①

1)

$\varphi = p \wedge (q \wedge r) \vee ((\neg p) \wedge ((\neg q) \wedge (\neg r)))$

$= (\neg p \Rightarrow (q \wedge r)) \vee \neg(p \vee \neg((\neg q) \wedge (\neg r)))$

$= (\neg p \Rightarrow (q \wedge r)) \vee \neg(p \vee q \vee r)$

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$= (p \wedge (q \wedge r)) \vee \neg(p \vee q \vee r)$

So φ is true if all p, q, r are true or none of them are true

$\psi = p \Leftrightarrow (q \Leftrightarrow r)$

Now

$q \Leftrightarrow r \Big|_{q=0, r=0, p=0} = 0$

So $p \Leftrightarrow (q \Leftrightarrow r) \Big|_{p=0, q=0, r=0} = 0$

Hence $\varphi \models \psi$ is false

now $q \Leftrightarrow r \Big|_{q=0, r=1, p=0} = 0$

$p \Leftrightarrow (q \Leftrightarrow r) \Big|_{p=0, r=1, q=0} = 1$

But $(p=0, r=1, q=0)$ will not satisfy φ

3)

a)

if

$X \models \alpha$

then $\forall v$ s.t. v satisfies X, $v(\alpha) = 1$

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$\Rightarrow v(\alpha \vee \beta) = v(\alpha) \vee v(\beta) = 1$

$\Rightarrow v$ satisfies $\{\alpha \vee \beta\}$

$\Rightarrow X \models \{\alpha \vee \beta\}$

same for $X \models \beta$

b)

Spse $X \models \alpha \vee \beta$ but $X \not\models \alpha$

then $\forall v$ s.t. v satisfies X, $v(\alpha \vee \beta) = 1$

X

$\Rightarrow v(\alpha) \vee v(\beta) = 1$

$\Rightarrow 0 \vee v(\beta) = 1$

$\Rightarrow v(\beta) = 1$

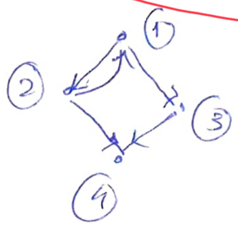
$\Rightarrow X \models \beta$

This is not true. You can construct a counter example. Also try to see where exactly your proof is wrong.

I am assumming ③ It usually does mean no. of edges

6) path length is no. vertices in the PATH

Note no. edges



① → ③ → ④

2^{nd}/first one, both true

① → ② → ④

2nd not Induced ✓

I am assumig

① → ② → ④ is not a path but induced SG

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for only subgraph

$\exists x, y, z \; (\underbrace{E(x,y) \wedge E(y,z)}_{\phi :=})$

Add $\neg(x=y) \wedge \neg(y=z)$

So φ says that $(x \to y)$ connected ✓

$(y \to z)$ connected

Now for induced we need to say $\forall g$ in G

g is connected to $x \iff g = y$, or, $g \neq z, x$ ——— (i)

g " " " $y \iff g = z$ or, $g \neq x, y$ ——— (ii)

We need one more statement

g is connected to z $\Leftrightarrow$ $g \neq x$ and $g \neq y$ ——— (iii)

Say $\varphi_1(x,y,z,g) := \not\exists\ E(x,g) \Leftrightarrow (y \equiv g) \vee \big(\neg(g \equiv x) \wedge \neg(g \equiv z)\big)$

Clearly $\varphi_1(x,y,g)$ is same as statement (i)

So $\varphi_1(y,z,x,g)$ is same as statement (ii)

Say $\varphi_2(x,y,z,g)$ $:=$ ~~$E(g,z) \Leftrightarrow \neg(g \equiv x) \wedge \neg($~~

$:= E(g,z) \Leftrightarrow \big(\neg(g \equiv x) \wedge \neg(g \equiv y)\big)$

will be same as (iii)

To make the G induced SG, we need to have (i), (ii), (iii) both true

$\therefore\ \exists x,y,z\ \forall g \left[\varphi_1(x,y,z,g) \wedge \varphi_1(y,z,x,g) \wedge \varphi_2(x,y,z,g) \wedge \varphi(x,y,z)\right]$

difference is for φ, (i), (ii), (iii) may not be true

7) a) ~~take~~ the universe $\mathbb{R}$

take $A(x)$ is true $\forall x$ s.t. $x \geq 0$

take $B(x)$ " " $\forall x$ s.t. $x - 1 \geq 0$

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Now

$$\exists x\, A(x) \Rightarrow \exists x\, B(x)$$

is true since the witness will be $A(x)+1$

But $\forall x\, (A(x) \Rightarrow B(x))$ is not true

take $A(x) = 0.5$

b) take $(\mathbb{R}, +, \times)$

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take $A(x)$: x is multiplicative identity

$B(x)$: x is additive "

$\forall x\, A(x)$ is false

$\forall x\, B(x)$ " "

$\therefore$ $\forall x\, A(x) \Rightarrow \forall x\, B(x)$ is true

$\forall x (A(x) \Rightarrow B(x))$ is false [take $x = 1$
$A(x) = 1$
$B(x) = 0$]

(6)

4)

take $\Sigma = \{P_n\}$

define $\Gamma = \{\ \varphi_n = P_1 \wedge \dots \wedge P_n$

take $\Gamma = \{\varphi_n\}$

assume Γ has ind. eq. subset T

Claim: $\varphi_{n+k} \Rightarrow \varphi_n$ always true

Easily to see bcz. $\varphi_{n+k} = \varphi_n \wedge (\dots)$

So if T is independent it can only have at max one element say φ_n

$T = \{\varphi_n\}$

or $T = \{\}$

to.

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i) $T = \{\}$ clearly not equivalent to Γ

~~take~~ ~~Set~~

ii) $T = \{\varphi_n\}$

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clearly $T \not\models \varphi_{n+1}$

s take $v := p_1 = \dots = p_n = 1$

$p_k = 0 \ \forall k > n$

$\therefore v(\varphi_{n+1}) = 0$ but v satisfi[es]

But Γ satisfies φ_{n+1} surely

$\rightarrow \leftarrow$

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2) $\varphi: (p \wedge q) \Rightarrow r$

$= \neg(p \wedge q) \vee r$

5

$\psi = (p \Rightarrow r) \vee (q \Rightarrow r)$

$\equiv (\neg p \vee r) \vee (\neg q \vee r)$

$\equiv (\neg p \vee \neg q) \vee r$

deduction

$\equiv \neg(p \wedge q) \vee r$

clearly it is logical consequence

by soundness

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5) ax1: $\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_1)$

ax2: $(\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \varphi_3)) \Rightarrow ((\varphi_1 \Rightarrow \varphi_2) \Rightarrow (\varphi_1 \Rightarrow \varphi_3))$

ax3: $(\neg\varphi_1 \Rightarrow \neg\varphi_2) \Rightarrow ((\neg\varphi_1 \Rightarrow \varphi_2) \Rightarrow \varphi_1)$

a) 1. assump.

2. assump

3. ax 1 $\varphi_1 := A$ $\varphi_2 := \neg B$

4. ax 1 $\varphi_1 := \neg A$ $\varphi_2 := B$

5. MP (2)(3)

6. MP (1)(4)

7. ax 3 $\varphi_1 = A$

$\varphi_2 = B$

8. MP (6)(7)

9. MP (5)(8)

10. We proved $\dfrac{\Gamma, \alpha \vdash B}{\Gamma \vdash \alpha \Rightarrow B}$ in class

Deduction theorem we say it ~~P1 (say)~~

(in ND it is $\Rightarrow$i)

10. P_1

11. P_1

5

7 5>6)

1. assump.

2. ax 3 $\varphi_1 = B$ $\varphi_2 = A$

3. MP 1 2

4. assump.

5

5. ax 1. $\varphi_1 = A$ $\varphi_2 = \neg B$

6. MP 4 5

7. MP 3 5

8. P_1

9) P_1

11

9) a)

$\varphi(x) := \cancel{\varphi}\ x + x \equiv x$

5

clearly only $x = 0$ is possible assignment

b)

5

$less(x, y) := (\exists z\ \ x + z \times z \equiv y) \wedge \neg(x \equiv y)$ (clearly $x < y \Leftrightarrow \sqrt{y - x}$ exists)

c) $nonneg(x) := \exists z\ \cancel{st}.\ x \equiv z \times z$

3

(,)

8)

$\varphi : \exists x\ (x \times x \equiv 3)$ L does not have constant symbols. Instead use

φ is true for $(\mathbb{R}, +, \times)$ $\forall y \exists x (x \times x \times x = y)$

1.5

φ is not " " $(\mathbb{Q}, +, \times)$

($x^2 - 3$ has not sol. in $\mathbb{Q}$ but in $\mathbb{R}$ it has)