3

2) We will prove a stronger claim : if the larger universe has elements $\geq 2^k$ and ~~then~~ smaller one has $2^k - 1$ Spoiler wins

We will induct on k

When $k = 1$ trivially true 

assume true for some $k-1 \geq 1$

__for k__

Spoiler will pick the middle point of the smaller universe

Obviously dup. will choose something from the middle of larger universe

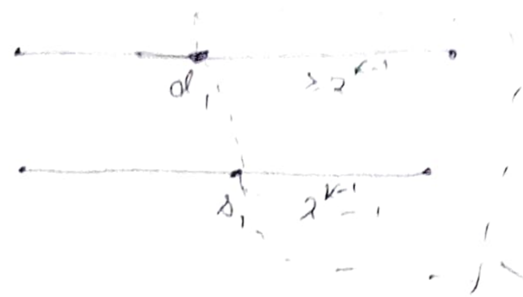
[if she chose some extreme point she will definitely lose]

So the scenario is

Note in the left or right, both side of s_1 there are $\frac{2^{k}-1-1}{2} = 2^{k-1}-1$ many points

One of the sides of d_1 must have $\geq \frac{2^k-1}{2} \geq 2^{k-1}$ many points. say this side is right [WLOG]

Now the its spoiler's turn. She will always pick points from right side of s_1, so that it can force dup. to pick from right side of d_1



5

→ New game is in the right side

and for the right side the case boils down to $k-1$. (IH)

1) a) Take universe $\mathbb{R}$, const $0, 1, -1$

relation $+$

$A(x)$: $x = 0$ → root of $p(x) = x$

$B(x)$: $x + 1 = 0$ → root of $p(x) = x + y$

So

$\exists x \, A(x) \Leftrightarrow \exists x \, B(x)$ is true

1

because

$A(x) = 1$ for $x = 0$

$B(x) = 1$ " $x = 1$

Now $\forall x \, (A(x) \Leftrightarrow B(x))$ is false

Take $x = 0$

$A(x)$ is true

but $B(x)$ is not.

b) Take the univers $\mathbb{R}_{\geq}$, Const. $0, 1$

1

$A(x)$: $x \geq 1$

$B(x)$: $x \geq 0$ Clearly $\forall x \, (A(x) \Rightarrow B(x))$ is true

Now $\exists x \, A(x)$ is true

But $\forall x \, B(x)$ is clearly false

hence $(\exists x \, A(x)) \Rightarrow \forall x \, B(x)$ is false

$FO[k+1]$ looks like

3) $\exists x \; \psi_{i_1}(x) \vee \cdots \vee \psi_{i_r}(x)$ $\quad 1 \le j \le N]$

X

$\psi_{i_1} \cdots \psi_{i_r} \in$

$\psi_{i_r} \in FO[k]$

So there are $2^N - 1$ many possibilities

There are much more