Programming Language Concepts: Lecture 18

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- Observe that if x does not occur free in M, then

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- Thus $\lambda x.(Mx)$ behaves just like M
- New reduction rule η (when $x \notin FV(M)$)

$$\lambda x.(Mx) \longrightarrow_{\eta} M$$

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$$(\lambda yz.z)((\lambda x.xx)(\lambda x.xx)) \longrightarrow_\beta \lambda z.z$$

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- Choice of reduction strategy may determine whether a normal form can be reached ...
- ...but can more than one normal form be reached?

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- Yes! We can do a breadth-first search of the reduction graph, and we are guaranteed to find a normal form eventually
- We could also reduce the term following the strategy of leftmost outermost reduction
- If a term has a normal form, leftmost outermost reduction will find it!

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- We have seen how to encode recursive functions in the λ-calculus
- We cannot in general determine if the computation of $f(n)$ terminates, given f and n
- But computing $f(n)$ is equivalent to asking if $[f][n]$ has a normal form
- So checking whether a given term has a normal form is undecidable

Church-Rosser theorem

Theorem (Church-Rosser)

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- Answer: No!

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  - Then $M \longleftrightarrow N$
  - Thus there is a P such that $M \xrightarrow{*} P$ and $N \xrightarrow{*} P$ (by Church-Rosser)
  - But since M and N are already in normal form, $M = P = N$ (upto renaming of bound variables)

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- Recall: $M \longleftrightarrow N$ iff there is a sequence $M = M_0, M_1, \ldots, M_k = N$ such that for all $i < k$: either $M_i \longrightarrow N$ or $N \longrightarrow M$

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  - If $M_{i+1} \longrightarrow M_i$, take $P_{i+1} = P_i$
  - If $M_i \longrightarrow M_{i+1}$, use the Diamond property to arrive at the desired P_{i+1}

$\square$

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- R has the Diamond property if

$$(\forall a, b, c)[(aRb \wedge aRc) \Rightarrow (\exists d)(bRd \wedge cRd)]$$

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The proof is by induction on length of R-chains

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Unfortunately, $\longrightarrow$ does not have the Diamond property!

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- Let $\Omega = \lambda x.xx$ and $\mathbf{I} = \lambda x.x$
- $\Omega(\mathbf{II}) \longrightarrow (\mathbf{II})(\mathbf{II})$ by outermost reduction and $\Omega(\mathbf{II}) \longrightarrow \Omega\mathbf{I}$ by innermost reduction

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- $\Omega(\mathbf{II}) \longrightarrow (\mathbf{II})(\mathbf{II})$ by outermost reduction and $\Omega(\mathbf{II}) \longrightarrow \Omega\mathbf{I}$ by innermost reduction
- $\Omega\mathbf{I} \longrightarrow \mathbf{II}$ but it takes two steps to go from $(\mathbf{II})(\mathbf{II})$ to $\mathbf{II}$!

Solution: Define a new "parallel reduction" $\Longrightarrow$ as follows

$$M \Longrightarrow M \qquad \frac{M \Longrightarrow M'}{\lambda x.M \Longrightarrow \lambda x.M'}$$

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  - if $M \Longrightarrow N$ then $M \xrightarrow{*}_\beta N$
  - Hence $M \overset{*}{\Longrightarrow} N$ iff $M \xrightarrow{*}_\beta N$

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  - if $M \Longrightarrow N$ then $M \xrightarrow{*}_\beta N$
  - Hence $M \stackrel{*}{\Longrightarrow} N$ iff $M \xrightarrow{*}_\beta N$
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- The proposition follows

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