Mathematical Logic

Chennai Mathematical Institute

End-Semester Exam, December 2021

1. Consider the first order language with three unary relation symbols C, D, F. Consider a structure for this language in which the universe of discourse is the set of all your classmates. Let C (resp. D, F) be the subset of classmates who have a pet cat (resp. dog, ferret). Express each of the following statements using first order logic.

   (a) A classmate has a cat, dog and a ferret. (3)

   (b) All your classmates have a cat, dog or a ferret. (3)

   (c) At least one of your classmates has a pet cat and a ferret, but not a dog. (3)

   (d) None of your classmates has a cat, dog and a ferret. (3)

   (e) For each of the three animals, there is a classmate of yours that has one. (3)

2. Consider the first order language with three binary relation symbols $=, <$ and kth. Consider structures for this language in which the universe is the set of natural numbers, $=$ is interpreted to be the equality relation on natural numbers, $<$ is the standard linear order on natural numbers and the binary relation kth represents a sequence of natural numbers — $(i, n) \in$ kth iff the i^{th} number of the sequence is n. For example, the sequence $(9, 3, 5, 2, 5)$ is represented by $\text{kth} = \{(0, 9), (1, 3), (2, 5), (3, 2), (4, 5)\}$. Translate the following statements into first order logic over such structures.

   (a) The sequence is finite. (3)

   (b) The sequence contains at least three distinct numbers, e.g., $(2, 6, 2, 4, 6)$ but not $(5, 8, 8, 5)$. (3)

   (c) The sequence is sorted in non-decreasing order. (4)

3. Let $k \in \mathbb{N}$. Assume that $\text{FO}[k]$ contains only finitely many formulas up to logical equivalence, if the free variables are among the fixed set $\{x_1, x_2, \ldots, x_m\}$. Suppose there is a property $\mathcal{P}$ satisfying the following condition: for any two structures A, B, if A has the property $\mathcal{P}$ and agrees on $\text{FO}[k]$ with B, then B also has the property $\mathcal{P}$. Prove that the property $\mathcal{P}$ is FO-definable. (10)

4. The finite satisfiability lemma for first-order logic says that a set X of formulas is satisfiable iff every finite subset of X is satisfiable. Is this statement true when restricted to finite structures? In other words, is it true that a set X of formulas is satisfiable in a finite structure iff all

finite subsets of X are satisfiable in finite structures? Your answer must be justified by a rigorous proof. (5)

5. Let σ be a logical language and $\forall x \exists y \phi$ be a formula over σ. Let σ' be the logical language obtained by adding a unary function symbol f to σ. Show how to construct a formula $\forall x \phi'$ over the language σ' such that $\forall x \exists y \phi$ is satisfiable iff $\forall x \phi'$ is satisfiable. Prove rigorously that your formula $\forall x \phi'$ works as intended. (5)

6. Recall that Presburger arithmetic is first-order logic over natural numbers with addition, successor and linear order (but no multiplication). Given a Presburger formula $\phi(x_1, x_2, x_3)$ with free variables x_1, x_2, x_3, let $[\![\phi(x_1, x_2, x_3)]\!] = \{(n_1, n_2, n_3) \in \mathbb{N}^3 \mid \mathbb{N} \models (\phi(n_1, n_2, n_3))\}$.

   Let $\Sigma = \{a, b, c\}$ and $w \in \Sigma^*$ be a string. The Parikh image of w is $\Pi(w) = (|w|_a, |w|_b, |w|_c)$, where for any letter $\sigma \in \{a, b, c\}$, $|w|_\sigma$ is the number of times σ appears in w. For a language $L \subseteq \Sigma^*$, it's Parikh image is $\Pi(L) = \{\Pi(w) \mid w \in L\}$. Suppose L is the language of the regular expression $ab(aabbbc)^*bc(bbccc)^*$. Write a Presburger formula $\phi(x_1, x_2, x_3)$ such that $[\![\phi(x_1, x_2, x_3)]\!] = \Pi(L)$. You don't need to prove rigorously that your formula works as intended, but write informally (and easily understandable) how it works. (5)