Programming in Haskell: Lecture 22

S P Suresh

October 30, 2019

Binary trees

- A binary tree data structure is defined as follows:

Binary trees

- A binary tree data structure is defined as follows:
- The empty tree is a binary tree

Binary trees

- A binary tree data structure is defined as follows:
- The empty tree is a binary tree
- A node containing an element with left and right subtrees is a binary tree

Binary trees

- A binary tree data structure is defined as follows:
- The empty tree is a binary tree
- A node containing an element with left and right subtrees is a binary tree
- Type constructor BTree

```
data BTree a = Nil
             | Node (BTree a) a (BTree a)
```

Binary trees

- A binary tree data structure is defined as follows:
- The empty tree is a binary tree
- A node containing an element with left and right subtrees is a binary tree
- Type constructor BTree

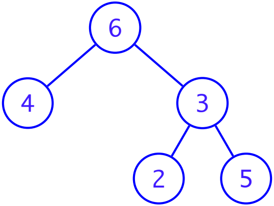
```
data BTree a = Nil
             | Node (BTree a) a (BTree a)
```

- Two value constructors:

```
Nil  :: BTree a
Node :: BTree a -> a -> BTree a -> BTree a
```

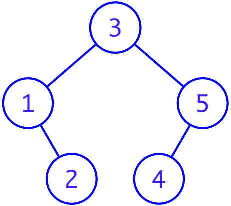
Binary trees

```
Node (Node Nil 4 Nil) 6
     (Node (Node Nil 2 Nil) 3
           (Node Nil 5 Nil))
```



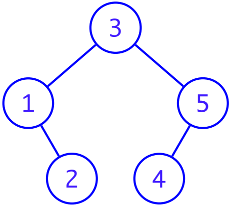
Binary trees

- Yet another binary tree



Binary trees

- Yet another binary tree



- Corresponding BTree

```
Node
    (Node Nil 1 (Node Nil 2 Nil))
3
    (Node (Node Nil 4 Nil) 5 Nil)
```

Functions on binary trees

- Number of nodes in a tree

```
size :: BTree a -> Int
size Nil             = 0
size (Node tl x tr) = 1 + size tl + size tr
```

Functions on binary trees

- Number of nodes in a tree

```
size :: BTree a -> Int
size Nil              = 0
size (Node tl x tr) = 1 + size tl + size tr
```

- **Height**: number of nodes on longest path from root

```
height :: BTree a -> Int
height Nil              = 0
height (Node tl x tr) = 1 + max (height tl) (height tr)
```

Creating a binary tree

- Create a binary tree from a list

Creating a binary tree

- Create a binary tree from a list
- As **balanced** as possible

Creating a binary tree

- Create a binary tree from a list
- As **balanced** as possible
- **Strategy**:

Creating a binary tree

- Create a binary tree from a list
- As **balanced** as possible
- **Strategy**:
  - Split the list in two halves

Creating a binary tree

- Create a binary tree from a list
- As **balanced** as possible
- **Strategy**:
    - Split the list in two halves
    - Recursively create a binary tree from each half

Creating a binary tree

- Create a binary tree from a list
- As **balanced** as possible
- **Strategy**:
  - Split the list in two halves
  - Recursively create a binary tree from each half
  - Join them together

Creating a binary tree

- Creating a balanced tree from a list

```
createTree :: [a] -> BTree a
createTree [] = Nil
createTree xs = Node
                (createTree front) x (createTree back)
    where
        n = length xs
        (front, x:back) = splitAt (n `div` 2) xs
```

Showing a binary tree

- To be able to show a binary tree, we need to derive a **Show** instance

```
data BTree a = Nil | Node (BTree a) a (BTree a)
    deriving Show

createTree [0..14] =
Node (Node (Node (Node Nil 0 Nil) 1 (Node Nil 2 Nil))
     3 (Node (Node Nil 4 Nil) 5 (Node Nil 6 Nil)))
7 (Node (Node (Node Nil 8 Nil) 9 (Node Nil 10 Nil))
     11 (Node (Node Nil 12 Nil) 13 (Node Nil 14 Nil)))
```

Showing a binary tree

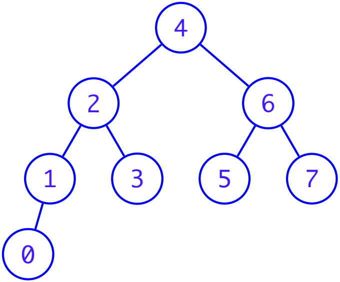
- To be able to show a binary tree, we need to derive a **Show** instance

```
data BTree a = Nil | Node (BTree a) a (BTree a)
    deriving Show

createTree [0..14] =
Node (Node (Node (Node Nil 0 Nil) 1 (Node Nil 2 Nil))
     3 (Node (Node Nil 4 Nil) 5 (Node Nil 6 Nil)))
7 (Node (Node (Node Nil 8 Nil) 9 (Node Nil 10 Nil))
     11 (Node (Node Nil 12 Nil) 13 (Node Nil 14 Nil)))
```

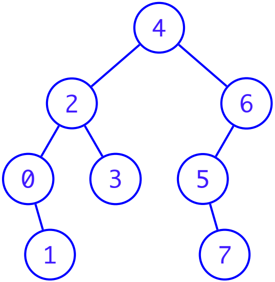
- Not particularly readable!

A custom show




```
4
  2
    1
      0
      *
    3
  6
    5
    7
```

A custom show

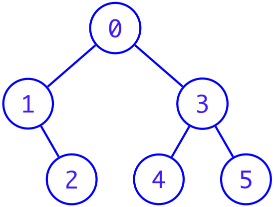



```
   4
     2
       0
         *
         1
       3
     6
       5
         *
         7
       *
```

A custom show

```
instance Show a => Show (BTree a) where
    show = intercalate "\n" (draw t)
draw :: Show a => BTree a -> [String]
draw Nil = ["*"]
draw (Node Nil x Nil) = [show x]
draw (Node tl x tr) = [show x] ++
                      shift (draw tl) ++
                      shift (draw tr)
   where shift = zipWith (++) (repeat "  ")
```

A custom show




```
0
|
+-1
| |
| +-*
| |
| `-2
|
`-3
  |
  +-4
  |
  `-5
```

A custom show

```
instance Show a => Show (BTree a) where
    show = intercalate "\n" (draw2 t)
draw2 :: Show a => BTree a -> [String]
draw2 Nil = ["*"]
draw2 (Node Nil x Nil) = [show x]
draw2 (Node tl x tr) = [show x] ++
                      shiftl (draw2 tl) ++
                      shiftr (draw2 tr)
   where shiftl = zipWith (++) ("+-":repeat "| ")
         shiftr = zipWith (++) ("`-":repeat "  ")
```

Creating a binary tree

- Creating a balanced tree from a list

```
createTree :: [a] -> BTree a
createTree [] = Nil
createTree xs = Node
                (createTree front) x (createTree back)
    where
        n = length xs
        (front, x:back) = splitAt (n `div` 2) xs
```

Creating a binary tree

- `length` and `splitAt` take linear time

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$
- Can we improve?

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$
- Can we improve?
  - Culprit is the repeated use of `length` and `splitAt`

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$
- Can we improve?
  - Culprit is the repeated use of `length` and `splitAt`
  - Can we avoid that?

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$
- Can we improve?
  - Culprit is the repeated use of `length` and `splitAt`
  - Can we avoid that?
- Consider the following function:

```
go :: Int -> [a] -> (BTree a, [a])
```

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$
- Can we improve?
  - Culprit is the repeated use of `length` and `splitAt`
  - Can we avoid that?
- Consider the following function:

```
go :: Int -> [a] -> (BTree a, [a])
```

- `go n l == (createTree (take n l), drop n l)`

Creating a binary tree

- `length` and `splitAt` take linear time
- $T(n) = 2T(n/2) + O(n)$ and hence $T(n) = O(n \log n)$
- Can we improve?
  - Culprit is the repeated use of `length` and `splitAt`
  - Can we avoid that?
- Consider the following function:

```
go :: Int -> [a] -> (BTree a, [a])
```

- `go n l == (createTree (take n l), drop n l)`
- Can we do this efficiently?

Creating a binary tree

- Creating a tree in linear time

```
createTree :: [a] -> BTree a
createTree l = fst (go (length l) l)
    where
        go :: Int -> [a] -> (BTree a, [a])
        go 0 xs = (Nil, xs)
        go n xs = (Node tl y tr, zs)
            where m = n `div` 2
                  (tl, y:ys) = go m xs
                  (tr, zs) = go (n-m-1) ys
```

Creating a binary tree

- Creating a tree in linear time

```
createTree :: [a] -> BTree a
createTree l = fst (go (length l) l)
    where
        go :: Int -> [a] -> (BTree a, [a])
        go 0 xs = (Nil, xs)
        go n xs = (Node tl y tr, zs)
            where m = n `div` 2
                  (tl, y:ys) = go m xs
                  (tr, zs) = go (n-m-1) ys
```

- $T(n) = 2T(n/2) + c$ and hence $T(n) = (2n-1)c$

The `Set` ADT

- Maintain a collection of distinct elements and support the following operations

The `Set` ADT

- Maintain a collection of distinct elements and support the following operations
  - `insertInto` – insert a given value into the set

The `Set` *ADT*

- Maintain a collection of distinct elements and support the following operations
  - `insertInto` – insert a given value into the set
  - `deleteFrom` – delete a given value from the set

The `Set` ADT

- Maintain a collection of distinct elements and support the following operations
  - `insertInto` – insert a given value into the set
  - `deleteFrom` – delete a given value from the set
  - `search` – check whether a given value is an element of the set

The Set ADT

- Maintain a collection of distinct elements and support the following operations
  - `insertInto` – insert a given value into the set
  - `deleteFrom` – delete a given value from the set
  - `search` – check whether a given value is an element of the set
- Straightforward implementation

```
module Set(Set, insertInto, deleteFrom, search) where
data Set a = Set [a]
```

The Set ADT

```haskell
data Set a = Set [a]

search :: Eq a => a -> Set a -> Bool
search x (Set xs) = x `elem` xs

insertInto :: Eq a => a -> Set a -> Set a
insertInto x (Set xs) = if x `elem` xs then Set xs
                        else Set (x:xs)

deleteFrom :: Eq a => a -> Set a -> Set a
deleteFrom x (Set xs) = Set (filter (/=x) xs)
```

`Set`: *complexity*

- `search` takes $O(n)$ time

`Set`: *complexity*

- `search` takes $O(n)$ time
- `insertInto` takes $O(n)$ time

`Set`: *complexity*

- `search` takes $O(n)$ time
- `insertInto` takes $O(n)$ time
- `deleteFrom` takes $O(n)$ time

`Set`: *complexity*

- `search` takes $O(n)$ time
- `insertInto` takes $O(n)$ time
- `deleteFrom` takes $O(n)$ time
- A sequence of n operations takes $O(n^2)$ time

`Set`: *complexity*

- `search` takes $O(n)$ time
- `insertInto` takes $O(n)$ time
- `deleteFrom` takes $O(n)$ time
- A sequence of n operations takes $O(n^2)$ time
- We can do better if the elements admit an order

Binary search trees

- A **binary search tree** is another way of implementing the `Set` ADT

Binary search trees

- A **binary search tree** is another way of implementing the `Set` ADT
- A binary search tree is a binary tree

Binary search trees

- A **binary search tree** is another way of implementing the `Set` ADT
- A binary search tree is a binary tree
- Stores values of type `a` such that `Ord a`

Binary search trees

- A **binary search tree** is another way of implementing the `Set` ADT
- A binary search tree is a binary tree
- Stores values of type `a` such that `Ord a`
- In a binary search tree:

Binary search trees

- A **binary search tree** is another way of implementing the `Set` ADT
- A binary search tree is a binary tree
- Stores values of type `a` such that `Ord a`
- In a binary search tree:
    - values in the left subtree are smaller than the root

Binary search trees

- A **binary search tree** is another way of implementing the `Set` ADT
- A binary search tree is a binary tree
- Stores values of type `a` such that `Ord a`
- In a binary search tree:
  - values in the left subtree are smaller than the root
  - values in the right subtree are larger than the root

Binary search trees

- The binary search tree

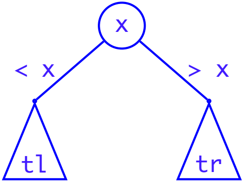
```
Node tl x tr
```

Binary search trees

- The binary search tree

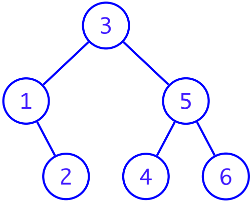
```
Node tl x tr
```

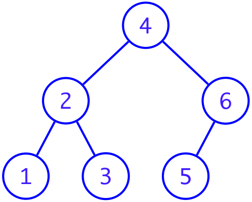
- Pictorially …



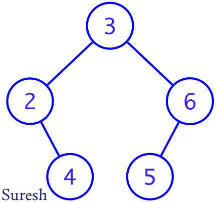
Binary search trees

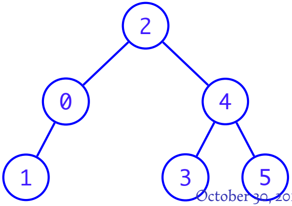
- Examples





- Non-examples





BST

- Binary search trees in Haskell:

```
data BST a = Nil | Node (BST a) a (BST a)
    deriving (Eq, Ord)
```

BST

- Binary search trees in Haskell:

```
data BST a = Nil | Node (BST a) a (BST a)
    deriving (Eq, Ord)
```

- The empty tree:

```
emptyBST :: BST a
emptyBST = Nil
```

BST

- Binary search trees in Haskell:

```
data BST a = Nil | Node (BST a) a (BST a)
    deriving (Eq, Ord)
```

- The empty tree:

```
emptyBST :: BST a
emptyBST = Nil
```

- Is a tree empty?

```
isEmpty :: BST a -> Bool
isEmpty Nil = True
isEmpty _   = False
```

Is it a search tree?

- Just naming it `BST` does not make it a binary search tree

```
isBST :: Ord a => BST a -> Bool
isBST Nil              = True
isBST (Node tl x tr) = isBST tl && isBST tr &&
                        (isEmpty tl || maxt tl < x) &&
                        (isEmpty tr || x < mint tr)
```

Minimum in a tree

- `mint` gives the minimum value in a non-empty tree:

```
mint :: Ord a => BST a -> a
mint (Node tl x tr)  = min x (min y z)
where y = if isEmpty tl then x else mint tl
      z = if isEmpty tr then x else mint tr
```

Minimum in a tree

- `mint` gives the minimum value in a non-empty tree:

```
mint :: Ord a => BST a -> a
mint (Node tl x tr)  = min x (min y z)
where y = if isEmpty tl then x else mint tl
      z = if isEmpty tr then x else mint tr
```

- `maxt` is similar (uses **`max`** instead of **`min`**)

Searching in a tree

- Searching for value v in a search tree

Searching in a tree

- Searching for value `v` in a search tree
- If the tree is empty, report **`False`**

Searching in a tree

- Searching for value `v` in a search tree
- If the tree is empty, report **`False`**
- If the tree is nonempty

Searching in a tree

- Searching for value `v` in a search tree
- If the tree is empty, report **`False`**
- If the tree is nonempty
    - If `v` is the value at the root, report **`True`**

Searching in a tree

- Searching for value `v` in a search tree
- If the tree is empty, report **`False`**
- If the tree is nonempty
    - If `v` is the value at the root, report **`True`**
    - If `v` is smaller than the value at the root, search in left subtree

Searching in a tree

- Searching for value `v` in a search tree
- If the tree is empty, report **`False`**
- If the tree is nonempty
  - If `v` is the value at the root, report **`True`**
  - If `v` is smaller than the value at the root, search in left subtree
  - If `v` is larger than the value at the root, search in right subtree

Searching in a tree

- Haskell code transcribe the search strategy:

```
searchBST :: Ord a => a -> BST a -> Bool
searchBST v Nil        = False
searchBST v (Node tl x tr)
    | v == x           = True
    | v < x            = searchBST v tl
    | v > x            = searchBST v tr
```

Searching in a tree

- Haskell code transcribe the search strategy:

```
searchBST :: Ord a => a -> BST a -> Bool
searchBST v Nil      = False
searchBST v (Node tl x tr)
    | v == x         = True
    | v < x          = searchBST v tl
    | v > x          = searchBST v tr
```

- Worst case running time: Height of the tree

Inserting in a tree

- Inserting an element into a tree follows a similar strategy:

```
insertBST :: Ord a => a -> BST a -> BST a
insertBST v Nil = Node Nil v Nil
insertBST v t@(Node tl x tr)
    | v < x     = Node (insertBST v tl) x tr
    | v > x     = Node tl x (insertBST v tr)
    | v == x    = t
```

Inserting in a tree

- Inserting an element into a tree follows a similar strategy:

```
insertBST :: Ord a => a -> BST a -> BST a
insertBST v Nil = Node Nil v Nil
insertBST v t@(Node tl x tr)
    | v < x     = Node (insertBST v tl) x tr
    | v > x     = Node tl x (insertBST v tr)
    | v == x    = t
```

- **Worst case running time**: Height of the tree

Inserting in a tree

- Inserting an element into a tree follows a similar strategy:

```
insertBST :: Ord a => a -> BST a -> BST a
insertBST v Nil = Node Nil v Nil
insertBST v t@(Node tl x tr)
    | v < x     = Node (insertBST v tl) x tr
    | v > x     = Node tl x (insertBST v tr)
    | v == x    = t
```

- Worst case running time: Height of the tree
- `@` allows one to refer to a data value by a name

Deleting from a tree

- We first tackle a simpler problem

Deleting from a tree

- We first tackle a simpler problem
    - Delete the maximum value

Deleting from a tree

- We first tackle a simpler problem
  - Delete the maximum value
  - and return the value as well as the modified tree

Deleting from a tree

- We first tackle a simpler problem
  - Delete the maximum value
  - and return the value as well as the modified tree
- Maximum is the rightmost node:

```
deleteMax :: Ord a => BST a -> (a, BST a)
deleteMax (Node tl x Nil) = (x, tl)
deleteMax (Node tl x tr) = let (y, ty) = deleteMax tr
                           in (y, Node tl x ty)
```

Deleting from a tree

- We first tackle a simpler problem
  - Delete the maximum value
  - and return the value as well as the modified tree
- Maximum is the rightmost node:

```
deleteMax :: Ord a => BST a -> (a, BST a)
deleteMax (Node tl x Nil) = (x, tl)
deleteMax (Node tl x tr) = let (y, ty) = deleteMax tr
                           in (y, Node tl x ty)
```

- Worst case running time: Height of the tree

Deleting from a tree

- Deleting a value `v` from a search tree

Deleting from a tree

- Deleting a value v from a search tree
- If the tree is empty, nothing to do

Deleting from a tree

- Deleting a value v from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty

Deleting from a tree

- Deleting a value `v` from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty
    - If `v` is smaller than the value at the root, delete from left subtree

Deleting from a tree

- Deleting a value v from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty
    - If v is smaller than the value at the root, delete from left subtree
    - If v is larger than the value at the root, delete from right subtree

Deleting from a tree

- Deleting a value v from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty
  - If v is smaller than the value at the root, delete from left subtree
  - If v is larger than the value at the root, delete from right subtree
  - If v is equal to the value at the root, remove root and

Deleting from a tree

- Deleting a value `v` from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty
    - If `v` is smaller than the value at the root, delete from left subtree
    - If `v` is larger than the value at the root, delete from right subtree
    - If `v` is equal to the value at the root, remove root and
        - If left subtree is empty, just slide the right subtree up a level

Deleting from a tree

- Deleting a value `v` from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty
    - If `v` is smaller than the value at the root, delete from left subtree
    - If `v` is larger than the value at the root, delete from right subtree
    - If `v` is equal to the value at the root, remove root and
        - If left subtree is empty, just slide the right subtree up a level
        - If left subtree is non-empty, replace root by maximum element of left subtree

Deleting from a tree

- Deleting a value v from a search tree
- If the tree is empty, nothing to do
- If the tree is nonempty
  - If v is smaller than the value at the root, delete from left subtree
  - If v is larger than the value at the root, delete from right subtree
  - If v is equal to the value at the root, remove root and
    - If left subtree is empty, just slide the right subtree up a level
    - If left subtree is non-empty, replace root by maximum element of left subtree
    - Search tree property is preserved

Deleting from a tree

- Code for delete:

```
deleteBST :: Ord a => a -> BST a -> BST a
deleteBST v Nil = Nil
deleteBST v (Node tl x tr)
| v < x           = Node (deleteBST v tl) x tr
| v > x           = Node tl x (deleteBST v tr)
| v == x          = if isEmpty tl then tr else Node ty y tr
    where (y, ty) = deleteMax tl
```

Deleting from a tree

- Code for delete:

```
deleteBST :: Ord a => a -> BST a -> BST a
deleteBST v Nil = Nil
deleteBST v (Node tl x tr)
| v < x          = Node (deleteBST v tl) x tr
| v > x          = Node tl x (deleteBST v tr)
| v == x         = if isEmpty tl then tr else Node ty y tr
    where (y, ty) = deleteMax tl
```

- Worst case running time: Height of the tree