Programming in Haskell
Aug-Nov 2015

LECTURE 11

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Measuring efficiency

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- Computation is reduction
    - Application of definitions as rewriting rules

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  - Application of definitions as rewriting rules
- Count the number of reduction steps
  - Running time is T(n) for input size n

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  `1:(2:(3:([4,5,6])))`
- `l1 ++ l2` : use the second rule `length l1` times, first rule once, always

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elem :: Int -> [Int] -> Bool
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- `elem 3 [3,7,8,9]` ⇛ `True`

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- `elem 3 [3,7,8,9]` ⇛ `True`
- Complexity depends on input size and value

Variation across inputs

- Worst case complexity
  - Maximum running time over all inputs of size n
  - Pessimistic: may be rare
- Average case
  - More realistic, but difficult/impossible to compute

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- Ignore constant factors, lower order terms
    - $O(n)$, $O(n \log n)$, $O(n^k)$, $O(2^n)$, …

Asymptotic complexity …

- Complexity of ++ is O(n), where n is the length of the first list
- Complexity of elem is O(n)
  - Worst case!

Complexity of reverse

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  - `T(0) = 1`
    `T(n) = T(n-1) + n`

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- Analyze directly (like ++), or write a recurrence for T(n)

  -
    ```
    T(0) = 1
    T(n) = T(n-1) + n
    ```
- Solve by expanding the recurrence

Complexity of `reverse` …

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  ...

  = T(0) + 1 + 2 + ... + n

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  ...

  $= T(0) + 1 + 2 + \ldots + n$

  $= 1 + 1 + 2 + \ldots + n = 1 + n(n+1)/2$

  $= O(n^2)$

```
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- Transfer to a new stack, top to bottom

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- Can we do better?
- Imagine we are reversing a stack of heavy stack of books
- Transfer to a new stack, top to bottom
- New stack is in reverse order!

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```
transfer :: [a] -> [a] -> [a]
transfer [] l = l
transfer (x:xs) l = transfer xs (x:l)
```

Speeding up `reverse` …

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- Input size for `transfer l1 l2` is `length l1`

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T(0) = 1
T(n) = T(n-1) + 1
```

- Expanding: T(n) = 1 + 1 + ... + 1 = O(n)

Speeding up `reverse` ...

```
fastreverse :: [a] -> [a]
fastreverse l = transfer l []
```

- Complexity is O(n)
- Need to understand the computational model to achieve efficiency

Summary

- Measure complexity in Haskell in terms of reduction steps
- Account for input size and values
  - Usually worst-case complexity
- Asymptotic complexity
  - Ignore constants, lower order terms
  - $T(n) = O(f(n))$

Sorting

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- Insertion sort

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insert :: Int -> [Int] -> [Int]
insert x [] = [x]
insert x (y:ys)
  | (x <= y) = x:y:ys
  | otherwise y:(insert x ys)
```

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- Clearly T(n) = O(n)

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- Recurrence

```
  T(0) = 1
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```

- Complexity: $T(n) = O(n^2)$

A better strategy?

- Divide list in two equal parts
- Separately sort left and right half
- Combine the two sorted halves to get the full list sorted

Combining sorted lists

- Given two sorted lists `l1` and `l2`, combine into a sorted list `l3`
  - Compare first element of `l1` and `l2`
  - Move it into `l3`
  - Repeat until all elements in `l1` and `l2` are over
- Merging `l1` and `l2`

Merging two sorted lists

32 74 89

21 55 64

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~~21~~ 55 64

21

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- Sort `l!!0` to `l!!(n/2-1)`
- Sort `l!!(n/2)` to `l!!(n-1)`
- Merge sorted halves into `l'`
- How do we sort the halves?
  - Recursively, using the same strategy!

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43	32	22	78	63	57	91	13

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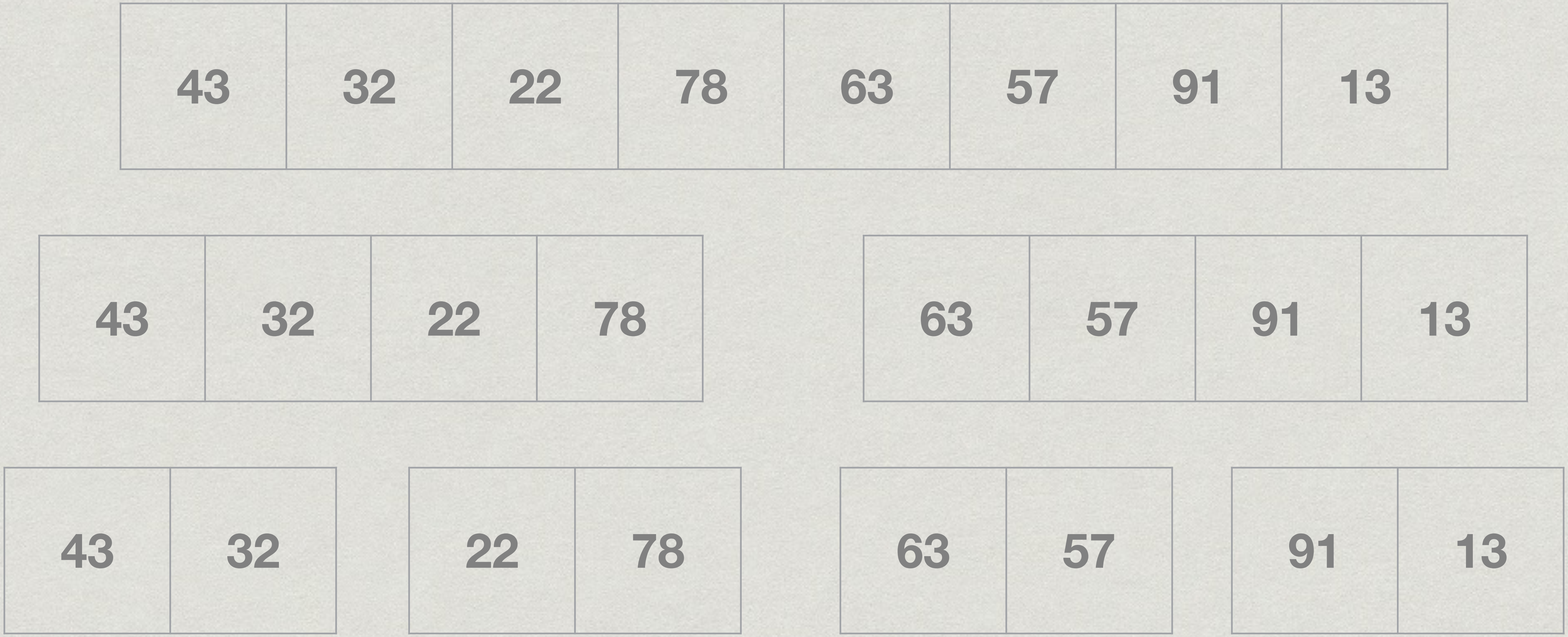
Merge Sort



43 | 32 | 22 | 78 | 63 | 57 | 91 | 13

43 | 32 | 22 | 78   63 | 57 | 91 | 13

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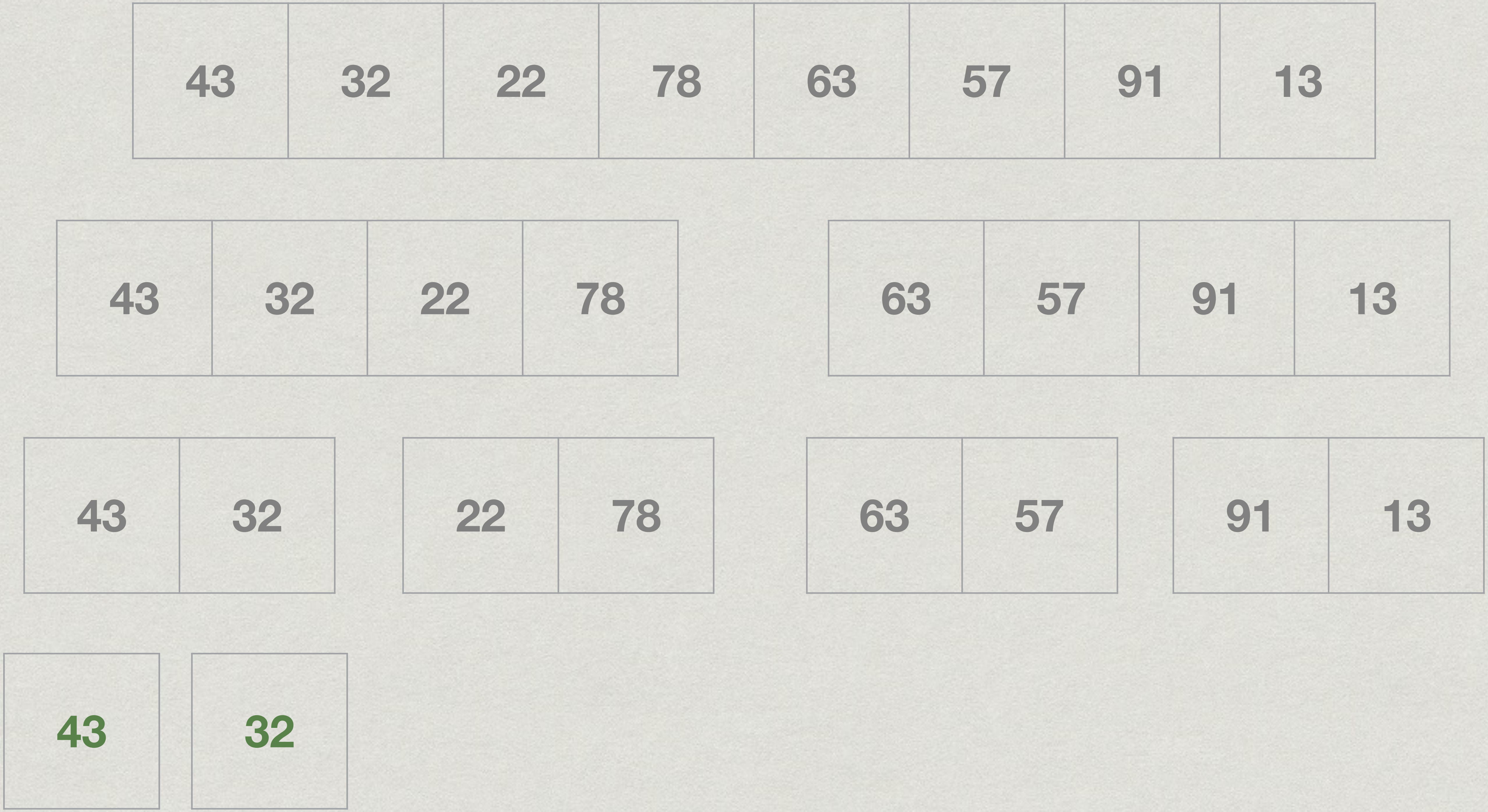
Merge Sort



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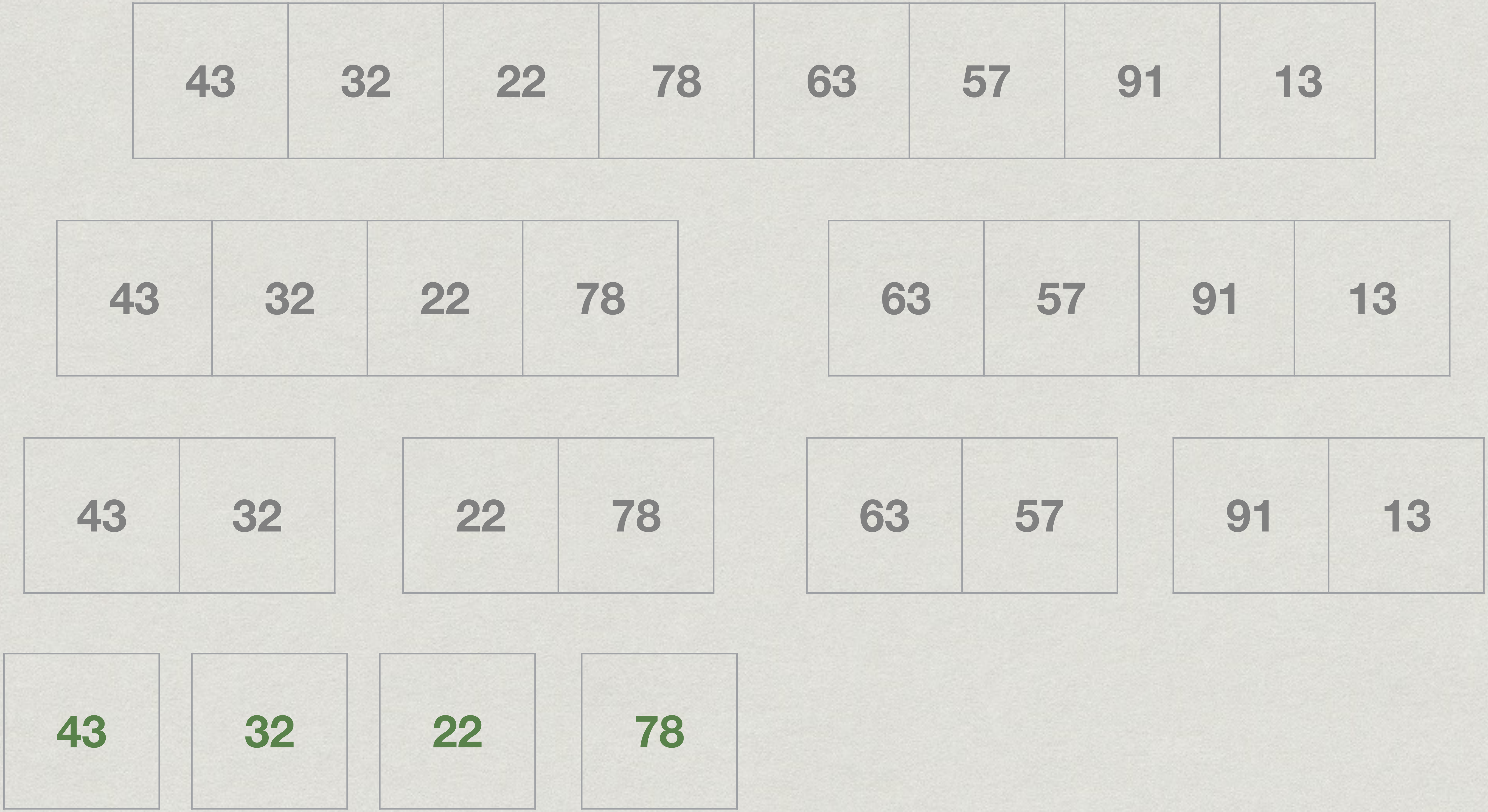

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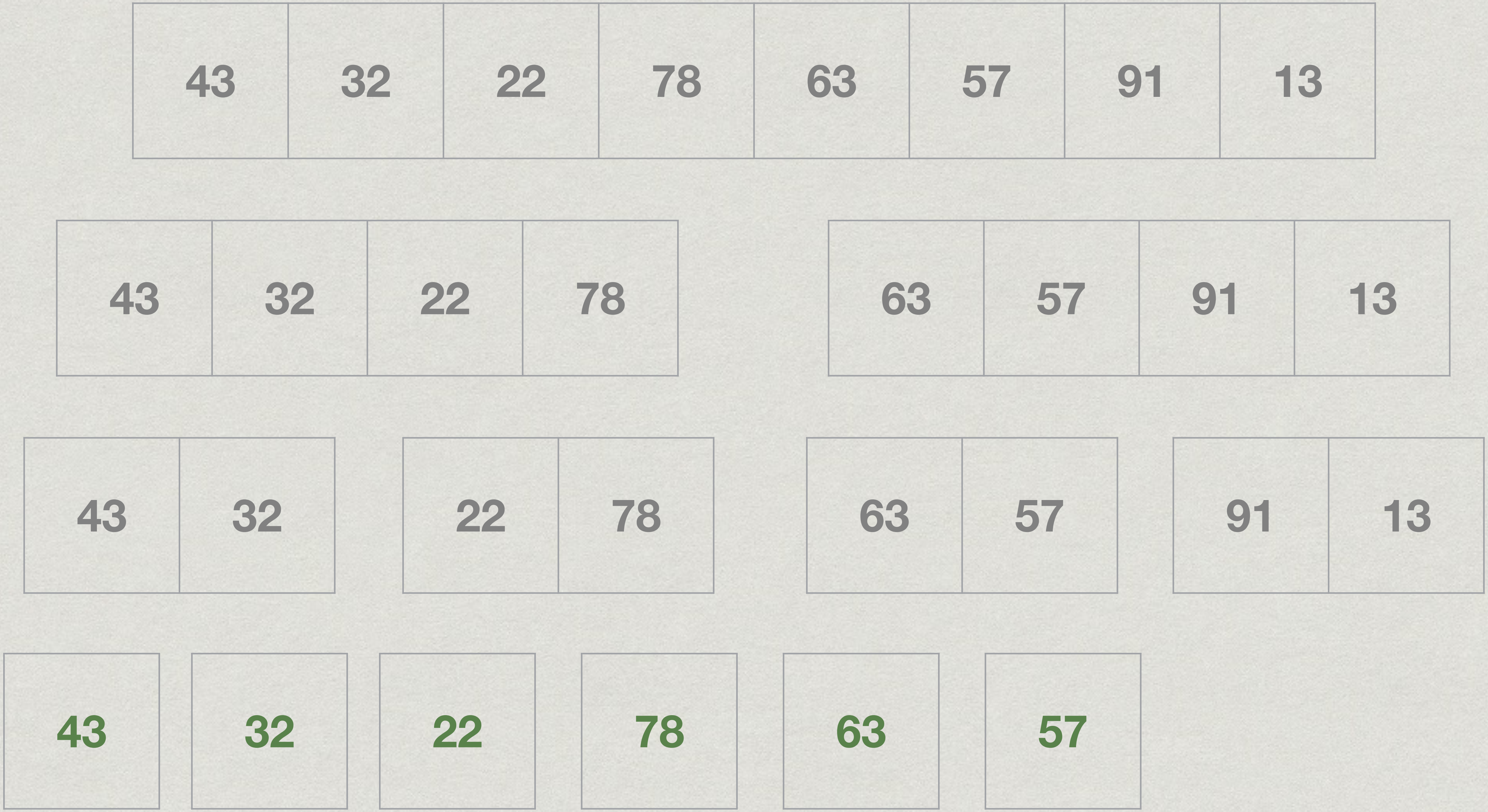

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43 32   22 78   63 57   91 13

43   32   22   78   63   57

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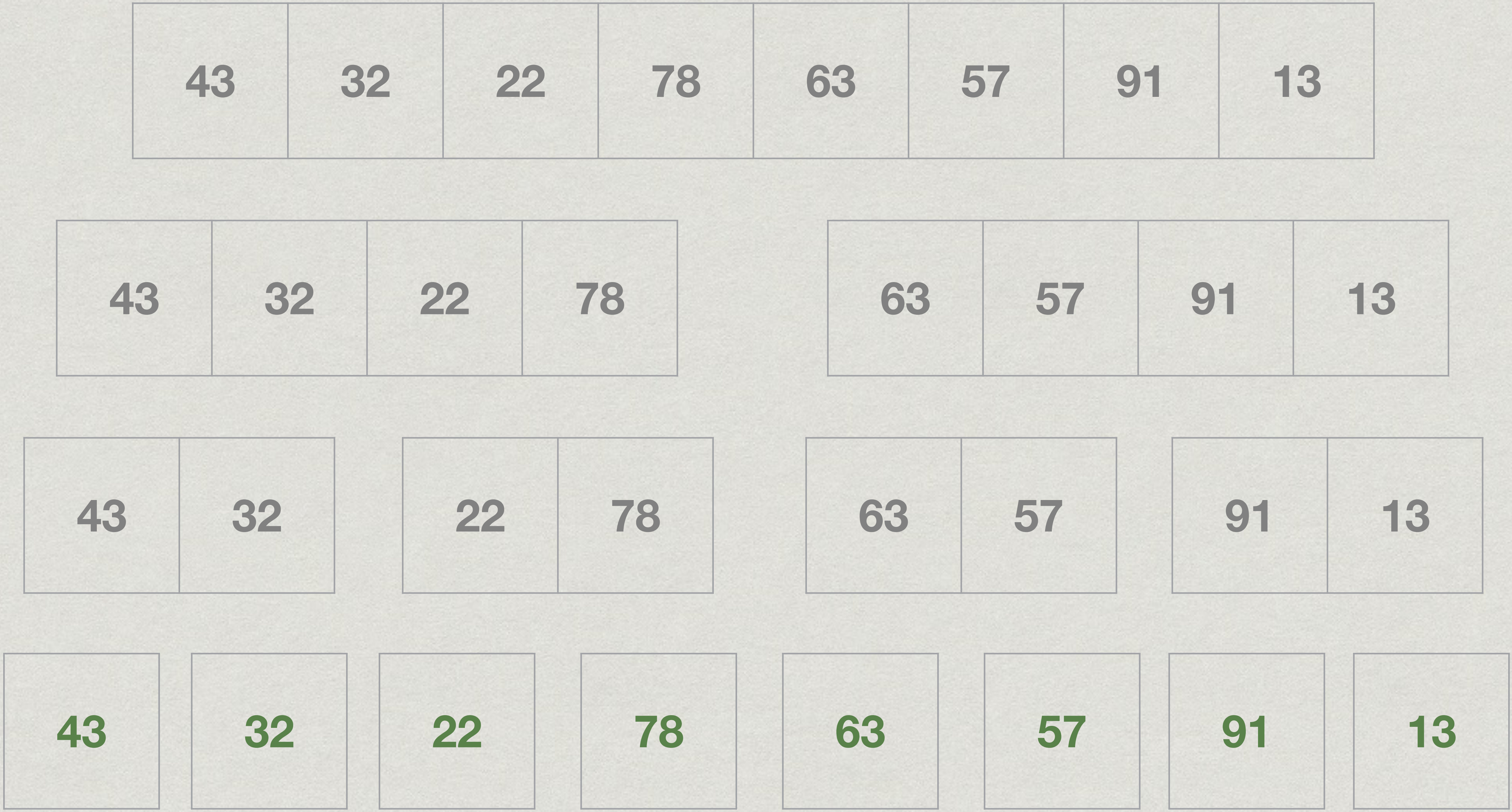

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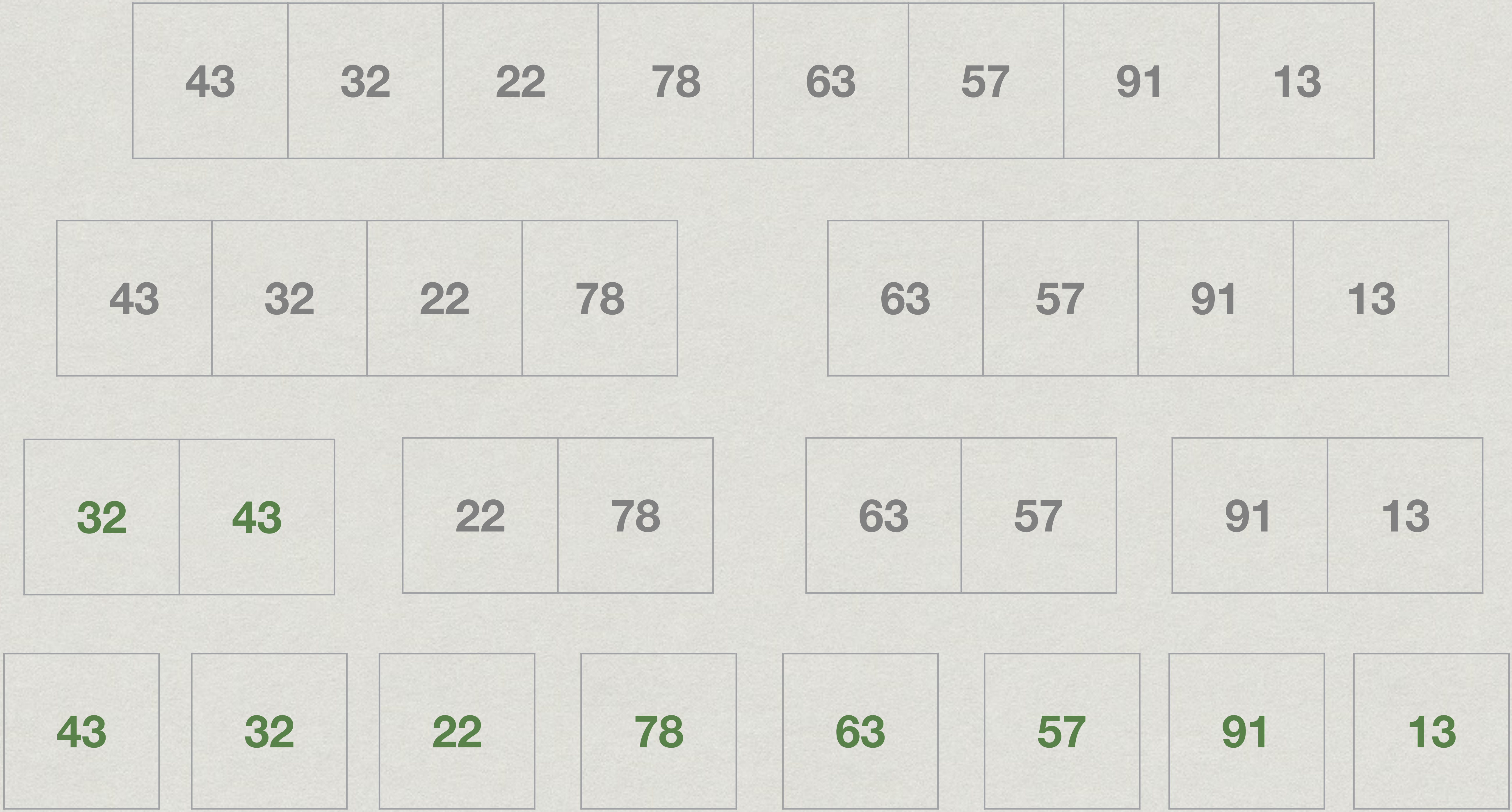

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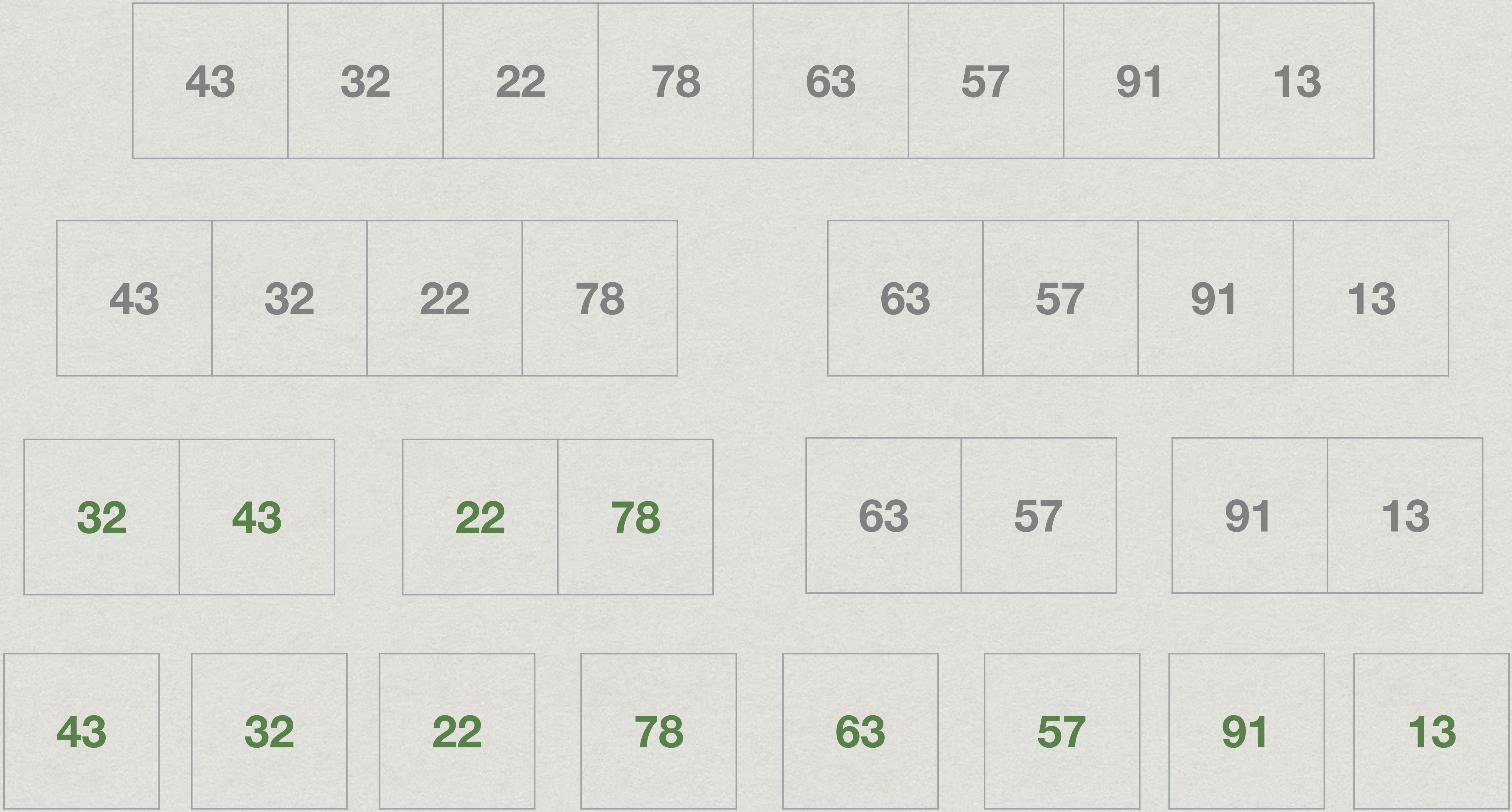

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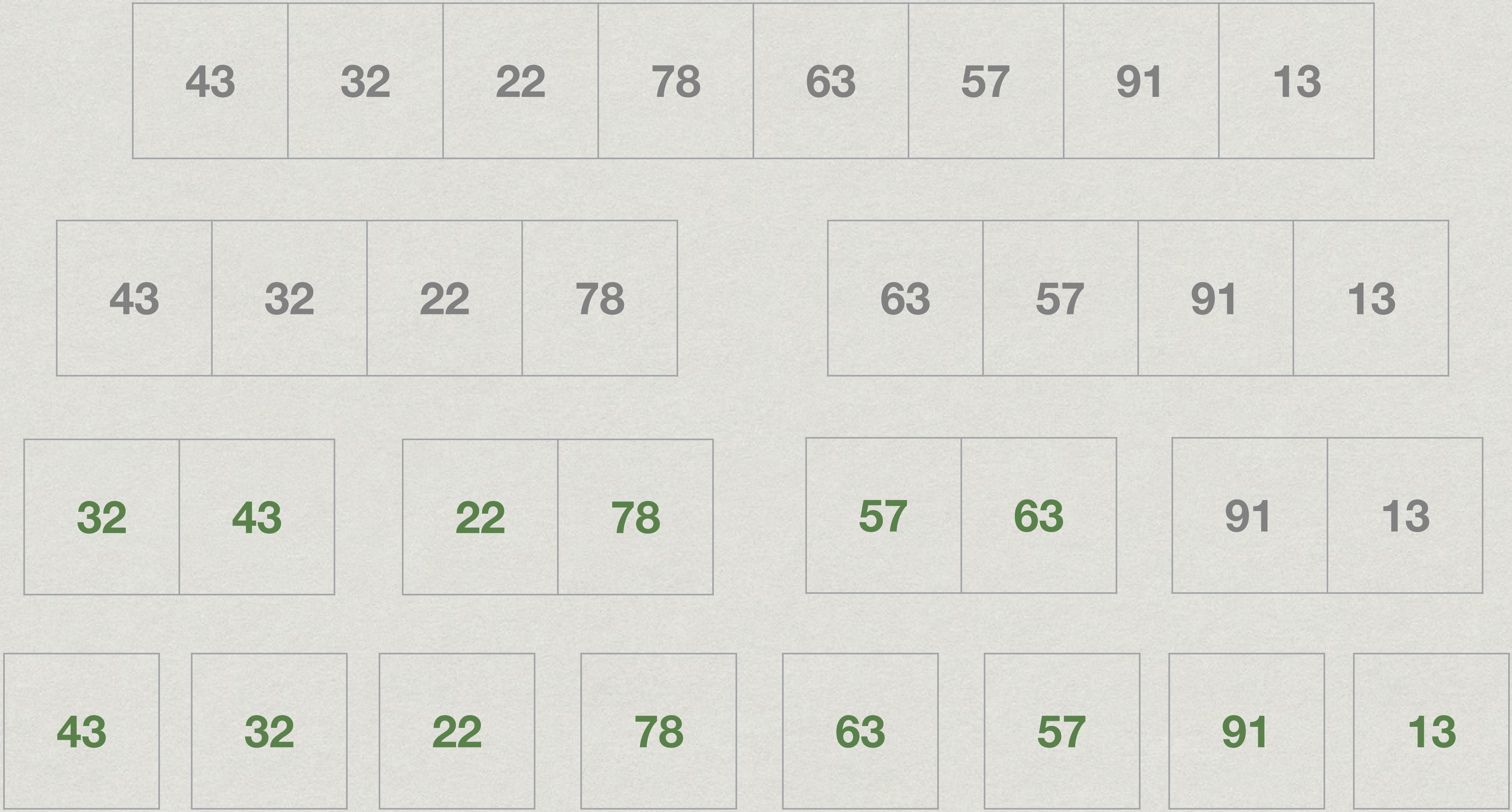

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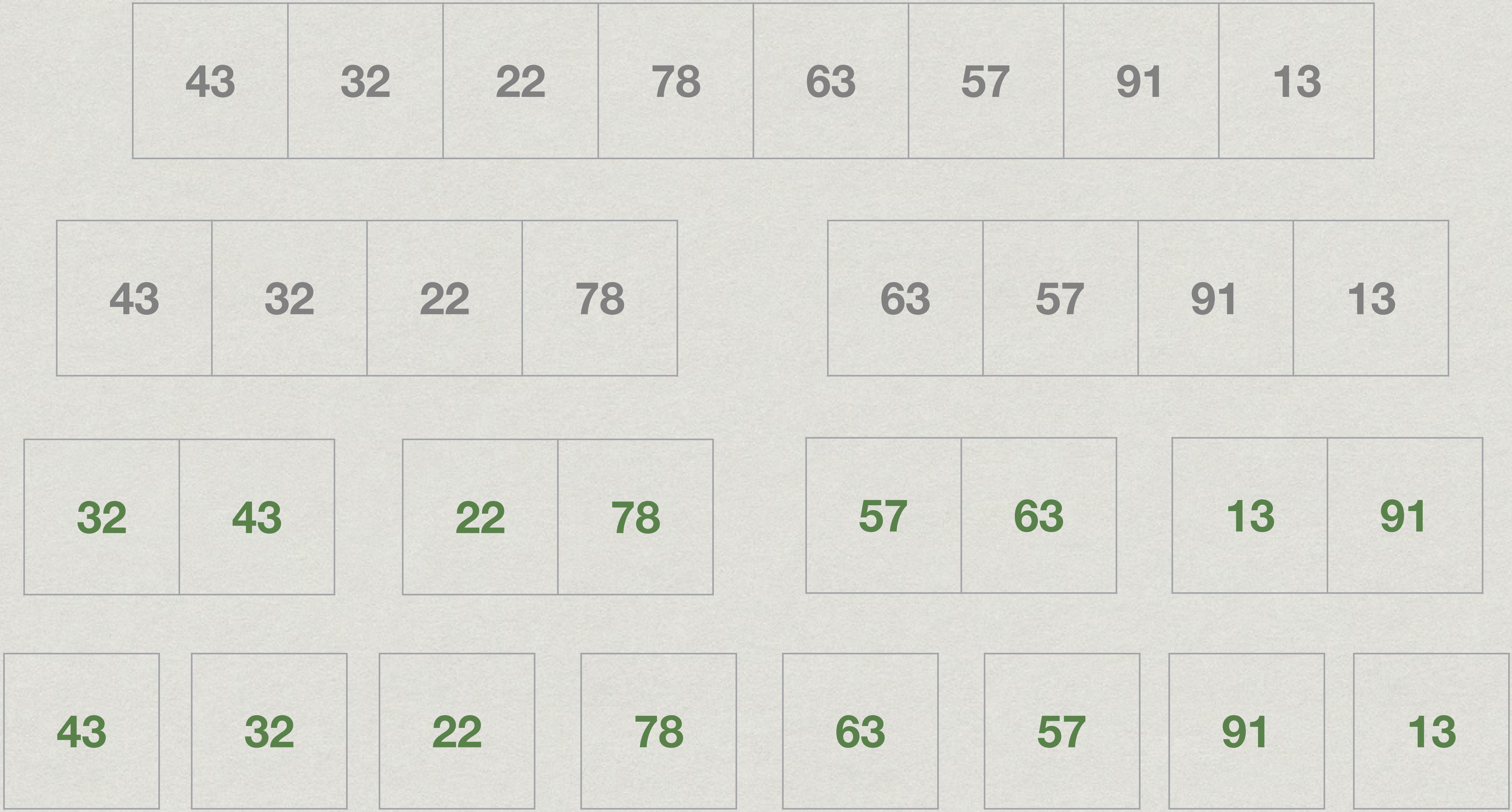

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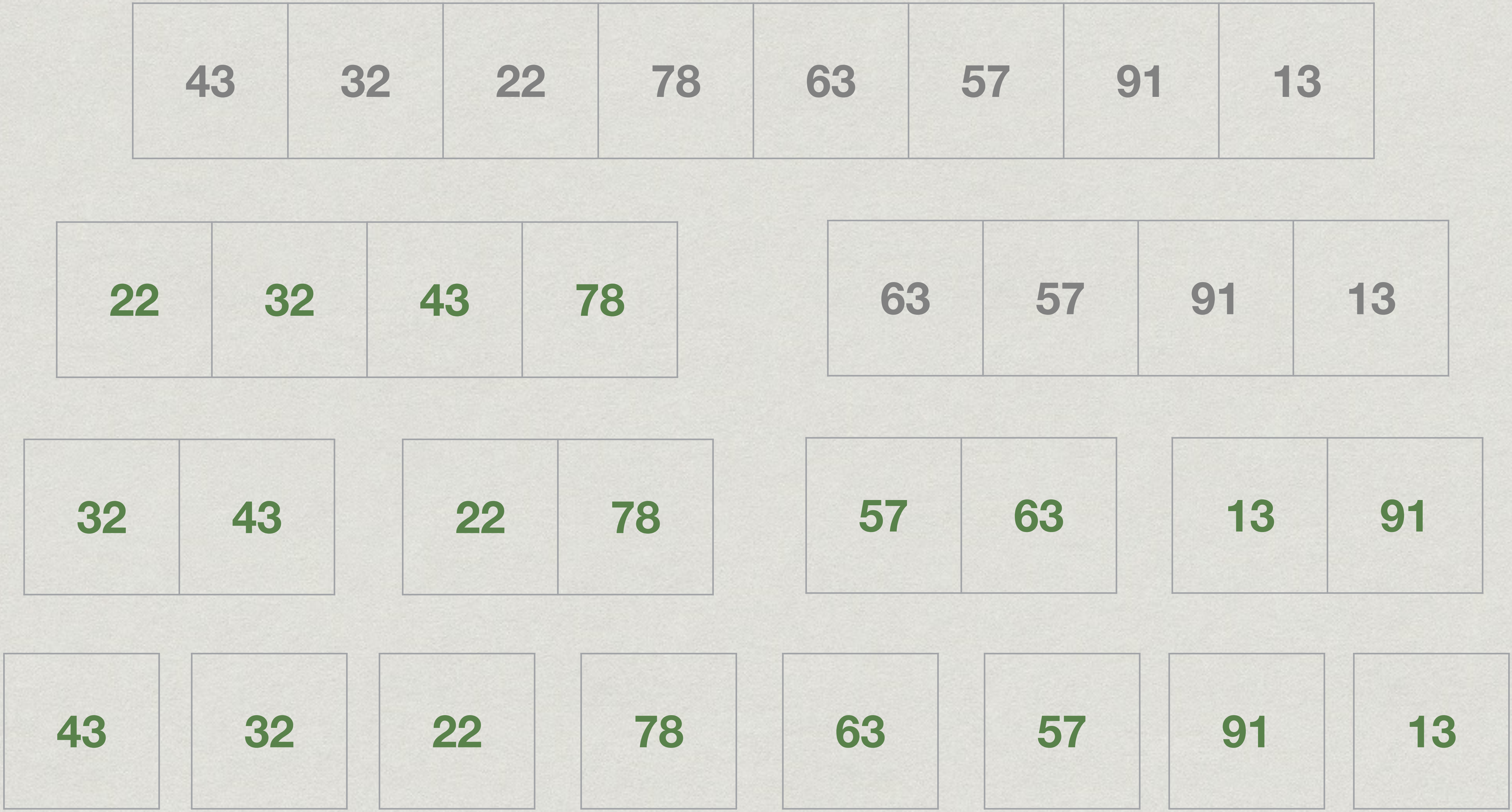

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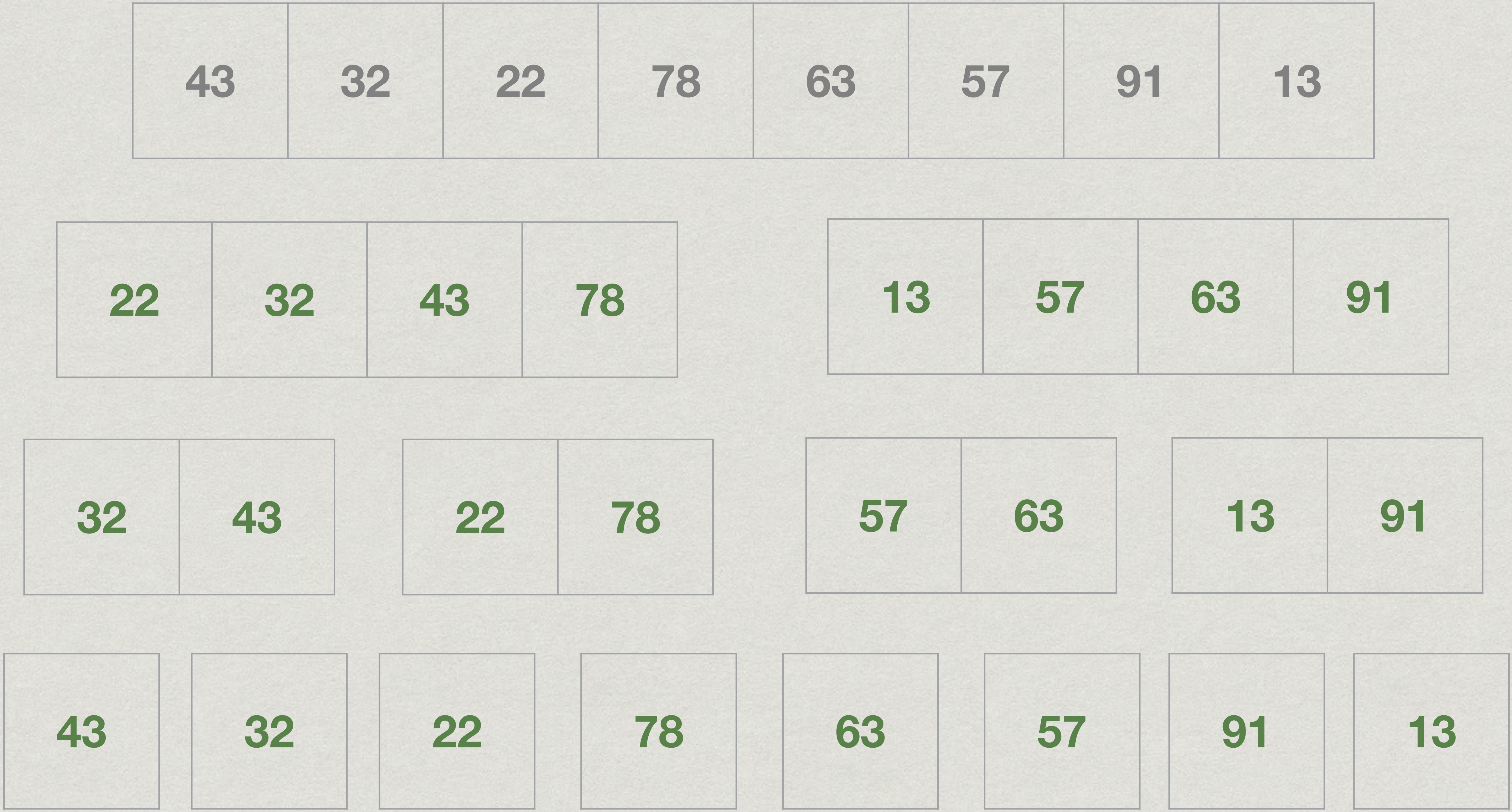

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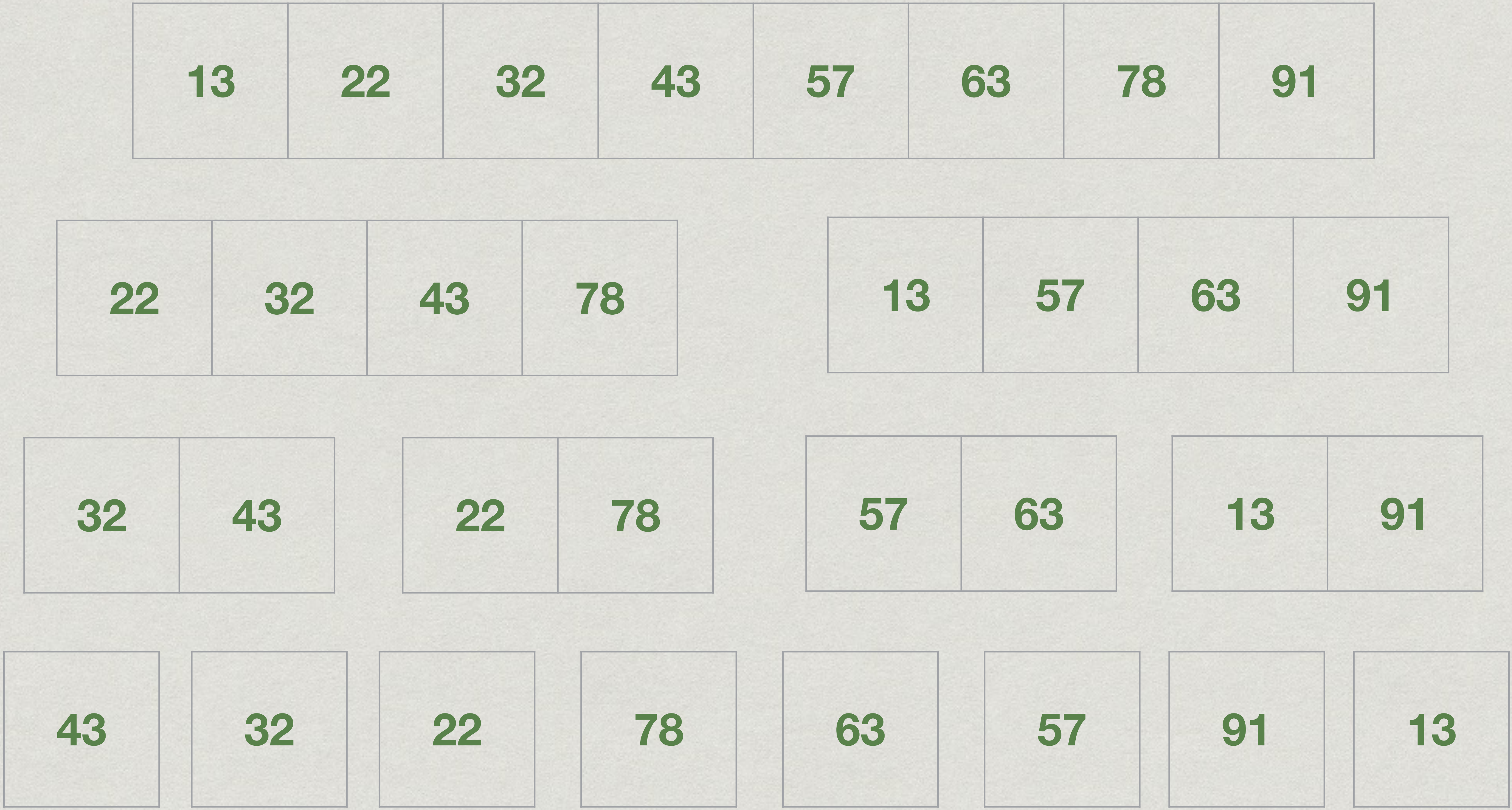

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merge xs [] = xs

merge (x:xs) (y:ys)
  | x <= y    = x:(merge xs (y:ys))
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- Each comparison adds one element to output
- $T(n) = O(n)$, where n is sum of lengths of input lists

Merge sort

```
mergesort :: [Int] -> [Int]
mergesort [] = []
mergesort [x] = [x]
mergesort l = merge (mergesort (front l))
                    (mergesort (back l))
  where
  front l = take ((length l) `div` 2) l
  back l = drop ((length l) `div` 2) l
```

Analysis of Merge Sort

- $T(n)$: time taken by Merge Sort on input of size n
  - Assume, for simplicity, that $n = 2^k$
  - $T(n) = 2T(n/2) + 2n$
  - Two subproblems of size $n/2$
  - Splitting the list into front and back takes n steps
  - Merging solutions requires time $O(n/2+n/2) = O(n)$
- Solve the recurrence by unwinding

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- $T(n) = 2^j\, T(n/2^j) + 2jn = 2^{\log n} + 2(\log n)\, n =$ $n + 2n \log n = O(n \log n)$

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- List is now sorted! No need to merge

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  - Split list with respect to this pivot element

Quicksort

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- Recursively sort the two partitions

Quicksort

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Quicksort

```
quicksort :: [Int] -> [Int]
quicksort [] = []
quicksort (x:xs) = (quicksort lower) ++
                   [splitter] ++
                   (quicksort upper)
   where
   splitter = x
   lower    = [ y | y <- xs, y <= x ]
   upper    = [ y | y <- xs, y > x ]
```

Analysis of Quicksort

Worst case

- Pivot is maximum or minimum
  - One partition is empty
  - Other is size $n-1$
  - $T(n) = T(n-1) + n = T(n-2) + (n-1) + n$
    $= \ldots = 1 + 2 + \ldots + n = O(n^2)$
- Already sorted array is worst case input!

Analysis of Quicksort

But …

- Average case is O(n log n)
  - Sorting is a rare example where average case can be computed
- What does average case mean?

Quicksort: Average case

- Assume input is a permutation of {1,2,…,n}
  - Actual values not important
  - Only relative order matters
  - Each input is equally likely (uniform probability)
- Calculate running time across all inputs
- **Expected** running time can be shown O(n log n)

Summary

- Sorting is an important starting point for many functions on lists
- Insertion sort is a natural inductive sort whose complexity is $O(n^2)$
- Merge sort has complexity $O(n \log n)$
- Quicksort has worst-case complexity $O(n^2)$ but average-case complexity $O(n \log n)$