Programming in Haskell: Lecture 15

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Fibonacci numbers

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- For $n > 2$, there is one call to `fib (n-1)` and one to `fib (n-2)`.
- So $G(n) = G(n-1) + G(n-2) = F(n-1) + F(n-2) = F(n)$.
- Effectively computing $F(n)$ by adding up so many 1s

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- $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887$
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- Thus $G(n) = F(n)$ is exponential in n

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- Multiple recursive calls with the same argument
- Wasteful recomputation!
- Suffices to keep track of two values:

```
fib = go (0,1)
    where
        go (a,b) 0 = a
        go (a,b) n = go (b, a+b) (n-1)
```

Fibonacci numbers

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- `z = zipWith (+) (0:1:z) (1:z)`

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- Then `fibs = 0:1:z`
- Substituting, we can define `z` without referring to `fibs`
- `z = zipWith (+) (0:1:z) (1:z)`
- Thus `z = 1:zipWith (+) (1:z) z`

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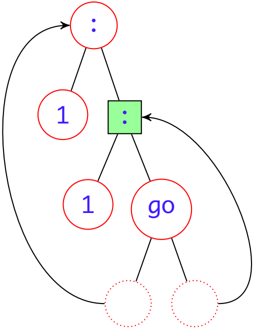
- Let `go = zipWith (+)` and remember the list is infinite (hence nonempty)
- Final code:

```
fib = (!!) fibs
fibs = 0:1:z
    where z = 1:go (1:z) z
          go (x:xs) (y:ys) = x+y: go xs ys
```

Computing `fibs`



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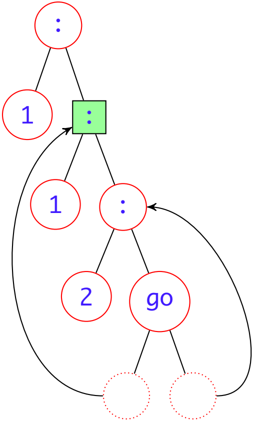

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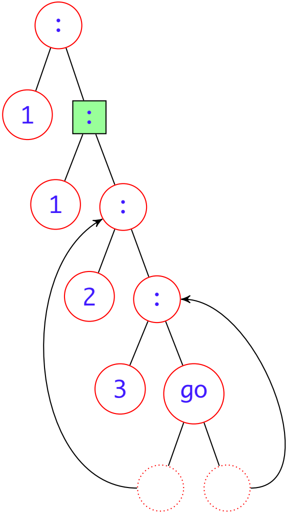

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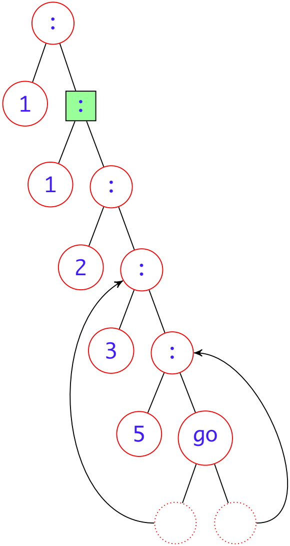

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- The pointers move down as `go` is evaluated more and more
- To compute `fib n` we expand the tree to `n` levels

Dynamic programming

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- Another example next

Longest common subsequence

- Given two strings `as` and `bs`, find the **length** of the **longest common subsequence** of `as` and `bs`

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- Haskell function `lcs`:

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    lcs "agcat" "gact" = 3 -- subsequence "gat"
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    - … and compute recursively

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lcs "" _  = 0
lcs _  "" = 0
lcs as bs = if a == b then 1 + lcs as' bs'
            else max (lcs as' bs) (lcs as bs')
    where (a, as') = (head as, tail as)
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- Many recursive calls repeated with same arguments

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- Important exercise in reasoning about programs
- First step: express the recursion in terms of prefixes

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lcs as bs = if a == b then 1 + lcs as' bs'
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    where (as', a) = (init as, last as)
          (bs', b) = (init bs, last bs)
```

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- Let `length as = m` and `length bs = n`
- For `i <- [0..m]` and `j <- [0..n]`, let

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    f i j = lcs (take i as) (take j bs)
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- Then we can define `f` directly as follows:

```
    f 0 _ = 0
    f _ 0 = 0
    f i j = g (as!!(i-1)) (bs!!(j-1))
              (f (i-1) (j-1), f (i-1) j, f i (j-1))
        where g a b (d,u,l) =
              if a == b then 1+d else max l u
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- `l 0 = replicate (n+1) 0`
- For `i > 0`,

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l i = 0: [g (as!!(i-1)) (bs!!(j-1))
            (f (i-1) (j-1), f (i-1) j, f i (j-1))
               | j <- [1..n]]
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- Observe that:

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    zip3 (l (i-1)) (tail (l (i-1))) (l i) =
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- So

```
l i = 0 : zipWith (g (as!!(i-1))) bs
             (zip3 (l (i-1)) (tail (l (i-1))) (l i))
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- Can clean it further:

```
l i = nextList (as!!(i-1)) (l (i-1))
    where nextList a l = 0 : zipWith (g a) bs
          (zip3 l (tail l) (nextList a l))
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- Recall that `l 0` is just a list of `0`s
- The final answer we want is `f m n = last (last lcsTab)`

Putting it all together

```haskell
lcs :: String -> String -> Int
lcs as bs = last (last lcsTab)
    where
    lcsTab        = firstList : zipWith nextList as lcsTab
    firstList     = replicate (length bs + 1) 0
    nextList a l  = 0: zipWith (g a) bs
                            (zip3 l (tail l) (nextList a l))
    g a b (d,u,l) = if a == b then 1 + d else (max u l)
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- Sample runs on the strings `"abababababababababababababababababababab"` and `"bbbbbbbbbbbbbbbbbbbbbbbbb"`
  - Answer is `19`
  - Naive recursion: `(32.35 secs, 19,422,476,336 bytes)`

Complexity of `lcs`

- Laziness ensures that `lcsTab` is expanded as needed
- An analysis similar to `fib` can be performed
- `lcsTab` is computed completely in $O(m \cdot n)$ time
- Sample runs on the strings `"abababababababababababababababababababab"` and `"bbbbbbbbbbbbbbbbbbbbbbbbb"`
    - Answer is `19`
    - Naive recursion: `(32.35 secs, 19,422,476,336 bytes)`
    - DP version: `(0.01 secs, 3,504,504 bytes)`

Computing the subsequence itself

```
lcs :: String -> String -> (Int, String)
lcs as bs = last (last lcsTab)
    where
    lcsTab          = firstList : zipWith nextList as lcsTab
    firstList       = replicate (length bs + 1) (0, "")
    nextList a l    = (0, ""): zipWith (g a) bs
                           (zip3 l (tail l) (nextList a l))
    g a b (d,u,l)   = if a == b
                      then (1 + fst d, snd d ++ [b])
                      else
                          if fst u > fst l then u else l
```