Logic mid-semester exam

Chennai Mathematical Institute

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1. Show that neither of the following two formulas is a logical consequence of the other: (5)

$$(p \Leftrightarrow (q \Leftrightarrow r))$$
$$(p \wedge (q \wedge r)) \vee ((\neg p) \wedge ((\neg q) \wedge (\neg r)))$$

2. Determine whether or not $(p \Rightarrow r) \vee (q \Rightarrow r)$ is a logical consequence of $(p \wedge q) \Rightarrow r$. (5)

3. Let X be a set of formulas in propositional logic and let α be a formula. Prove or disprove the following.

   (a) If either $X \models \alpha$ or $X \models \beta$, then $X \models \alpha \vee \beta$. (5)

   (b) If $X \models \alpha \vee \beta$, then $X \models \alpha$ or $X \models \beta$. (5)

4. Say that a set X_1 of formulas is equivalent to a set X_2 of formulas iff for any formula α, $X_1 \models \alpha$ iff $X_2 \models \alpha$. A set X is independent if no member of X is a logical consequence of the remaining members in X. Show that there is an infinite set of formulas in propositional logic which doesn't have an independent equivalent subset. (15)

5. For the following derivations in Hilbert's proof system, explain how each step is obtained. Your explanation should be short and precise, like application of deduction theorem to a previous step, application of modus ponens to two previously derived formulas, instance of an axiom etc.

   (a) 1. $\{A, \neg A\} \vdash \neg A$
   2. $\{A, \neg A\} \vdash A$
   3. $\{A, \neg A\} \vdash A \Rightarrow (\neg B \Rightarrow A)$
   4. $\{A, \neg A\} \vdash \neg A \Rightarrow (\neg B \Rightarrow \neg A)$
   5. $\{A, \neg A\} \vdash \neg B \Rightarrow A$
   6. $\{A, \neg A\} \vdash \neg B \Rightarrow \neg A$
   7. $\{A, \neg A\} \vdash (\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$
   8. $\{A, \neg A\} \vdash (\neg B \Rightarrow A) \Rightarrow B$
   9. $\{A, \neg A\} \vdash B$

10. $\{\neg A\} \vdash A \Rightarrow B$
11. $\vdash \neg A \Rightarrow (A \Rightarrow B)$ (5)

(b) 1. $\{A, \neg B \Rightarrow \neg A\} \vdash \neg B \Rightarrow \neg A$
2. $\{A, \neg B \Rightarrow \neg A\} \vdash (\neg B \rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$
3. $\{A, \neg B \Rightarrow \neg A\} \vdash (\neg B \Rightarrow A) \Rightarrow B$
4. $\{A, \neg B \Rightarrow \neg A\} \vdash A$
5. $\{A, \neg B \Rightarrow \neg A\} \vdash A \Rightarrow (\neg B \Rightarrow A)$
6. $\{A, \neg B \Rightarrow \neg A\} \vdash (\neg B \Rightarrow A)$
7. $\{A, \neg B \Rightarrow \neg A\} \vdash B$
8. $\{\neg B \Rightarrow \neg A\} \vdash A \Rightarrow B$
9. $\vdash (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ (5)

6. Let $G = (V, E)$ be any directed graph and let $S \subseteq V$ be any subset of vertices of G. Then the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of *all* of the edges in E that have both endpoints in S. If we remove zero or more edges from $G[S]$, the resulting graph is said to be a *subgraph* of G.

   Let L be a first-order language consisting of one binary relation symbol E. An L-structure can be thought of as a directed graph where the set of vertices is the universe of discourse and the set of directed edges is the binary relation interpreting E. Write a first-order sentence that is true in exactly those directed graphs that have a path of length three as a subgraph. Write a first-order sentence that is true in exactly those directed graphs that have a path of length three as an induced subgraph. Explain the difference between the two first-order sentences. (10)

7. Consider the first-order language consisting of two unary relation symbols A and B. For each of the following formulas, give an interpretation that will make the formula *false*.

   (a) $(\exists x A(x) \;\Rightarrow\; \exists x B(x)) \;\Rightarrow\; \forall x(A(x) \;\Rightarrow\; B(x))$ (2)

   (b) $(\forall x A(x) \;\Rightarrow \forall x B(x)) \;\Rightarrow \forall x(A(x) \;\Rightarrow\; B(x))$ (3)

8. Consider the first-order language consisting of one binary functional symbol $\times$. In one structure, the universe of discourse is the set of rational numbers and $\times$ is interpreted to be the standard multiplication function on rational numbers. In the second structure, the universe of discourse is the set of real numbers and $\times$ is interpreted to be the standard multiplication function on real numbers. Write a first-order sentence that is true in the second structure but false in the first one. (2)

9. Let L be a first-order language consisting of two binary function symbols $+$ and $\times$. Let $(\mathcal{M}, \iota)$ be the L-structure whose underlying universe of discourse is the set $\mathbb{R}$ of real numbers and the interpretation for $+$ and $\times$ are the standard addition and multiplication functions over real numbers.

Suppose $\phi(x)$ is a first-order formula with the free variable x, σ is any variable assignment and r is any real number, then we write $(\mathcal{M}, \iota) \models \phi(r)$ as a shorthand for $((\mathcal{M}, \iota), \sigma[x \mapsto r]) \models \phi(x)$.

(a) Write a first-order formula $\phi(x)$ with one free variable x such that $(\mathcal{M}, \iota) \models \phi(r)$ iff $r = 0$. (5)

(b) Write a first-order formua $less(x, y)$ with two free variables x, y such that $(\mathcal{M}, \iota) \models less(r_1, r_2)$ iff the real number r_1 is less than the real number r_2. (5)

(c) Write a first-order formula $nonneg(x)$ such that $(\mathcal{M}, \iota) \models nonneg(r)$ iff $r \in [0, \infty)$. (3)