Programming in Haskell: Lectures 23 & 24

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November 4 & 6, 2019

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- In general, a tree might not be balanced
- Inserting in ascending or descending order results in highly skewed trees

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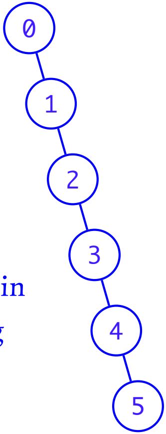
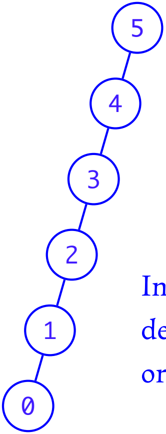


Inserting in ascending order



Inserting in descending order

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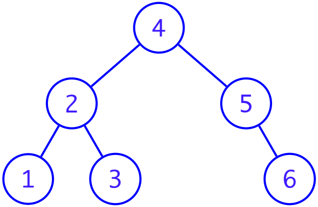

Height-balanced
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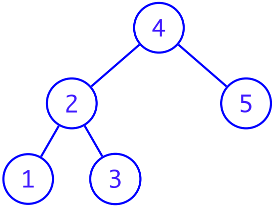

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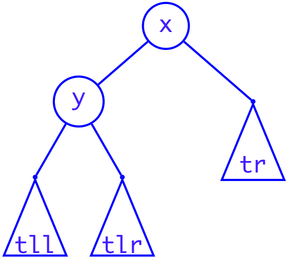
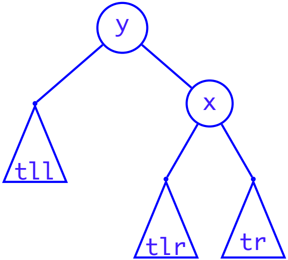
Tree rotations – rotate right





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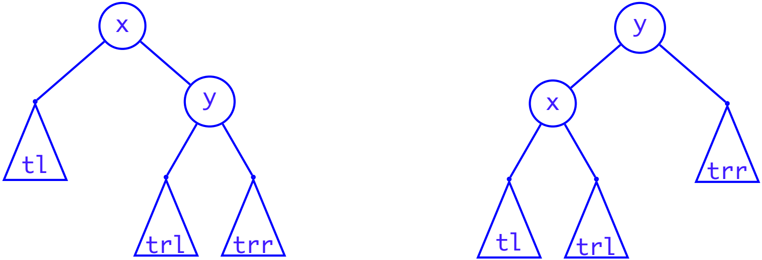

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- Need to update height after each operation

AVL trees

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- Also need a measure of how skewed a tree is – its slope

```haskell
slope :: AVL a -> Int
slope Nil              = 0
slope (Node _ _ tl tr) = height tl - height tr
```

- Check if `t` is an AVL tree:

```
isAVL :: Ord a => AVL a -> Bool
isAVL Nil        = True
isAVL t@(Node _ x tl tr)
                 = abs (slope t) < 2 &&
                     isAVL tl && isAVL tr &&
                     (isEmpty tl || maxt tl < x) &&
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AVL trees – rotates

- Since we maintain height at each node, we need to adjust it after each operation:

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rotateRight :: AVL a -> AVL a
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- We rebalance each node on the path visited by operation

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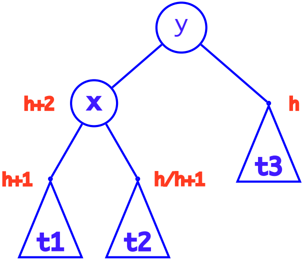

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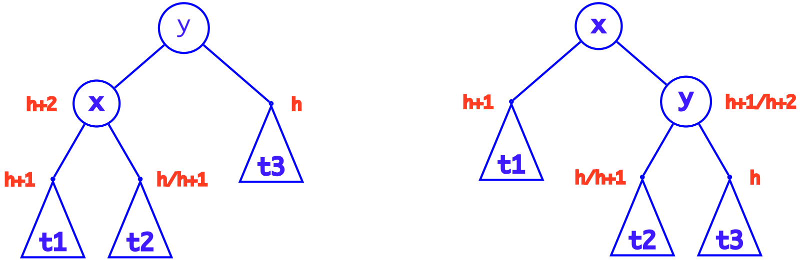

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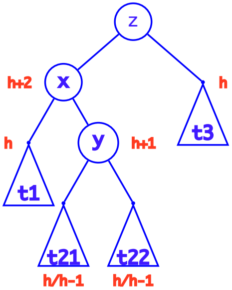

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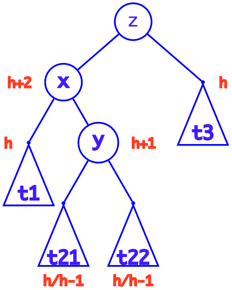

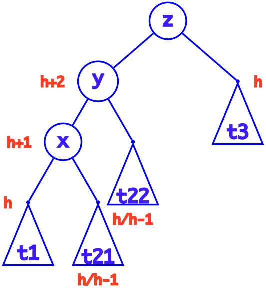

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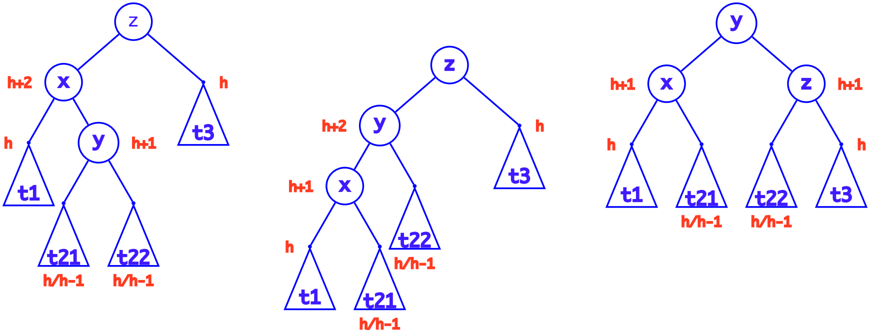

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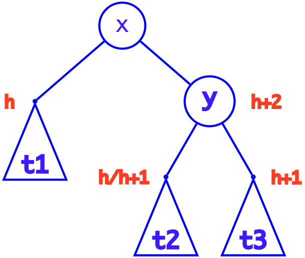

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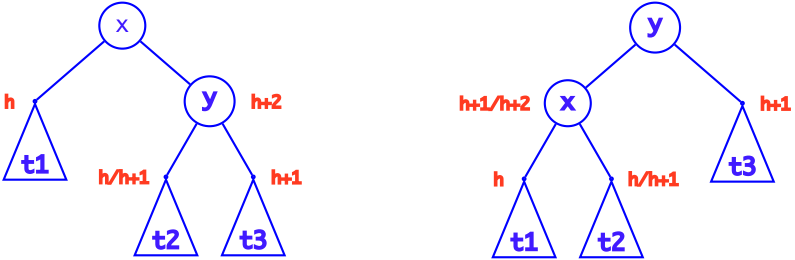

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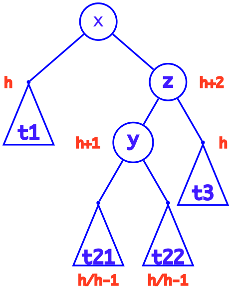

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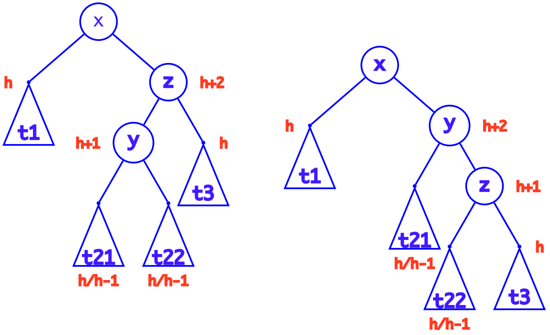

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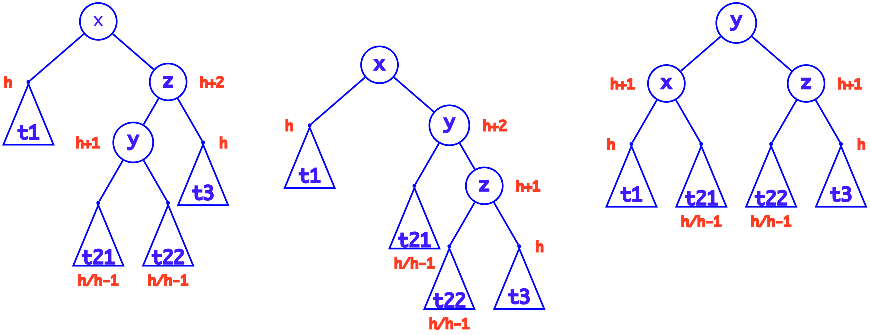

Rebalancing in Haskell

- The `rebalance` function

```haskell
rebalance :: Ord a => AVL a -> AVL a
rebalance t@(Node h x tl tr)
    | abs st < 2 = t
    | st == 2 = if stl == -1 then
                rotateRight (Node h x (rotateLeft tl) tr)
                else rotateRight t
    | st == -2 = if str == 1 then
                 rotateLeft (Node h x tl (rotateRight tr))
                 else rotateLeft t
    where (st, stl, str) = (slope t, slope tl, slope tr)
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- Constant time operation

Searching in an AVL tree

```
searchAVL :: Ord a => a -> AVL a -> Bool
searchAVL v Nil = False
searchAVL v (Node _ x tl tr)
    | v == x = True
    | v < x  = searchAVL v tl
    | v > x  = searchAVL v tr
```

Inserting in a tree

```haskell
insertAVL :: Ord a => a -> AVL a -> AVL a
insertAVL v Nil = Node 1 v Nil Nil
insertAVL v t@(Node h x tl tr)
    | v < x  = rebalance (Node nhl x ntl tr)
    | v > x  = rebalance (Node nhr x tl ntr)
    | v == x = t
    where
        ntl = insertAVL v tl
        ntr = insertAVL v tr
        nhl = 1 + max (height ntl) (height tr)
        nhr = 1 + max (height tl) (height ntr)
```

Deleting the maximum element

```
deleteMax :: Ord a => AVL a -> (a, AVL a)
deleteMax (Node _ x tl Nil) = (x, tl)
deleteMax (Node h x tl tr)  = (y, rebalance (Node nh x tl ty))
    where
        (y, ty) = deleteMax tr
        nh      = 1 + max (height tl) (height ty)
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    | v == x    = if isEmpty tl then tr
                  else rebalance (Node nhy y ty tr)
  where
    (y, ty)     = deleteMax tl
    (ntl, ntr) = (deleteAVL v tl, deleteAVL v tr)
    nhl         = 1 + max (height ntl) (height tr)
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- Search takes time proportional to height of the tree
- Insert and delete spend constant time on each node of some maximal path
- Take time proportional to height of the tree
- Height of a tree with n nodes is $\leq 2\log n$

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- Height of a tree with n nodes is $\leq 2 \log n$
- Thus each operation takes $O(\log n)$ time

Tree operations – complexity

- Left and right rotates take constant time
- Rebalance of a tree takes constant time, when both subtrees are balanced
- Search takes time proportional to height of the tree
- Insert and delete spend constant time on each node of some maximal path
- Take time proportional to height of the tree
- Height of a tree with n nodes is $\leq 2\log n$
- Thus each operation takes $O(\log n)$ time
- A sequence of n operations take at most $O(n \log n)$ time

Other useful functions

- Create an empty AVL tree:

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emptyAVL :: AVL a
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- Create an AVL tree from a list ($O(n \log n)$ time):

```
createAVL :: Ord a => [a] -> AVL a
createAVL = foldl' (flip insertAVL) emptyAVL
```

Other useful functions

- Create a sorted list from an AVL tree:

```
inorder :: Ord a => AVL a -> [a]
inorder Nil                = []
inorder (Node _ x tl tr) = inorder tl ++ [x] ++ inorder tr
```

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- $T(n) = 2T(n/2) + O(n)$, so $T(n) = O(n \log n)$
- Culprit is `++`, which takes $O(n)$ time

Other useful functions

- Smarter `inorder`:

```
inorder :: Ord a => AVL a -> [a]
inorder t                       = go t []
    where go Nil l              = l
          go (Node _ x tl tr) l = go tl (x:go tr l)
```

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- $T(n) = 2T(n/2) + c$, so $T(n) = O(n)$
- We can sort a list in $O(n \log n)$ time by:

```
treesort :: Ord a => [a] -> [a]
treesort = inorder . createAVL
```

Other useful functions

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- If `l` itself is sorted, `t` is search tree
- This is just the smart `createTree` we saw in a previous class
- Works in $O(n)$ time

inorderTree

```haskell
inorderTree :: [a] -> AVL a
inorderTree l = fst (go (length l) l)
    where
        go :: Int -> [a] -> (AVL a, [a])
        go 0 xs = (Nil, xs)
        go n xs = (Node h y tl tr, zs)
            where
                m = n `div` 2
                (tl, y:ys) = go m xs
                (tr, zs) = go (n-m-1) ys
                h = 1 + max (height tl) (height tr)
```

A module for AVL trees

- Saved in `AVL.hs`

```
module AVL(AVL, emptyAVL, isEmpty, isAVL,
           insertAVL, deleteAVL, searchAVL,
           createAVL, inorder, inorderTree) where

data AVL a = Nil | Node Int a (AVL a) (AVL a)
    deriving (Eq, Ord)

instance Show a => Show (AVL a) where
    show t = intercalate "\n" (draw t)
```

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```

- Can be used to define the `Set` ADT

The Set ADT again

```haskell
module Set(Set, emptySet, createSet,
           insertInto, deleteFrom, search,
           union, intersect, diff) where
import AVL

data Set a = Set (AVL a)
instance (Ord a, Show a) => Show (Set a) where
    show (Set t) = show (inorder t)

emptySet :: Ord a => Set a
emptySet = Set emptyAVL
```

The Set ADT again

```haskell
createSet :: Ord a => [a] -> Set a
createSet = Set . createAVL

search :: Ord a => a -> Set a -> Bool
search x (Set t) = searchAVL x t

insertInto :: Ord a => a -> Set a -> Set a
insertInto x (Set t) = Set (insertAVL x t)

deleteFrom :: Ord a => a -> Set a -> Set a
deleteFrom x (Set t) = Set (deleteAVL x t)
```

More set operations

```haskell
union :: Ord a => Set a -> Set a -> Set a
union (Set t1) (Set t2) = Set $ inorderTree $
                unionMerge (inorder t1) (inorder t2)


unionMerge :: Ord a => [a] -> [a] -> [a]
unionMerge [] ys = ys
unionMerge xs [] = xs
unionMerge (x:xs) (y:ys)
    | x < y  = x:unionMerge xs (y:ys)
    | y < x  = y:unionMerge (x:xs) ys
    | x == y = x:unionMerge xs ys
```

More set operations

```haskell
intersect :: Ord a => Set a -> Set a -> Set a
intersect (Set t1) (Set t2) = Set $ inorderTree $
                   intersectMerge (inorder t1) (inorder t2)

intersectMerge :: Ord a => [a] -> [a] -> [a]
intersectMerge [] ys = []
intersectMerge xs [] = []
intersectMerge (x:xs) (y:ys)
    | x < y  = intersectMerge xs (y:ys)
    | y < x  = intersectMerge (x:xs) ys
    | x == y = x:intersectMerge xs ys
```

More set operations

```
diff :: Ord a => Set a -> Set a -> Set a
diff (Set t1) (Set t2) = Set $ inorderTree $
              diffMerge (inorder t1) (inorder t2)

diffMerge :: Ord a => [a] -> [a] -> [a]
diffMerge [] ys = []
diffMerge xs [] = xs
diffMerge (x:xs) (y:ys)
    | x < y  = x:diffMerge xs (y:ys)
    | y < x  = diffMerge (x:xs) ys
    | x == y = diffMerge xs ys
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- An illustration of the power of Haskell
- Mathematical definitions almost directly transcribed to code
- Pattern matching is very powerful
- Allows us to easily specify complex transformations on data