Programming in Haskell: Lecture 26

S P Suresh

November 13, 2019

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PRGH 2019: Lecture 26

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Sudoku

		4			5	7		
					9	4		
3	6							8
7	2			6				
			4		2			
				8			9	3
4							5	6
		5	3					
		6	I			9		

Basic structures

• Basic data structures:

type Digit = Char
type Row a = [a]
type Matrix a = [Row a]
type Grid = Matrix Digit

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• Choices for each cell:

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• Grid entries:

```
digits = "123456789"
blank :: Digit -> Bool
blank = (== '-')
```

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- Expand the given grid to all possible valid complete grids
 - Fill in each empty cell with all choices
 - Expand a matrix of choices to a list of complete grids
 - Choose all valid grids from this list
- solve implements this strategy:

```
solve :: Grid -> [Grid]
solve = filter valid . expand . choices
```

Filling in all choices

• Filling a cell with choices:

choice :: Digit -> [Digit]
choice d = if blank d then digits else [d]

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- map choice fills all cells in a row with choices
- To fill all cells in a grid with choices:

choices :: Grid -> Matrix Choices
choices = map (map choice)

Expanding list of choices

• We take cartesian product of the matrix of choices using:

cp :: [[a]] -> [[a]] cp [] = [[]] cp (xs:xss) = [x:ys | x <- xs, ys <- cp xss]

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- cp [[1,2], [3,4]] = [[1,3], [1,4], [2,3], [2,4]]
- expand computes the list of all complete grids:

```
expand :: Matrix Choices -> [Grid]
expand = cp . map cp
```

• In a valid complete grid:

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 - Each row has distinct entries

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 - Each 3×3 box has distinct entries
- Checking for distinct entries in a list:

```
nodups :: Eq a => [a] -> Bool
nodups [] = True
nodups (x:xs) = all (/= x) xs && nodups xs
```

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 - Each row has distinct entries
 - Each column has distinct entries
 - Each 3 × 3 box has distinct entries
- Checking for distinct entries in a list:

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• all is a built-in function:

all p [] = True
all p (x:xs) = p x && all p xs

• A grid is a list of 9 rows

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- Extracting all rows of a grid:

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• Extracting all columns:

cols [xs] = [[x] | x <- xs] cols (xs:xss) = zipWith (:) xs (cols xss)

• cols [[1,2], [3,4], [5,6]] = [[1,3,5], [2,4,6]]

• Extracting the 3 × 3 boxes:

а	b	С	d
е	f	g	h
i	j	k	l
т	п	0	p

ab cd ef gh ij kl mn op

ab cd ef gh ij kl mn op group

ab cd ef gb ij kl mn op

ab cd ef gh ij kl mn op map cols

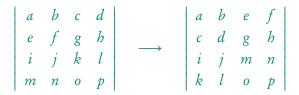
ab ef cd gh ij mn kl op

ab ef cd gh ij mn kl op ungroup

ab ef cd gh ij mn kl op

ab ef cd gh ij mn kl op

map ungroup



All valid solutions

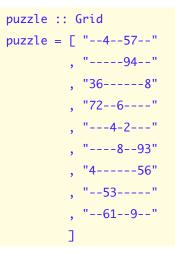
```
boxs :: Matrix a -> Matrix a
boxs = map ungroup . ungroup . map cols . group . map group
```

```
valid :: Grid -> Bool
valid g = all nodups (rows g) &&
    all nodups (cols g) &&
    all nodups (boxs g)
```

solve = filter valid . expand . choices
solution = head . solve

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All valid solutions



Sudoku example

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4							5	6
		5	3					
		6	I			9		

puzzle

Sudoku example

I	8	4	6	2	5	7	3	9
5	7	2	8	3	9	4	6	Ι
3	6	9	7	4	Ι	5	2	8
7	2	8	9	6	3	Ι	4	5
9	5	3	4	Ι	2	6	8	7
6	4	Ι	5	8	7	2	9	3
4	I	7	2	9	8	3	5	6
2	9	5	3	7	6	8	I	4
8	3	6	I	5	4	9	7	2

solution puzzle

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 - $9^{40} = 147808829414345923316083210206383297601$

- Our program is useless
- Even with half the grid filled, we have to check 9⁴⁰ grids for validity
 - 9⁴⁰ = 147808829414345923316083210206383297601
- Takes forever even for a grid with only 9 blank cells

```
["652439817", "879165432", "413872596"
,"548697321", "931524768", "267381945"
,"795213684", "184956273", "326748159"]
(946.02 secs, 703,822,023,848 bytes)
```

A better strategy?

• **Obvious improvement**: Try to prune the choices even before expanding to a list of grids

solve = filter valid . expand . prune . choices
solution = head . solve

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• **Obvious improvement**: Try to prune the choices even before expanding to a list of grids

solve = filter valid . expand . prune . choices
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• We would like prune to satisfy:

filter valid . expand . prune = filter valid . expand

• How do we prune the choices?

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- Consider a row with three entries and six blank cells
- Let the entries be 1, 5 and 9
- Then the list of choices for the blank cells is [234678]
- If some column has entries 1, 4 and 7, choices further pruned to [2368]
- Similar pruning based on entries in 3×3 box
- Potentially huge savings!

- Note that:
 - rows . rows = id
 cols . cols = id
 boxs . boxs = id

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- Now each row is a box of the original grid
- Prune each row
- Apply boxs again to restore order of cells
- Similarly with cols

prune	<pre>= pruneBy boxs .</pre>				
	pruneBy cols . pruneBy rows				
pruneBy f	= f . map pruneRow . f				
pruneRow row	<pre>= map (remove fixed) row</pre>				
where fixed	= [d [d] <- row]				
remove xs ds	= if (length ds == 1) then ds				
	else ds \\ xs				

Performance

• This program performs much better on very easy puzzles

```
["652439817","879165432","413872596"
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(0.01 secs, 513,832 bytes)
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- But this struggles on even puzzles with 39 entries
 - Aborted after running it for 6 hours on my laptop

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Further improvements?

• Improved strategy

Further improvements?

- Improved strategy
 - Expand one cell at a time

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Improved strategy

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- Interleave expansion and pruning

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- We now get a list of matrices
 - Each of these matrices contain singleton as well as non-singleton cells
- Prune each of these matrices
- And expand one cell in each incomplete matrix that results

Expanding one cell

• Expanding the smallest non-singleton cell:

```
expand1 :: Matrix Choices -> [Matrix Choices]
expand1 rows
                           -
    [rows1 ++ [row1 ++ [c]:row2] ++ rows2 | c <- cs]</pre>
    where
        (rows1, row:rows2) = break (any smallest) rows
        (row1, cs:row2) = break smallest row
        smallest cs
                           = length cs == n
                           = minimum (counts rows)
        n
                           = filter (/=1).
        counts
                             map length . concat
```

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Safe grids, complete grids

• A matrix of choices is safe if none of the singleton cells clash

Safe grids, complete grids

• A matrix of choices is safe if none of the singleton cells clash

We can stop expanding if the matrix consists only of singleton entries

```
complete :: Matrix Choices -> Bool
complete = all (all singleton)
where singleton l = length l == 1
```

Searching for safe, complete grids

• Recall:

choices = map (map choice)
choice d = if blank d then digits else [d]

Searching for safe, complete grids

• Recall:

choices = map (map choice)
choice d = if blank d then digits else [d]

• To find all solutions, we search after creating a matrix of choices

solve :: Grid -> [Grid]
solve = search . choices

Searching for safe, complete grids

• We alternate pruning and expanding a cell for incomplete grids

Performance

• Works very well on easy inputs (39 cells filled in):

solution ["-5-43-81-", "-----3-", "-13--2---", "--8-9---1", "9-15-4-68", "-67---945", "795----84", "-8-956---", "32-748-59"]

["652439817","879165432","413872596" ,"548697321","931524768","267381945" ,"795213684","184956273","326748159"] (0.02 secs, 3,129,904 bytes)

Performance

• Quite well on puzzles of higher difficulty (only 25 cells filled in):

["261935874","957418236","843276591"
,"329681457","674359182","185742963"
,"732894615","418563729","596127348"]
(0.02 secs, 9,082,360 bytes)

Performance

• Holds its own on against puzzles of very high difficulty (only 17 entries):

["812975364","946381572","537264819"
,"751849623","384126957","269753148"
,"498612735","623597481","175438296"]
(14.42 secs, 9,823,352,008 bytes)

Summary

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- Incremental design of a non-trivial program
- Reasonably simple logic, close to how a human would solve
- Power of laziness backtracking is easily programmed