators presented in this section, as already pointed out, reduce to the corresponding ones in the previous section, without any problems, as $e^2 \to \infty$. In the process, however, it is important to note that the operators retain their commutation relations untarnished, although the two theories are completely unrelated through e at the level of the algebra of the basic fields. This limit, therefore, is actually smooth! What makes this possible is the fact that the statistics phase depends not on e, but, only on θ. This is a testimony of the fact that the Chern–Simons term alone is responsible for non-trivial statistics.

For e^2 finite, although the commutation relations for the gauge invariant fields presented for the two theories are identical, a fundamental difference persists. As discussed in the last section, following Eq. (3.32), in the pure Chern–Simons case, it is possible to eliminate the minimal interaction between the matter fields and the gauge fields in favour of non-trivial statistics for gauge-invariant operators. In the case of topologically massive scalar electrodynamics, however, the gauge-invariant operators, which have fractional statistics, constructed by us do not completely eliminate the original interaction in the theory. The reason for this complication can be traced to the fact that the addition of the Maxwell term does not merely alter the constraint structure of the theory. It leads to a short-range coulomb potential between the matter fields. As a consequence of this the gauge invariant fields have complicated interactions. The fact that the lagrangian density (4.1) cannot be expressed as a free theory, except for complicated statistical interactions between the gauge invariant fields implied by the commutation relations, is a reflection of this fact. It is worth stressing at this stage that there is a fundamental difference between the conclusions arrived at here and the corresponding ones known in literature [9,10]. It is well known that the gauge fields of the model discussed in this section, as opposed to the ones in the last section, are endowed with a topological mass. As a result of this the matter fields are now dressed by a cloud of gauge fields of finite extent rather than one of an infinite extent as in the pure Chern–Simons case. If the distance between two matter fields is less than the diameter of this cloud, the fields are interacting. If the distance is larger than the diameter then the interaction is screened and they behave like free particles. It was argued in Refs. [9, 10] that in the former case the fields have canonical statistics while in the latter case they exhibit anyonic behaviour. The gauge invariant operators we have constructed in this section have fractional statistics irrespective of whether the coulomb interaction is screened or not.

Before we conclude this section, a discussion of the spin–statistics theorem in topologically massive scalar electrodynamics is in order. The energy momentum tensor for this theory can also be worked out by coupling the theory to gravity and then varying the action with respect to the metric and then setting the metric flat. It works out to be

$$T_{\mu\nu} = (D_\mu\phi)^* (D_\nu\phi) + (D_\nu\phi)^* (D_\mu\phi) - \frac{1}{e^2} F^\sigma_{\ \mu} F_{\nu\sigma}$$
$$- g_{\mu\nu}\left\{(D_\sigma\phi)^* (D^\sigma\phi) - \frac{1}{4e^2} F_{\lambda\sigma} F^{\lambda\sigma}\right\}. \qquad (14.15)$$

Hence

$$T_{0i} = \pi(\partial_i\phi) + \pi^*(\partial_i\phi)^* + F_{ij}\left(\pi^j - \frac{\theta}{4\pi^2}\varepsilon^{jk} A_k\right)$$
$$+ iA_i(\pi\phi - \pi^*\phi^*). \qquad (14.16)$$

The angular momentum operator can be obtained by simply plugging in the above expression in the definition

$$J = \int d^2x \varepsilon_{ij} x_i T_{0j}.$$

It is easy to show after some straightforward algebra that the angular momentum operator can be written as

$$J = \int d^2\mathbf{x} \left[\pi^k(\varepsilon^{ij} x_i \partial_j \delta_{lk} + \varepsilon_{lk}) A^l + \varepsilon^{ij} x_i \left\{ (\pi\, \partial_j\phi + \pi^*\, \partial_j\phi^*) \right.\right.$$
$$\left.\left. + A_j\left(-j_0 + \frac{\theta}{4\pi^2} \varepsilon^{mn}\, \partial_m A_n + \partial^m \pi_m \right) \right\} - \partial^m(\pi_m \varepsilon^{ij} x_i A_j) \right]. \qquad (14.17)$$

The first two terms represent the angular momentum carried by the gauge fields and the canonical terms that appear for complex scalar fields, respectively, apart from a term that is proportional to the Gauss law constraint. Once again the third term is anomalous and, as in the last section, we denote it by J_s. It is interesting to note, however, that since this is a boundary term, we can freely use the asymptotic solutions of the gauge fields. Towards such an end note that we can try the general ansatz

$$A_i(\mathbf{x}) = C_1 K_i(0, \mathbf{x}) + C_2 G_i(0, \mathbf{x})$$

with the functions K_i and G_i defined as in Eqs. (3.6), (3.7), and (3.8). Requiring the above ansatz to satisfy the gauge-fixing condition (2.11b) yields

$$0 = \int d^3\mathbf{x}\, K_i(\mathbf{y}, \mathbf{x})\, A_i(\mathbf{x}) = C_1 \int d^3\mathbf{x}\, K_i(\mathbf{y}, \mathbf{x})\, K_i(\mathbf{0}, \mathbf{x}),$$

upon using Eq. (3.16). Hence $C_1 = 0$ and

$$A_i(\mathbf{x}) = C_2 \varepsilon_{ij} K_j(\mathbf{0}, \mathbf{x}) = C_2 \varepsilon_{ij}(x^j/x^2). \tag{4.18}$$

In order to fix the constant C_2 we can now mimic the arguments of Semenoff and Sodano, Ref. [10]. It is easy to check that the hamiltonian (4.4) yields, upon integrating out A_0, a term which depends on the infrared cutoff μ. The fact that the energy is independent of this cutoff is ensured by the neutrality condition

$$\int d^2\mathbf{x} \left(-j_0 + \frac{\theta}{2\pi^2} \varepsilon^{ij} \partial_i A_j \right) = 0. \tag{4.19}$$

Substituting the expression for the gauge field from Eq. (4.18) into the above equation produces

$$C_2 = \frac{\pi}{\theta} Q.$$

Hence,

$$A_i(\mathbf{x}) = \frac{\pi}{\theta} Q \varepsilon_{ij} \frac{x^j}{x^2}. \tag{4.20}$$

In conjuction with the Gauss law the neutrality condition also implies

$$\int d^2\mathbf{x}\, \partial^i F_{0i} = 0.$$

Thus there is no net flux escaping across the boundary of the two-dimensional space which means that the electric field is screened, thus lending credence to the qualitative arguments of the last paragraph. The asymptotic form of the momentum conjugate to the gauge field can also be worked out to be

$$\pi_m = -\frac{Q}{4\pi}\frac{x_m}{x^2}. \tag{4.21}$$

The boundary term in the expression for angular momentum can now be explicitly evaluated and it reduces to the expression, after some simple algebra,

$$J_s = -\frac{\pi}{2\theta} Q^2$$

which is the same as expression (3.27). It is also easy to check that the commutator of the charge operator with the gauge invariant operator (4.7) is conventional. Therefore all the arguments delineated in the discussion of spin in the last section follow here as well. The generalised spin statistics connection found in the last section, therefore, holds in the present section too. It is worth mentioning that in deriving the results in this section we have not committed to any particular length scale. The asymptotic solutions of the gauge fields have been used only because the anomalous term appearing in the definition of the angular momentum is a boundary term. Since the gauge fields vanish at spatial infinity, normally such a term would not have contributed to the angular momentum. However, as discussed in this section, they do not vanish sufficiently rapidly for the boundary term under consideration to be unimportant. In fact this very term gives the all-important fractional spin to the one-particle state. This result, therefore, like the one for statistics, is exact. Thus the gauge invariant operators presented in this section authentically represent anyons which do not undergo any spin or statistics transmutations as the length scale is changed.

5. Algebraic Consistency Conditions

In the last few sections we have carefully built up a structure of manifestly gauge-invariant anyon operators within the framework of a prototype relativistic field theory we have chosen to work with. Although this edifice was constructed by a straightforward application Dirac's method of quantization, there are a few questions on whose answers it is delicately poised. This section is devoted towards answering these nagging questions.

The foremost question in this regard concerns the fractional spin which follows from the anomalous piece in the definition of the angular momentum. While it is important to have this to have a consistent spin–statistics connection, it is equally important to know whether this anomalous Lorentz generator leaves the theory Poincaré invariant—a requirement any sensible relativistic quantum field theory must satisfy. It may be worth mentioning that Poincaré invariance is by no means obvious in both the Chern–Simons scalar electrodynamics and topologically massive scalar electrodynamics, given the unusual commutation relations of the basic fields in the theories, presented in Section 2 and 4.

Poincaré generators can be defined from the energy momentum tensor through the following relations:

$$P_\mu = \int d^2\mathbf{x}\, T_{0\mu} \tag{5.1a}$$

$$J = \int d^2\mathbf{x}\, \varepsilon_{0ij} x_i T_{0j} \tag{5.1b}$$

$$K_\mu = \int d^2\mathbf{x}\, x_\mu T_{00}. \tag{5.1c}$$

The P_μ, J, and K_μ in the above expressions represent translation, rotation, and boost generators, respectively. The entire Poincaré algebra is given by the commutation relations between the above generators,

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho})$$

$$[P_\mu, P_\nu]_- = 0$$

and

$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho),$$

where

$$J_{\mu\nu} = \begin{cases} J_{ij} = -J_{ji} = \varepsilon_{0ij} J_0 \\ J_{0i} = -J_{i0} = K_i. \end{cases}$$

As was first demonstrated by Schwinger [22], all these relations follow simply by integrating the three independent relations given below in terms of the unintegrated components of the energy momentum tensor:

$$[T_{00}(\mathbf{x}), T_{00}(\mathbf{y})]_- = i(T_{0i}(\mathbf{x}) + T_{0i}(\mathbf{y}))\, \partial^i_x\, \delta^2(\mathbf{x}-\mathbf{y}) \tag{5.3a}$$

$$[T_{00}(\mathbf{x}), T_{0i}(\mathbf{y})]_- = i(T_{ij}(\mathbf{x}) - g_{ij} T_{00}(\mathbf{y}))\, \partial^j_x\, \delta^2(\mathbf{x}-\mathbf{y}) \tag{5.3b}$$

$$[T_{0i}(\mathbf{x}), T_{0j}(\mathbf{y})]_- = i(T_{0j}(\mathbf{y})\, \partial^i_x + T_{0j}(\mathbf{x})\, \partial^i_x)\, \delta^2(\mathbf{x}-\mathbf{y}). \tag{5.3c}$$

We first concentrate on the Chern–Simons scalar electrodynamics case. Recalling the expression for the energy momentum tensor for this theory from Eq. (3.22), we have

$$T_{\mu\nu} = (D_\mu\phi)^* (D_\nu\phi) + (D_\nu\phi)^* (D_\mu\phi) - g_{\mu\nu}(D_\lambda\phi)^* (D^\lambda\phi).$$

Hence,

$$T_{00} = \pi^*(\mathbf{x})\, \pi(\mathbf{x}) - (D_i\phi(\mathbf{x}))^* (D^i\phi(\mathbf{x})) \tag{5.4a}$$

$$T_{0i} = \pi(\mathbf{x})(D_i\phi(\mathbf{x})) + \pi^*(\mathbf{x})(D_i\phi(\mathbf{x}))^*. \tag{5.4b}$$

The corresponding expressions for topologically massive scalar electrodynamics are

$$\begin{aligned}T_{\mu\nu} &= (D_\mu\phi)^* (D_\nu\phi) + (D_\nu\phi)^* (D_\mu\phi) - \frac{1}{e^2} F^\sigma_\mu F_{\nu\sigma} \\ &\quad - g_{\mu\nu}\left\{(D_\sigma\phi)^* (D^\sigma\phi) - \frac{1}{4e^2} F_{\lambda\sigma} F^{\lambda\sigma}\right\} \\ T_{00} &= \pi^*\pi - (D_i\phi)^* (D^i\phi) + \frac{1}{4e^2} F_{ij} F^{ij} \\ &\quad - \frac{e^2}{2}\left(\pi_i - \frac{\theta}{4\pi^2}\varepsilon_{ij}A^j\right)\left(\pi^i - \frac{\theta}{4\pi^2}\varepsilon^{ik}A_k\right)\end{aligned} \tag{5.5a}$$

$$T_{0i} = \pi(D_i\phi) + \pi^*(D_i\phi)^* + F_{ij}\left(\pi^j - \frac{\theta}{4\pi^2}\varepsilon^{jk}A_k\right). \tag{5.5b}$$

It is now straightforward to compute the commutators between the components of the energy momentum tensor for each of the two sets (5.4) and (5.5) by using the commutation relations for the basic fields presented in Sections 2 and 4, respectively. The resulting terms precisely yield the Schwinger conditions in Eq. (5.3), upon using the various properties of the kernels $K_i(\mathbf{x}, \mathbf{y})$ and the Gauss law. Thus, the unconventional commutation relations for the basic fields notwithstanding, the quantization scheme presented in the previous sections leaves both Chern–Simons and topologically massive scalar electrodynamics Poincaré invariant, at least on the physical subspace of the theory which is projected out by the Gauss law.

We next demonstrate that the Ward identity for two anyons coupled to a vector current holds at the three level. Consider, therefore, the three-point function

$$\begin{aligned}G_\mu(p, q) &= \int d^3x \int d^3y \exp\{-iq\cdot x - ip\cdot y\} \langle 0| \\ &\quad \times T(j_\mu(x)\,\hat{\phi}(y)\,\hat{\phi}^*(0))\,|0\rangle.\end{aligned} \tag{5.6}$$

Upon using standard current algebra manipulations this equation leads to

$$\begin{aligned}q^\mu G_\mu(p, q) &= -i\int d^3x \int d^3y \exp\{-iq\cdot x - ip\cdot y\}\, \partial^\mu_x \langle 0| \\ &\quad \times T(j_\mu(x)\,\hat{\phi}(y)\hat{\phi}^*(0))\,|0\rangle \\ &= -i\int d^3x \int d^3y \exp\{-iq\cdot x - ip\cdot y\} \\ &\quad \times \{\langle 0|\, T(\partial^\mu_x j_\mu(x)\,\hat{\phi}(y)\,\hat{\phi}^*(0))\,|0\rangle \\ &\quad + \langle 0|\, T(\delta(x_0 - y_0)[j_0(x), \hat{\phi}(y)]\,\hat{\phi}^*(0))\,|0\rangle \\ &\quad + \langle 0|\, T(\delta(x_0)[j_0(x), \hat{\phi}^*(0)]\,\hat{\phi}(y))\,|0\rangle\}.\end{aligned} \tag{5.7}$$

The first term on the right-hand side vanishes because of current conservation. The other two terms can be simplified by substituting for the commutator which can be worked out easily. For both Chern–Simons and topologically massive scalar electrodynamics this works out to be

$$[j_0(\mathbf{x}), \hat{\phi}(\mathbf{y})]_- = \delta^2(\mathbf{x}-\mathbf{y})\,\hat{\phi}(\mathbf{x}) \tag{5.8a}$$

and, similarly,

$$[j_0(\mathbf{x}), \hat{\phi}^*(\mathbf{y})]_- = -\delta^2(\mathbf{x}-\mathbf{y})\,\hat{\phi}^*(\mathbf{x}). \tag{5.8b}$$

Plugging the above commutators into Eq. (5.7) yields the desired Ward identity,

$$iq^\mu G_\mu(p, q) = \Delta(p+q) - \Delta(p), \tag{5.9}$$

where

$$\Delta(p) = \int d^3x \exp(-ip\cdot x)\langle 0|\, T(\hat{\phi}(x)\,\hat{\phi}^*(0))\,|0\rangle \tag{5.10}$$

is the anyon propagator. It is trivial to check that an exactly identical result holds for the basic complex scalar fields. Thus, although the algebraic structures of the basic commutation relations of the two theories are vastly different and both are even more drastically different from conventional canonical structures in theories without Chern–Simons terms, the Ward identies are preserved. This is a consequence of the fact that both models admit a conserved vector current and the all-important commutator between the zeroth component of the current and a matter field is conventional.

We will now show that the gauge fixing conditions in Eq. (2.11) completely fix the local gauge invariances in the theory. It is easy to check that the actions corresponding to the lagrangian densities in Eqs. (2.1) and (4.1) are invariant under the simultaneous gauge transformations,

$$\phi(x) \to \phi'(x) = \exp\{i\varepsilon(x)\}\,\phi(x) \tag{5.11a}$$

$$\phi^*(x) \to \phi^{*\prime}(x) = \phi^{*\prime}(x)\exp\{-i\varepsilon(x)\} \tag{5.11b}$$

and

$$A_\mu(x) \to A'_\mu(x) = A_\mu - i\,\partial_\mu\varepsilon(x). \tag{5.11c}$$

Requiring the transformed variables, A'_i, to satisfy the gauge fixing condition (2.11b), we obtain

$$\begin{aligned} 0 &= \int d^2y K_i(\mathbf{x}, \mathbf{y})\, A'_i(x_0, \mathbf{y}) \\ &= \int d^2y K_i(\mathbf{x}, \mathbf{y})\, A_i(x_0, \mathbf{y}) - i\int d^2y K_i(\mathbf{x}, \mathbf{y})\,\partial_i\,\varepsilon(x_0, \mathbf{y}) \end{aligned}$$

which implies

$$\varepsilon(x) = i \int d^2y K_i(\mathbf{x}, \mathbf{y})\, A_i(x_0, \mathbf{y}) + \omega(x_0), \tag{5.12}$$

where $\omega(x_0)$ is purely a function of time and where we have once again done an integration by parts and used Eq. (2.14) in arriving at the last equality. The above requirement, therefore, completely fixes the spatial dependence of the gauge transformations parameter. The temporal behaviour of ε can similarly be fixed by requiring the transformed variables, A_0', to satisfy the other gauge fixing condition, Eq. (2.11a), from which we obtain

$$\begin{aligned} 0 &= \int d^2y K_0(\mathbf{x}, \mathbf{y})\, A_0'(x_0, \mathbf{y}) \\ &= \int d^2y K_0(\mathbf{x}, \mathbf{y})\, A_0(x_0, \mathbf{y}) - i \int d^2y K_0(\mathbf{x}, \mathbf{y})\, \partial_0 \varepsilon(x_0, \mathbf{y}). \end{aligned}$$

Subtituting for ε from Eq. (5.12) we obtain

$$\begin{aligned} 0 &= \int d^2y K_0(\mathbf{x}, \mathbf{y})\, A_0(x_0, \mathbf{y}) \\ &\quad + \int d^2y K_0(\mathbf{x}, \mathbf{y}) \int d^2z K_i(\mathbf{y}, \mathbf{z})\, \partial_0 A_i(x_0, \mathbf{z}) \\ &\quad + \int d^2y K_0(\mathbf{x}, \mathbf{y})(\partial_0 \omega(x_0)) \\ &= \int d^2y K_0(\mathbf{x}, \mathbf{y})\, A_0(x_0, \mathbf{y}) + \int d^2y K_0(\mathbf{x}, \mathbf{y})(\partial_0 \omega(x_0)) \\ &\quad + \int d^2y K_0(\mathbf{x}, \mathbf{y}) \int d^2z K_i(\mathbf{y}, \mathbf{z})\, F_{0i}(x_0, \mathbf{z}) \\ &\quad + \int d^2y K_0(\mathbf{x}, \mathbf{y}) \int d^2z K_i(\mathbf{y}, \mathbf{z})\, \partial_i A_0(x_0, \mathbf{z}). \end{aligned}$$

After doing an integration by parts in the last term and using the property (2.14), the above equation reduces to

$$\int d^2y K_0(\mathbf{x}, \mathbf{y}) \left[\int d^2z K_i(\mathbf{y}, \mathbf{z})\, F_{0i}(x_0, \mathbf{z}) - (\partial_0 \omega(x_0)) \right] = 0. \tag{5.13}$$

Now, because the kernels $K_i(\mathbf{x}, \mathbf{y})$ are time independent, we have, furthermore,

$$\dot{\chi}(\mathbf{x}) = \partial_0 \chi(\mathbf{x}) = \int d^2y K_i(\mathbf{x}, \mathbf{y})\, \partial_0 A_i(x_0, \mathbf{y})$$

$$= \int d^2y [K_i(\mathbf{x}, \mathbf{y})\, F_{0i}(x_0, \mathbf{y}) + K_i(\mathbf{x}, \mathbf{y})\, \partial_i^y A_0(x_0, \mathbf{y})]$$

$$= -A_0(\mathbf{x}) + \int d^2y K_i(\mathbf{x}, \mathbf{y})\, F_{0i}(x_0, \mathbf{y}),$$

where we have done an integration by parts and used Eq. (2.14) in arriving at the last equality and the definition of F_{0i}. On comparing with Eq. (2.12b) the above equation yields

$$A_0(\mathbf{x}) = \int d^2y K_i(\mathbf{x}, \mathbf{y})\, F_{0i}(x_0, \mathbf{y}). \tag{5.14}$$

Plugging the above expression into $\chi_0(\mathbf{x})$ produces the identity

$$\int d^2y K_0(\mathbf{x}, \mathbf{y}) \int d^2z K_i(\mathbf{y}, \mathbf{z})\, F_{0i}(x_0, \mathbf{z}) \approx 0. \tag{5.15}$$

Upon using this identity, Eq. (5.13) implies that the $\partial_0 \omega(x_0)$ is given by a linear combination of the constraints in the theory. On the physical subspace, therefore, this condition fixes ω to be a constant in time.

Thus, as advertised, the two gauge-fixing conditions completely fix the local gauge invariance in the theory. The above analysis gives us an opportunity to compare our work with related past work. Note that if we naively choose

$$K_i(\mathbf{x}, \mathbf{y}) = \partial_i^y \delta^2(\mathbf{x}, \mathbf{y}) \tag{5.16a}$$

and

$$K_0(\mathbf{x}, \mathbf{y}) = \delta^2(\mathbf{x} - \mathbf{y}), \tag{5.16b}$$

the gauge-fixing conditions used by us reduce to the ones used in Ref. [9]. If these gauge-fixing conditions are to completely eliminate the gauge invariance in the theory, they must obey the above consistency checks. Substituting them in Eq. (5.15) we immediately find that

$$\partial_i^x F_{0i}(x) = 0.$$

It is only too well known that such an equation is not possible in any theory which has couplings with external sources such as the ones dealt with in this paper. It is gratifying to note that the constraint on the kernels $K_i(\mathbf{x}, \mathbf{y})$ in Eq. (2.14) does not allow solutions of the form (5.16) and hence there is no scope for a travesty of Eq. (5.13). Equations (5.16) are, therefore, naturally forbidden on grounds of algebraic consistency.

6. Conclusions and Outlook

In this paper we have tried to address some issues regarding fractional spin and statistics in (2+1)-dimensional space-time. It may be recalled that these possibilities within the realm of quantum mechanics have been studied for quite some time now. All the intricacies of these two fundamental concepts can, however, emerge only from a thorough understanding of at least one completely relativistic quantum field theory which supports physical excitations which are anyonic. Unfortunately, attempts at gaining such an insight through canocical quantization, in the past, have been either incomplete or have led to contradictory results. The most glaring intellectual loophole in these attempts owes its existence to the fact that the operators purportedly representing anyons in these theories do not seem to be any way related to Wilczek's holistic approach towards the construction of an anyon which is at the heart of all quantum mechanical results. Moreover, these operators are not manifestly gauge invariant, which casts doubts over their physical status. Other technical problems associated with such attempts have been mentioned in Section 1. Further details can be obtained in Ref. [12].

It is worth mentioning that there have been other approaches to field theoretic construction of anyons as well. Notable amongst them are a path-integral approach developed by Forte and Jolicoeur [23] which mimics the quantum mechanical construction of Wu [15] and the lattice approach dealt with by Luscher *et al.* [24]. These lattice models are still incomplete because construction of the continuum limit and of interpolating operators for asymptotic states are problems which are as yet unsolved. The solitonic models fashioned by Wilczek and Zee [25] have also been studied in some detail, but suffer from non-renormalizability. Constructive field theoretic methods have been developed by Frohlich and Marchetti and by Schroer [26].

This paper, however, concentrates on the canonical quantization method with the objective of plugging in the loophole referred to above which renders obscure any connection between known quantum mechanical results for anyons and the candidate relativistic quantum field theories advanced to explain them. We have demonstrated by explicit construction that operators defined according to Schwinger's line integral prescription are anyon operators. They are manifestly gauge invariant and are obvious generalizations of Wilczek's prototype quantum mechanical anyons. This is a remarkable departure from the results obtained in related past work. These operators also create and annihilate particles from the vacuum through traditional Fock space methods. The multi-particle state constructed this way is shown to be related to the Laughlin state for fractional quantum Hall effect. We have also discussed various algebraic consistency conditions like Poincaré invariance, Ward identity, and the proof that the local gauge invariance in the theory is completely fixed by the gauge-fixing conditions used. We have also argued that, although the theory becomes drastically changed by the addition of the Maxwell term because of the altered constraint structure, gauge-invariant anyon operators can still be constructed which smoothly reduce to the corresponding ones in the pure Chern–Simons case as the coupling of the Maxwell term goes to zero. All other results are checked against these operators and they confirm the role played by the Chern–Simons term in imparting fractional spin and statistics. Of significant interest is the fact that all the technical difficulties faced in related past work involving the exchange of derivatives and integrals of multivalued objects are bypassed by using gauge-fixing conditions involving non-local kernels. It is therefore logical to conclude that this improvement, along with the intrinsic merit of our anyon operators being manifestly gauge invariant is a pointer in the direction that one can construct consistent relativistic field theories of anyons.

Pursuing the path shown by this pointer leads to several interesting questions which deserve further study. An obvious possibility concerns a non-abelian generalization of the studies made in this paper. Some preliminary results in this direction have already been obtained by two of us in Ref. [27]. In the following, however, we list a few of the more important issues which are closer to the abelian theory discussed here.

It is worth understanding in the first place whether it would still be possible for us to construct gauge invariant anyon operators similar to the ones presented in this paper if a mass term and a polynomial potential term for the basic matter fields are included in the definition of the model. Clearly all the results in this paper would hold in such a case if there is no spontaneous symmetry breaking. In the broken phase, however, the gauge fields would pick up a non-topological mass term and consequently the constraint structure of the theory is likely to be altered. The impact of such a change on the possibility of constructing anyon operators remains to be examined. The second question is regarding the addition of fermions. It appears straightforward to incorporate them within the framework of this paper and we do not foresee any significant departure from the results obtained here. The third important issue concerns the working out of gauge invariant anyon correlation functions in this theory. This problem is presently being pursued by us and we hope to report the result in the near future. The most promising avenue that opens out in the wake of the results in this paper is related to the possibility of constructing and studying theories of anyon electrodynamics. It is tempting to envisage the following scenario in this context. Suppose the complex scalar fields in the original model we have considered are coupled to another gauge field in addition to the Chern–Simons gauge field and let the former gauge field's dynamics be governed by the Maxwell term. Using the Chern–Simons interaction to impart fractional spin and statistics, can one develop a perturbation theory for the interaction of anyons with the dynamical gauge field so that one can calculate cross sections for realistic processes involving creation, annihilation, and scattering of anyons? Presumably the first step towards writing down the Feynman rules for such processes would require us to develop a method of handling the exponential of the line integral of the gauge field in the definition of the anyon operator through some non-perturbative technique. We are currently exploring ways of doing this because it would be useful in the problem of computing gauge-invariant anyon correlation functions referred to above. Finally, properties of anyon S-matrices and questions regarding renormalization of anyon field theories should serve as a logical culmination of what appears to be a long impressive agenda.

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