ON SPECIAL VALUES OF DIRICHLET SERIES WITH PERIODIC COEFFICIENTS

ABHISHEK BHARADWAJ AND SIDDHI PATHAK

Abstract. Let f be an algebraic valued periodic arithmetical function and $L(s,f)$, defined as $L(s,f) := \sum_{n=1}^{\infty} f(n)/n^s$ for $\Re(s) > 1$, be the associated Dirichlet series. In this paper, we study the vanishing and arithmetic nature of the special values $L(k,f)$ when $k > 1$ is a positive integer. We prove a generalization of the Baker-Birch-Wirsing theorem conditional on the Polylog conjecture. Adopting a new approach, we define an induction operator on the space of periodic arithmetic functions, which makes precise the notion of an "imprimitive" arithmetic function. This enables us to obtain an analog of Okada's criterion for $L(1,f) = 0$ and derive a natural decomposition of the vector space

$$\mathcal{O}_k(N) = \{f : \mathbb{Z} \to \mathbb{Q} \,|\, f(n+N) = f(n) \text{ for all } n \in \mathbb{Z},\, L(k,f) = 0\}.$$

1. Introduction

Let f be an arithmetical function, periodic with period $N \geq 2$. The L-series attached to f is defined as

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{N^s} \sum_{a=1}^{N} f(a)\, \zeta\left(s, \frac{a}{N}\right),$$

which converges absolutely for $\Re(s) > 1$. Here $\zeta(s,x) := \sum_{n=0}^{\infty} (n+x)^{-s}$ denotes the Hurwitz zeta-function. Using properties of the Hurwitz zeta-function, one can see that $L(s,f)$ has analytic continuation to the entire complex plane except for a simple pole at $s = 1$ with residue $\frac{1}{N}\sum_{a=1}^{N} f(a)$. Thus, $L(s,f)$ is entire if and only if $\sum_{a=1}^{N} f(a) = 0$, in which case, $L(1,f)$ exists.

Motivated by Dirichlet's theorem of non-vanishing of $L(1,\chi)$, S. Chowla [6] initiated the study of non-vanishing of $L(1,f)$ in 1964. In an answer to a question proposed by Chowla, A. Baker, B. Birch and E. Wirsing [2] applied Baker's theory of linear forms in logarithm of algebraic numbers and proved the following general theorem.

Theorem (Baker, Birch, Wirsing). *If f is a non-vanishing function defined on the integers with algebraic values and period N such that (i) $f(n) = 0$ whenever $1 < (n,N) < N$ and (ii) the N^{th} cyclotomic polynomial Φ_N is irreducible over $\mathbb{Q}(f(1), f(2), \cdots, f(N))$, then*

$$L(1,f) \neq 0,$$

when the series converges.

If either of the two conditions (i) or (ii) on the function f are relaxed, then there exist a plethora of examples such that $L(1,f) = 0$. For instance, let f be the arithmetical function

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periodic modulo p^2 defined such that

$$L(s,f) = \left(1 - \frac{p}{p^s}\right)^2 \zeta(s). \tag{1}$$

Then f is $\mathbb{Q}$-valued, periodic modulo p^2, does not satisfy (i) and $L(1,f) = 0$.

A natural question then is to classify all $\mathbb{Q}$-valued periodic functions f such that $L(1,f) = 0$. In 1982, Okada [21] translated this problem into a linear algebraic setting and obtained a criterion for the vanishing of $L(1,f)$. Going further, in [16], R. Murty and the second author introduced the Okada space, namely,

$$\mathcal{O}(N) := \big\{f : \mathbb{Z} \to \mathbb{Q} \,|\, f(n+N) = f(n) \text{ for all } n \in \mathbb{Z},\, L(1,f) = 0\big\}$$

and constructed an explicit basis for $\mathcal{O}(N)$ using Okada's criterion. As a consequence, they obtained generalizations of the Baker-Birch-Wirsing theorem and connected this problem to the arithmetic nature of Euler's constant γ.

The aim of this paper is to study the values $L(k,f)$ for $k > 1$ in a similar spirit as above. For a positive integer $N \geq 2$, and a number field K, we define

$$F(N;K) = \big\{f : \mathbb{Z} \to K \,|\, f(N+n) = f(n), \text{ for all } n \in \mathbb{Z}\big\},$$

$$F_0(N;K) = \left\{f \in F(N;K) \,\Big|\, \sum_{a=1}^{N} f(a) = 0\right\},$$

$$F_D(N;K) = \big\{f \in F(N;K) \,|\, f(a) = 0 \text{ for } (a,N) \neq 1\big\},$$

$$\mathcal{O}_k(N;K) = \{f \in F(N;K) \,|\, L(k,f) = 0\}.$$

If $f \in F_D(N;K)$, we will say that f is of *Dirichlet type*. As we will mostly focus on *rational* valued periodic functions, we let $F(N) := F(N;\mathbb{Q})$, $F_0(N) := F_0(N;\mathbb{Q})$, $F_D(N) := F_D(N;\mathbb{Q})$ and $\mathcal{O}_k(N) := \mathcal{O}_k(N;\mathbb{Q})$ for brevity.

One can express the special value $L(k,f)$ as

$$L(k,f) = \frac{1}{N^k}\sum_{a=1}^{N} f(a)\,\zeta\left(k, \frac{a}{N}\right) = \frac{(-1)^k}{(k-1)!\,N^k}\sum_{a=1}^{N} f(a)\,\psi_{k-1}\left(\frac{a}{N}\right),$$

where

$$\psi_m(z) = \frac{d^m}{dz^m}\frac{\Gamma'(z)}{\Gamma(z)}$$

is the m-th polygamma function. Furthermore, if $\widehat{f}(b) := N^{-1}\sum_{a=1}^{N} f(a)\,e^{-2\pi i ab/N}$, then

$$L(k,f) = \sum_{b=1}^{N} \widehat{f}(b)\,\mathrm{Li}_k\left(e^{2\pi i b/N}\right), \tag{2}$$

with $\mathrm{Li}_m(z) := \sum_{n=1}^{\infty} z^n\, n^{-m}$ being the m-th polylog function. For details, we refer the reader to [18, Section 2].

Therefore, the non-vanishing of $L(k,f)$ entails the understanding of $\overline{\mathbb{Q}}$-linear relations among values of the polylog functions at roots of unity. However, even the irrationality of these polylog values remains unknown. Towards this goal, the following conjecture was proposed by S. Gun, M. R. Murty and P. Rath [9].

Conjecture 1 (Polylog Conjecture)**.** *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be algebraic numbers with $|\alpha_i| \le 1$ such that $\mathrm{Li}_k(\alpha_1), \mathrm{Li}_k(\alpha_2), \cdots, \mathrm{Li}_k(\alpha_n)$ are $\mathbb{Q}$-linearly independent. Then, $\mathrm{Li}_k(\alpha_1), \mathrm{Li}_k(\alpha_2), \cdots, \mathrm{Li}_k(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.*

In [10, Theorem 1.9], the authors showed that under the assumption of the Polylog conjecture, an analog of the Baker-Birch-Wirsing theorem holds. We record below a generalization of [10, Theorem 1.9], which will be proved in Section 2.

Theorem 1.1. *Let $N > 1$ be an integer, $\mathbb{F}$ be a number field and $f \in F_D(N;\mathbb{F})$. Let $K := \mathbb{F} \cap \mathbb{Q}(e^{2\pi i/N})$ and $H := \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/N})/K) \subseteq (\mathbb{Z}/N\mathbb{Z})^*$. Assume that supp$(f)$, the support of f in $(\mathbb{Z}/N\mathbb{Z})^*$, is contained in H. Then*

$$L(k,f) \neq 0$$

unless f is identically 0, conditional on the Polylog conjecture (Conjecture 1).

In the above theorem, Polylog conjecture plays the crucial role of reducing a problem involving values of transcendental functions to relations among algebraic numbers. This is similar to that of Baker's theorem in the Baker-Birch-Wirsing result. Therefore, proving Theorem 1.1 unconditionally would necessarily require one to establish the Polylog conjecture with α_i's being roots of unity.

It is evident from Theorem 1.1 that the two crucial conditions required to obtain the non-vanishing of $L(k,f)$ rely on (i) the support of the function f and (ii) the number field generated by values of f. Theorem 1.1 presents the most general situation in which the non-vanishing of $L(k,f)$ can be established.

Proceeding analogously as in the case of $L(1,f)$, our next aim is to characterize periodic functions f such that $L(k,f) = 0$ for a fixed integer $k > 1$. In this paper, we present a comprehensive study of this problem, by focusing on relaxing the condition on the support of the function f in Theorem 1.1. More specifically, we consider functions $f \in \mathcal{O}_k(N;K)$ such that $K \cap \mathbb{Q}(e^{2\pi i/N}) = \mathbb{Q}$. For simplicity, we restrict to the case $K = \mathbb{Q}$. However, the theorems hold for all number fields K disjoint from the N-th cyclotomic field.

A special case of Theorem 1.1, namely when $f \in F_D(N;\mathbb{Q})$, was formulated as a conjecture independently by Milnor [12] regarding the $\mathbb{Q}$-linear independence of Hurwitz zeta-values. Since this conjecture was inspired by earlier work of S. Chowla and P. Chowla [7], we refer to it as the Chowla-Milnor conjecture, following the convention in [9].

Conjecture 2 (Chowla-Milnor Conjecture)**.** *If $f \in F_D(N)$, then $L(k,f) = 0 \iff f \equiv 0$.*

Thus, as a consequence of Theorem 1.1, we derive the Chowla-Milnor conjecture assuming the Polylog conjecture. This is also proved in [9, Theorem 4].

For any positive integer r, define the dilation operator, $\mathcal{D}_r : F(N;K) \to F(rN;K)$ as

$$\mathcal{D}_r(f)(n) = \begin{cases} f\left(\dfrac{n}{r}\right) & \text{if } r \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

and for a fixed positive integer $k \geq 1$, let

$$\mathrm{Ann}_r(f) := f - r^k \, \mathcal{D}_r(f)$$

denote the annihilation operator. Thus, we have

$$L(s, \mathrm{Ann}_r(f)) = \left(1 - \frac{r^k}{r^s}\right) L(s, f),$$

that is, Ann_r defines a linear map from $F(N)$ to $\mathcal{O}_k(N)$.

Towards exploring vanishing of the values $L(k, f)$, we prove Theorem 3.3, which is the analog of Theorems [21, Theorem 10] and [22, Theorem 1] for an integer $k > 1$ in Section 3. Additionally, we investigate the structure of the vector space $\mathcal{O}_k(N)$ and show that

Theorem 1.2. *If k, $N > 1$ are fixed positive integers, then assuming the Chowla-Milnor conjecture,*

$$\mathcal{O}_k(N) = \bigoplus_{\substack{d|N, \\ d>1}} \mathrm{Ann}_d \left(F_D \left(\frac{N}{d} \right) \right).$$

Equivalently, for every $f \in \mathcal{O}_k(N)$, there exist unique functions $g_d \in F_D(N/d)$ such that

$$L(s, f) = \sum_{\substack{d|N, \\ d>1}} \left(1 - \frac{d^k}{d^s}\right) L(s, g_d),$$

conditional on the Chowla-Milnor conjecture (Conjecture 2).

Obtaining an explicit expression for the functions g_d in terms of the function f appears to be a herculean task. However, when N is a product of two or three distinct primes, this can be done using the operators defined in Section 3. We include these computations in Section 5 and underline the difficulties that arise in the general case.

The Chowla-Milnor conjecture remains open. However, we prove that it enjoys the following anatomical property. We say that the Chowla-Milnor conjecture is true modulo N if Conjecture 2 holds for all $f \in F_D(N)$. In Section 4, we prove that

Theorem 1.3. *If Conjecture 2 is true modulo N for some integer $N \geq 2$, then it is true mod d, for all divisors $d > 1$ of N.*

Using the tools developed in Section 4, we show that the functions appearing in Theorem 3.3 are intimately connected to the imprimitivity of Dirichlet characters.

Focusing on the classical theory of values of Dirichlet L-series, we recall here that

$$L(k, \chi) \in \pi^k \overline{\mathbb{Q}}^*,$$

that is, $L(k, \chi)$ is a non-zero algebraic multiple of π^k, when k and χ are either both even or both odd (see [13, Section 5]). However, when k and χ have opposite parity, the transcendental nature of $L(k, \chi)$ is still unproved. Using (2), it is clear that $L(k, \chi)$ is a linear combination of polylogarithms evaluated at roots of unity. Naturally, the Polylog conjecture is relevant in this study. Although Conjecture 1 is compelling, it is insufficient to imply the transcendence (or even irrationality) of the values $L(k, \chi)$ when k and χ have opposite parity. For this purpose, one requires the Strong Polylog conjecture (see [3]), which predicts that a non-vanishing linear form in polylogarithm of algebraic numbers is transcendental. In the penultimate section, we adopt a complementary approach and study the consequences of the Polylog conjecture on the *algebraicity* of the special values $L(k,\chi)$.

Let $\mathcal{S}_k := \{\chi \bmod N \mid L(k,\chi) \in \overline{\mathbb{Q}}, N \geq 2 \text{ squarefree}\}$ consist of the set of all distinct characters with squarefree period such that $L(k,\chi)$ is algebraic. Then the elements of $\mathcal{S}_k$ are necessarily characters that satisfy $\chi(-1) \neq (-1)^k$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define $\chi^\sigma(n) := \sigma(\chi(n))$ for all $n \in \mathbb{Z}$. It is not evident that if $\chi \in \mathcal{S}_k$, then $\chi^\sigma \in \mathcal{S}_k$, even if one assumes the Polylog conjecture. We show that this holds under an additional mild hypothesis.

Theorem 1.4. *Suppose that there exists a squarefree integer $N > 1$, and two distinct characters χ and Ψ mod N such that*

$$\chi, \Psi \in \mathcal{S}_k \text{ and } \mathbb{Q}(\chi) \cap \mathbb{Q}(\Psi) = \mathbb{Q}.$$

Then conditional on the Polylog conjecture (Conjecture 1), $\mathcal{S}_k$ is closed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, that is, if $\eta \in \mathcal{S}_k$, then η^σ is also in $\mathcal{S}_k$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The condition restricting the characters of $\mathcal{S}_k$ to have squarefree period is of technical nature, and can probably be relaxed. The absence of the disjointness hypothesis implies that a positive proportion of characters χ of squarefree period (including the trivial and quadratic characters) do not belong to $\mathcal{S}_k$.

2. A general Baker-Birch-Wirsing type theorem

In this section, we prove Theorem 1.1, which is a general version of the Baker-Birch-Wirsing theorem. Our proof follows along the lines of [10, Theorem 1.9]. However, we highlight that the structure of the proof indicates the validity of the statement in a broader setup.

We first prove the lemma below which forms one of the two fundamental ideas in the proof of Baker-Birch-Wirsing type theorems.

Lemma 2.1. *Fix a positive integer $k > 1$. Let $f \in F(N;\mathbb{F})$ for a number field $\mathbb{F}$ with $K := \mathbb{F} \cap \mathbb{Q}(e^{2\pi i/N})$. Let $H := \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/N})/K) \subseteq (\mathbb{Z}/N\mathbb{Z})^*$. For $h \in H$, let $\sigma_h(f)(n) := f(h^{-1}n)$. Then*

$$L(k,f) = 0 \iff L(k,\sigma_a(f)) = 0,$$

for all $a \in H$, conditional upon the Polylog conjecture.

Proof. Suppose that $L(k,f) = 0$. Let $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r \in \{e^{2\pi i a/N} : 1 \leq a \leq N\}$ be such that $\{\mathrm{Li}_k(\alpha_b) : 1 \leq b \leq r\}$ is a maximal $\mathbb{Q}$-linearly independent subset of $\{\mathrm{Li}_k(e^{2\pi i a/N}) : 1 \leq a \leq N\}$. Thus, we can write

$$\mathrm{Li}_k(e^{2\pi i a/N}) = \sum_{b=1}^{r} A_{ab}\,\mathrm{Li}_k(\alpha_b), \qquad 1 \leq a \leq N.$$

Substituting this in (2), the value $L(k,f)$ becomes

$$L(k,f) = \sum_{a=1}^{N} \widehat{f}(a) \sum_{b=1}^{r} A_{ab}\,\mathrm{Li}_k(\alpha_b) = \sum_{b=1}^{r} \mathrm{Li}_k(\alpha_b) \sum_{a=1}^{N} A_{ab}\,\widehat{f}(a). \tag{3}$$

By the choice of α_b's we get that under the Polylog conjecture

$$\sum_{b=1}^{r} \mathrm{Li}_k(\alpha_b) \sum_{a=1}^{N} A_{ab}\,\widehat{f}(a) = 0 \implies \sum_{a=1}^{N} A_{ab}\,\widehat{f}(a) = 0, \text{ for all } 1 \leq b \leq r.$$

Since $\widehat{f}(a) \in \mathbb{F}(e^{2\pi i/N})$, we obtain that for $\sigma \in \mathrm{Gal}(\mathbb{F}(e^{2\pi i/N})/\mathbb{F})$,

$$\sum_{a=1}^{N} A_{ab}\, \sigma\left(\widehat{f}(a)\right) = 0.$$

We can identify $\mathrm{Gal}(\mathbb{F}(e^{2\pi i/N})/\mathbb{F})$ with H via the restriction map and thus, $\sigma = \sigma_h$ for some $h \in H$ such that $\sigma_h(e^{2\pi i/N}) := e^{2\pi i h/N}$. Then from the definition of the Fourier transform, we have that

$$\sigma_h\left(\widehat{f}(a)\right) = \sum_{\substack{n=1,\\ (n,N)=1}}^{N} f(n)\, e^{-2\pi i a n h/N} = \sum_{\substack{m=1,\\ (m,N)=1}}^{N} f(h^{-1}m)\, e^{-2\pi i a m/N} = \widehat{\sigma_h(f)}(a). \tag{4}$$

Thus, we have the relation

$$\sum_{a=1}^{N} A_{ab}\, \widehat{\sigma_h(f)}(a) = 0,$$

which together with (3) gives $L(k, \sigma_h(f)) = 0$ for all $h \in H$. □

Remark. *Theorem 1.1 holds under the weaker assumption of an analog of the Polylog conjecture for roots of unity instead of all algebraic numbers. Moreover, note that the above proof holds more generally. Indeed, let $f \in F(N;\mathbb{F})$, and let σ be an automorphism of $\mathbb{F}(e^{2\pi i/N})/\mathbb{Q}$ such that $\sigma_c := \sigma\big|_{\mathbb{Q}(e^{2\pi i/N})}$ with $c \in (\mathbb{Z}/N\mathbb{Z})^*$ and $\sigma_c(e^{2\pi i/N}) = e^{2\pi i c/N}$. Then equation (4) can be replaced with*

$$\sigma\left(\widehat{f}(a)\right) = \sum_{\substack{n=1,\\ (n,N)=1}}^{N} \sigma\left(f(n)\right)\, e^{-2\pi i a n c/N} = \sum_{\substack{m=1,\\ (m,N)=1}}^{N} \sigma\left(f(c^{-1}m)\right) e^{-2\pi i a m/N} = \widehat{\sigma_c\left(f^{\sigma}\right)}(a),$$

where $\sigma_c\left(f^{\sigma}\right)(m) := \sigma(f(c^{-1}m))$. Hence we deduce that $L(k,f) = 0$ implies that $L(k, \sigma_c(f^{\sigma})) = 0$.

The second input integral to the proof of Baker-Birch-Wirsing type theorems is the evaluation of a Dedekind determinant.

Lemma 2.2. *Let G be a finite abelian group and $f : G \to \mathbb{C}$ be a complex-values function on G. Suppose that $\mathfrak{M} = \left[f(xy^{-1})\right]_{x,y\in G}$ is the corresponding Dedekind matrix. Then*

$$\det \mathfrak{M} = \prod_{\chi} \Bigg(\sum_{x \in G} f(x)\, \chi(x) \Bigg), \tag{5}$$

where the product is over all characters χ of G.

For a proof of this fact, we refer the reader to [20].

Proof of Theorem 1.1. Suppose that $L(k,f) = 0$. Lemma 2.1 implies that $L(k, \sigma_h(f)) = 0$ for all $h \in H$. Using the expression for $L(k,f)$ in terms of values of the Hurwitz zeta-functions, we have

$$\sum_{a \in \mathrm{supp}(f)} f(ha)\, \zeta\left(k, \frac{a}{N}\right) = 0 \qquad \text{for all } h \in H.$$

Equivalently, we have the relations

$$\sum_{a\in H} f(a)\, \zeta\left(k, \frac{ah^{-1}}{N}\right) = 0 \qquad \text{for all } h \in H \tag{6}$$

as $\mathrm{supp}(f) \subseteq H$. This can be interpreted as $\mathfrak{M}\vec{v} = \vec{0}$ with

$$\mathfrak{M} = \left[\zeta\left(k, \frac{ah^{-1}}{N}\right)\right]_{a,h\in H} \text{ and } \vec{v} = \left[f(a)\right]_{a\in H}.$$

The theorem would be proved if we show that $\mathfrak{M}$ is invertible, that is, $\det(\mathfrak{M}) \neq 0$. It is evident that $\mathfrak{M}$ is a Dedekind matrix. Therefore by Lemma 2.2,

$$\det \mathfrak{M} = \prod_{\chi \in \widehat{H}} \left(\sum_{a \in H} \chi(a)\, \zeta\left(k, \frac{a}{N}\right) \right),$$

where $\widehat{H}$ is the group of characters of H. By Pontryagin duality, there is a unique subgroup $\mathcal{V} \subseteq (\mathbb{Z}/N\mathbb{Z})^*$ such that

$$\widehat{H} \simeq (\mathbb{Z}/N\mathbb{Z})^*/\mathcal{V}.$$

Thus there is a unique extension $K_{\mathcal{V}}/\mathbb{Q}$ such that

$$\mathbb{Q} \subseteq K_{\mathcal{V}} \subseteq \mathbb{Q}(e^{2\pi i/N}) \text{ and } \mathrm{Gal}(K_{\mathcal{V}}/\mathbb{Q}) \simeq H.$$

The characters of H can now be identified with characters of $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/N})/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^*$ that are trivial on $\mathcal{V}$. Hence by (2),

$$\sum_{a \in H} \chi(a)\, \zeta\left(k, \frac{a}{N}\right) = N^k\, L(k, \chi),$$

with χ being a Dirichlet character in the classical sense. Since $L(k, \chi) \neq 0$, we see that $\det \mathfrak{M} \neq 0$, proving the theorem. □

Remark. *The statement of Theorem 1.1 also holds if the support of the function f is contained in a coset of H in $(\mathbb{Z}/N\mathbb{Z})^*$. Indeed, suppose that bH is a coset of H in $(\mathbb{Z}/N\mathbb{Z})^*$ and that $f \in F_D(N; K)$ with supp$(f) \subseteq bH$ such that $L(k, f) = 0$. Set $g(n) := f(bn)$. Now $g \in F_D(N; K)$ with supp$(g) \subseteq H$ satisfying $L(k, g) = 0$. Hence applying Theorem 1.1, we can conclude that $g \equiv 0 \implies f \equiv 0$.*

3. **Structure of the Okada space**

In [22], T. Okada proved that for arithmetical functions, periodic mod N, taking values in a field disjoint from the N-th cyclotomic field, $L(1, f) = 0$ only if f is "induced from lower level periodic functions", that is, f is, in a sense, "imprimitive". In this section, we formulate these ideas in precise terms and discuss their consequences on the vanishing of the special value $L(k, f)$.

We first prove a generalization of Okada's result for $L(k, f)$ with $k > 1$. Using notation that is considerably simplified, we also deduce Okada's criterion for vanishing of $L(1, f)$. An alternate derivation of Okada's criterion can be found in [4]. We will then use the general Okada's criterion to prove Theorem 1.2. In contrast to Okada, our methods will rely on the properties of the L-function $L(s, f)$ rather than algebraic relations among the special values themselves.

The definition of the annihilator operator in the introduction implies that

$$L(s, \mathrm{Ann}_r(f)) = \left(1 - \frac{r^k}{r^s}\right) L(s, f).$$

Thus if $L(k, f) < \infty$, then $L(k, \mathrm{Ann}_r(f)) = 0$ for all positive integers $r > 1$. The natural question that this observation leads to is whether all periodic functions f such that $L(k, f) = 0$ are generated by $\mathrm{Ann}_r(g)$ for certain periodic function g and positive integers r. We answer this in the affirmative below.

Let $\mathcal{M}(N)$ be the monoid generated by prime divisors of N, that is, if $p_1, p_2, \cdots, p_t$ are all the distinct primes dividing N, then

$$\mathcal{M}(N) = \{p_1^{a_1}\, p_2^{a_2}\, \cdots\, p_t^{a_t} \mid a_1, \cdots a_t \text{ are non-negative integers}\}.$$

Note that

$$\sum_{m\in\mathcal{M}(N)} \frac{1}{m^s} = \prod_{j=1}^{t}\left(1-\frac{1}{{p_j}^s}\right)^{-1},$$

which is entire. Thus, $\sum_{m\in\mathcal{M}(N)} 1/m$ is absolutely convergent.

Fix a positive integer $k \geq 1$. For $f \in F(N;K)$ and a divisor d of N, let

$$f_d^{(k)}(n) = \sum_{\substack{m\in\mathcal{M}(N),\\ d\,|\,m}} \frac{f(mn)}{m^k} = \sum_{m\in\mathcal{M}(N)} \frac{f(dmn)}{(dm)^k}.$$

Then $f_d^{(k)}$ is periodic with period N/d, and $f_d^{(k)}(n) = f_1^{(k)}(dn)/d^k$. Moreover, $f_d^{(k)}(n) \in K$. Indeed, if a prime $p \mid N$, then the sequence $\{p^a\}_{a\in\mathbb{N}}$ is eventually periodic mod N. Therefore, $f_1^{(k)}(n)$ can be expressed as a sum of finitely many terms and finitely many geometric progressions, with values in K. Hence, the value of the series $f_1^{(k)}(n)$ (and in turn, $f_d^{(k)}(n)$) will lie in K. Thus, $f_d^{(k)} \in F(N/d;K)$. Furthermore, let

$$\widetilde{f_1^{(k)}}(n) := \begin{cases} f_1^{(k)}(n) & \text{if } (n,N)=1,\\ 0 & \text{otherwise.}\end{cases}$$

With this notation in place, we prove the following crucial proposition.

Proposition 3.1. *Fix an integer $k \geq 1$. Let $N \geq 2$ be a positive integer and $f \in F(N;K)$. Then*

$$f = \widetilde{f_1^{(k)}} + \sum_{d|N} \mu(d)\,\mathrm{Ann}_d\big(f_d^{(k)}\big),$$

where $\mu(\cdot)$ denotes the Möbius function.

Proof. We begin by noting that for any periodic function f, $\mathrm{Ann}_1(f)(n) = f(n) - \mathcal{D}_1(f)(n) = f(n) - f(n) = 0$. Recall that

$$\sum_{d|N} \mu(d) = \begin{cases} 1 & \text{if } N=1,\\ 0 & \text{otherwise.}\end{cases}$$

Now consider the sum

$$\mathfrak{F}(n) := \sum_{d|N} \mu(d)\,\mathrm{Ann}_d\big(f_d^{(k)}\big)(n). \tag{7}$$

Expanding the sum using the definition of the Ann_d operator, we have

$$\begin{aligned}
\mathfrak{F}(n) &= \Big(\sum_{d|N} \mu(d)\, f_d^{(k)}(n)\Big) - \Big(\sum_{d|N} \mu(d)\, d^k\, \mathcal{D}_d(f_d^{(k)})(n)\Big) \\
&= \Big(\sum_{d|N} \mu(d) \sum_{\substack{m\in\mathcal{M}(N),\\ d|m}} \frac{f(mn)}{m^k}\Big) - \Big(\sum_{\substack{d|N,\\ d|n}} \mu(d)\, d^k\, f_d^{(k)}\left(\frac{n}{d}\right)\Big) \\
&= \Big(\sum_{m\in\mathcal{M}(N)} \frac{f(mn)}{m^k} \sum_{\substack{d|N,\\ d|m}} \mu(d)\Big) - \Big(\sum_{\substack{d|N,\\ d|n}} \mu(d)\, d^k\, \frac{f_1^{(k)}(n)}{d^k}\Big) \\
&= f(n) - \Big(f_1^{(k)}(n) \sum_{d|(n,N)} \mu(d)\Big), \\
&= f(n) - \widetilde{f_1^{(k)}}(n).
\end{aligned}$$

□

Therefore, for every $f \in F(N;K)$,

$$L(s,f) = L(s,\widetilde{f_1^{(k)}}) + \sum_{d|N} \mu(d)\left(1 - \frac{d^k}{d^s}\right) L(s, f_d^{(k)}). \tag{8}$$

Note that in the above decomposition, $f_d^{(k)} \in F(N/d)$. If N is assumed to be squarefree, then one can further prove the following.

Lemma 3.2. *Let N be squarefree and $\widetilde{f_d^{(k)}} := f_d^{(k)}\chi_{0,N/d} \in F_D(N/d)$ where $\chi_{0,N/d}$ denotes the principal character mod N/d. Then*

$$L(s,\mathfrak{F}) = \sum_{\substack{d|N\\ d\neq 1}} \mu(d) \prod_{p|d}\left(1 - \frac{p^k}{p^s}\right) L(s,\widetilde{f_d^{(k)}}).$$

Proof. The function $L(s, f_d^{(k)})$ can be written as the linear combination

$$L(s, f_d^{(k)}) = \sum_{m|N/d} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{f_d^{(k)}(mn)\chi_{0,N/md}(n)}{m^s} = \sum_{m|N/d} \frac{m^k}{m^s} L(s,\widetilde{f_{md}^{(k)}}).$$

We get the second equality by observing that $f_d^{(k)}(mn) = m^k \widetilde{f_{dm}^{(k)}}(n)$ whenever $(n, N/dm) = 1$. Therefore, we have

$$L(s,\mathfrak{F}) = \sum_{d|N} \mu(d)\left(1 - \frac{d^k}{d^s}\right) \sum_{m|N/d} \frac{m^k}{m^s} L(s,\widetilde{f_{dm}^{(k)}}) = \sum_{d|N} L(s,\widetilde{f_d^{(k)}}) \sum_{m|d} \mu(m)\left(1 - \frac{m^k}{m^s}\right) \frac{(d/m)^k}{(d/m)^s}.$$

It remains to evaluate the second summand in the above sum. For $d \neq 1$, by applying the identity $\sum_{m|d} \mu(m) = 0$, we have

$$\sum_{m|d} \mu(m)\left(1 - \frac{m^k}{m^s}\right) \frac{(d/m)^k}{(d/m)^s} = \sum_{m|d} \mu(m) \frac{(d/m)^k}{(d/m)^s} = \prod_{p|d}\left(1 - \frac{p^k}{p^s}\right). \tag{9}$$

We obtain the final expression by noting that N (and hence d) is squarefree and using the Möbius inversion formula. □

3.1. **Vanishing criterion for $L(k,f)$ with $k \geq 2$.** We are now equipped to prove analogs of Okada's vanishing criteria, that is, [21, Theorem 10] and [22, Theorem 1]. More specifically, we prove the following.

Theorem 3.3. *Let $f \in \mathcal{O}_k(N)$. Then assuming the Chowla-Milnor conjecture, we have*

$$L(s,f) = \sum_{d|N} \left(1 - \frac{d^k}{d^s}\right) \mu(d)\, L(s, f_d^{(k)}).$$

Moreover, for $f \in F(N)$, $L(k,f) = 0$ if and only if $(f(1), f(2), \cdots, f(N))$ satisfies the following system of $\varphi(N)$ linear equations: for each $1 \leq n \leq N$ with $(n,N) = 1$,

$$X_n + \sum_{\substack{a=1,\\ 1<(a,N)}}^{N} X_a\, A_k(a,n) = 0,$$

where $A_k(a,n) \in \mathbb{Q}$ is defined by

$$A_k(a,n) = \sum_{\substack{m\in\mathcal{M}(N),\\ mn\equiv a \bmod N}} \frac{1}{m^k}.$$

Proof. By Lemma 3.1, the L-function associated to f has the form as in (8). Since $k > 1$, the functions $L(s, f_d^{(k)})$ are holomorphic around $s = k$ and hence

$$L(k,f) = L(k, \widetilde{f_1^{(k)}}).$$

Therefore if $L(k,f) = 0$, then $L(k, \widetilde{f_1^{(k)}}) = 0$. Recall that $\widetilde{f_1^{(k)}} \in F_D(N)$. Thus, the Chowla-Milnor conjecture (Conjecture 2) implies that $\widetilde{f_1^{(k)}} = 0$. This proves the first assertion.

Using the definition of $\widetilde{f_1^{(k)}}$, we obtain that $L(k,f) = 0$ if and only if for every $(n,N) = 1$,

$$0 = \sum_{m\in\mathcal{M}(N)} \frac{f(mn)}{m^k} = \sum_{a=1}^{N} f(a) \sum_{\substack{m\in\mathcal{M}(N),\\ mn\equiv a \bmod N}} \frac{1}{m^k}.$$

Note that if $(a,N) = 1$, then the congruence $mn \equiv a \bmod N$ has a solution in $\mathcal{M}(N)$ only if $n \equiv a \bmod N$, in which case, the only solution is $m = 1$. This proves the theorem. ☐

The corollary below follows from Theorem 3.3.

Corollary 3.4. *Let $f \in F(N)$ be such that $f(1) = 1$. Let $M_f := \max\{|f(n)| : 1 \leq n \leq q\}$. If $L(k,f) = 0$, then conditional on the Chowla-Milnor conjecture, M_f is not attained at a residue class n satisfying $(n,N) = 1$. Moreover,*

$$M_f \geq \frac{1}{(N^{-k} + (\zeta(k) - 1))},$$

which tends to infinity as k tends to infinity.

Proof. First suppose that there exists an n coprime to N such that $M_f = |f(n)|$ and $L(k,f) = 0$. Then we have by Theorem 3.3,

$$\sum_{m\in\mathcal{M}(N)} \frac{f(mn)}{m^k} = 0 \implies -f(n) = \sum_{\substack{m\in\mathcal{M}(N)\\ m\neq 1}} \frac{f(mn)}{m^k} \implies M_f = |f(n)| \leq M_f\,(\zeta(k) - 1) < M_f,$$

which is a contradiction.

On the other hand, with the normalization $f(1) = 1$ we have a lower bound on M_f. Indeed, let n be such that $|f(n)| = M_f$. Then we obtain that

$$f(1) + \frac{f(n)}{n^k} = -\sum_{m\neq 1,n} \frac{f(m)}{m^k} \implies \left| f(1) + \frac{f(n)}{n^k} \right| \leq M_f\ (\zeta(k) - 1) \implies 1 - \frac{M_f}{n^k} \leq M_f\,(\zeta(k) - 1)\,.$$

By choosing n larger than N, we obtain the desired result. □

3.2. **Vanishing criterion for $L(1, f)$.** With the framework set up earlier, we give a simplified treatment of Okada's Theorem [22, Theorem 1] below. The difference between the analysis for the vanishing of $L(k, f)$ with $k \geq 2$ and that of $L(1, f)$ comes from the possibility of the pole at $s = 1$ of $L(s, f)$. Indeed, it is evident from Theorem 3.3 that for $k \geq 2$,

$$\dim_{\mathbb{Q}} \mathcal{O}_k(N) = N - \varphi(N), \tag{10}$$

whereas it follows from [21, Theorem 10] that

$$\dim_{\mathbb{Q}} \mathcal{O}_1(N) = N - \varphi(N) - \omega(N),$$

where $\omega(N)$ denotes the number of distinct prime divisors of N.

We prove a proposition of independent interest which is used in the proof. Throughout this discussion, $\zeta_q = e^{2\pi i/q}$ and $\zeta_N = e^{2\pi i/N}$.

Proposition 3.5. *Let $q = p^r$ with p being a prime and let I be the index set of positive integers less than q that are co-prime to q. If $a_j \in \overline{\mathbb{Q}}$ such that $\sum_{j\in I} a_j = 0$ then*

$$\sum_{j\in I} a_j \log|1-\zeta_q^j| \text{ is a } \overline{\mathbb{Q}} \text{ linear combination of logarithm of units in } \mathbb{Z}[\zeta_q].$$

Proof. We know that the ideal (p) totally ramifies in $\mathbb{Q}(\zeta_q)$ and hence in $\mathbb{Q}(\zeta_q)^+ = \mathbb{Q}(\zeta_q + \overline{\zeta_q})$. Therefore, we have

$$(p) = (|1-\zeta_q)|)^{\varphi(q)/2}$$

and hence, $p = |1-\zeta_q|^{\varphi(q)/2}\, u$ for some unit $u \in \mathbb{Z}[\zeta_q]$. Applying the Galois action $\sigma_j : \zeta_q \to \zeta_q^j$ for $j \in I$, we obtain that $p = |1-\zeta_q^j|^{\varphi(q)/2}\, u_j$, where u_j is also an unit in $\mathbb{Z}[\zeta_q]$. Now,

$$\begin{aligned}\sum_{j\in I} a_j \log|1-\zeta_q^j| &= \frac{1}{\varphi(q)/2}\sum_{j\in I} a_j \log|1-\zeta_q^j|^{\varphi(q)/2} = \frac{1}{\varphi(q)/2}\sum_{j\in I} a_j \log\left(p u_j^{-1}\right)\\ &= \frac{1}{\varphi(q)/2}\sum_{j\in I} a_j(-\log u_j + \log p) = -\frac{1}{\varphi(q)/2}\sum_{j\in I} a_j \log u_j,\end{aligned}$$

where in the last step, we have used the hypothesis $\sum_{j\in I} a_j = 0$. □

The following lemma is a direct consequence of Baker's theorem of linear forms in logarithms of algebraic numbers and the fact that that prime ideals do not contain units.

Lemma 3.6. *Let $\mathbb{F}$ be a number field. Suppose that $u_1, \ldots, u_n \in \mathcal{O}_{\mathbb{F}}^*$ and let S be a finite set of rational primes. Then*

$$\overline{\mathbb{Q}}\,\langle \log u_i \mid 1 \leq i \leq n\rangle \cap \overline{\mathbb{Q}}\,\langle \log p \mid p \in S\rangle = \{0\}.$$

Okada's vanishing criteria [21, Theorem 10] follows easily from [22, Theorem 1]. Thus, we prove an equivalent formulation of [22, Theorem 1] below.

Theorem 3.7 (Okada, [22])**.** *Let $f \in \mathcal{O}(N)$. Then*

$$L(s,f) = \sum_{\substack{p|N,\\ p-prime}} \left(1 - \frac{p}{p^s}\right) L(s,h_p),$$

for certain $h_p \in F_0(N/p)$.

Proof. We first observe that for any $g \in F(N;K)$ and $r = l_1 l_2 \cdots l_t$ with l_i's being distinct primes,

$$\begin{aligned}
\mathrm{Ann}_r(g) &= g - r\,\mathcal{D}_r(g)\\
&= \Big(g - l_1\,\mathcal{D}_{l_1}(g)\Big) + \Big(l_1\,\mathcal{D}_{l_1}(g) - (l_1 l_2)\,\mathcal{D}_{l_1 l_2}(g)\Big)\\
&\qquad + \cdots + \Big((l_1 l_2 \cdots l_{t-1})\,\mathcal{D}_{l_1 l_2 \cdots l_{t-1}}(g) - (l_1 l_2 \cdots l_t)\,\mathcal{D}_{l_1 l_2 \cdots l_t}(g)\Big)\\
&= \mathrm{Ann}_{l_1}(g) + \mathrm{Ann}_{l_2}\Big(l_1\,\mathcal{D}_{l_1}(g)\Big) + \cdots + \mathrm{Ann}_{l_t}\Big((l_1 l_2 \cdots l_{t-1})\,\mathcal{D}_{l_1 l_2 \cdots l_{t-1}}(g)\Big).
\end{aligned}$$

Hence, for $k = 1$, the expression in (8) can be written as

$$L(s,f) = L(s,\widetilde{f_1}) + \sum_{\substack{p|N,\\ p-prime}} \left(1 - \frac{p}{p^s}\right) L(s,h_p),$$

for certain $h_p \in F(N/p)$ with $\widetilde{f_1} := \widetilde{f_1^{(1)}}$ as defined previously.

Let ρ_{h_p} denote the residue of $L(s,h_p)$ at $s = 1$. Then

$$L(1,f) = L(1,\tilde{f}_1) - \sum_{p|N} \rho_{h_p}\, \log p,$$

by Taylor's theorem. Now suppose that $L(1,f) = 0$. This implies that

$$L(1,\tilde{f}_1) = \sum_{p|N} \rho_{h_p}\, \log p. \tag{11}$$

We want to conclude that $\tilde{f}_1 = 0$ and that $\rho_{h_p} = 0$ for all $p \mid N$. Note that $\sum_{a=1}^{N} \tilde{f}_1(a) = 0$, as the sum $L(1,\tilde{f})$ converges to a finite value.

Now, note that $L(1,\tilde{f}_1)$ is a $\overline{\mathbb{Q}}$-linear combination of logarithm of units in $\mathbb{Z}[\zeta_N, i]$. Indeed, write $L(1,\tilde{f}_1) = L(1,\tilde{f}_1{}^o) + L(1,\tilde{f}_1{}^e)$ where

$$\tilde{f}_1{}^o := \frac{\tilde{f}_1(n) - \tilde{f}_1(-n)}{2} \qquad \text{and} \qquad \tilde{f}_1{}^o := \frac{\tilde{f}_1(n) - \tilde{f}_1(-n)}{2}$$

denote the odd and even parts of f respectively. We know that $L(1,\tilde{f}_1{}^o)$ is an algebraic multiple of $\pi = 2\log i$ (See [17]) and $L(1,\tilde{f}_1{}^e)$ is a $\overline{\mathbb{Q}}$-linear combination of logarithm of positive algebraic numbers (See equation (12) below). For $\tilde{f}_1$ even, we claim that $L(1,\tilde{f}_1)$ is a $\overline{\mathbb{Q}}$-linear form of logarithm of units in $\mathbb{Z}[\zeta_N]$. To see this, for a divisor d of N, we define the set $S_{d,N}$ as follows :

$$S_{d,N} := \{1 \le a \le N \mid a/N = c/d \text{ for } (c,d) = 1\}.$$

Thus,

$$L(1,\tilde{f}_1) = -\sum_{d|N} \sum_{a \in S_d} \widehat{\tilde{f}_1}(a) \log|1 - \zeta_N^a| \tag{12}$$

If d has at least two distinct odd prime factors or if $4p \mid d$ for some odd prime p, then the inner sum is a $\overline{\mathbb{Q}}$ linear combination of logarithms of units in $\mathbb{Z}[\zeta_N]$ as $|1-\zeta_d|$ is an unit in $\mathbb{Z}[\zeta_d]$. Hence it suffices to show that Proposition 3.5 can be applied for sets $S_{p^k,N}$ and $S_{2p^k,N}$ (which occur if N is even). Note that

$$\sum_{a\in S_{p^k,N}} \widehat{\tilde{f}}_1(a) = \sum_{\substack{j=1\\(j,p)=1}}^{p^k-1} \widehat{\tilde{f}}_1(Nj/p^k) = \frac{1}{N}\sum_{\substack{j=1\\(j,p)=1}}^{p^k-1}\sum_{b=1}^{N} \tilde{f}_1(b)\, e^{2\pi ijb/p^k}$$
$$= \sum_{\substack{b=1\\(b,N)=1}}^{N} \tilde{f}_1(b) \sum_{\substack{j=1\\(j,p)=1}}^{p^k-1} e^{2\pi ijb/p^k} = 0,$$

and Proposition 3.5 can be applied to the inner sum in (12) consisting of indices in $S_{p^k,N}$. The same proof works verbatim when we replace $S_{p^k,N}$ by $S_{2p^k,N}$ for primes p dividing N. Therefore, $L(1,\tilde{f}_1)$ is a $\overline{\mathbb{Q}}$ linear combination of logarithm in units in $\mathbb{Z}[\zeta_N]$.

We can now apply Lemma 3.6 to show that $\rho_{h_p} = 0$. Hence $L(1,\tilde{f}_1) = 0$ and by the Theorem of Baker, Birch and Wirsing, we have $\tilde{f}_1 = 0$. □

Remark. *It is possible to show that $L(1,\tilde{f}_1)$ is a $\overline{\mathbb{Q}}$ linear combination of logarithm of units in $\mathbb{Z}[\zeta_N, i]$ by following the steps as mentioned in the proof of* [2, Theorem 1] *or by appealing to Ramachandra units as mentioned in* [15, Section 4] *and applying Lemma 3.6. However, Proposition 3.5 is a more direct approach, which seems to be missing from the literature.*

3.3. **Proof of Theorem 1.2.** An alternate interpretation of Theorem 3.3 is that for k, $N \geq 2$,

$$\mathcal{O}_k(N) = \sum_{d\mid N} \mathrm{Ann}_d\left(F(N/d)\right).$$

However, it is clear by comparing dimensions of the vector spaces involved that the above sum is not direct. We address this issue in Theorem 1.2 by obtaining a decomposition of $\mathcal{O}_k(N)$ into disjoint subspaces, each of which can be identified with functions "induced from lower levels".

Proof of Theorem 1.2. Note that Theorem 1.2 is immediate from the following statement: if $f \in \mathcal{O}_k(N)$, then for each proper divisor d of N, we have a unique function $g_d \in F_D(N/d)$ such that

$$L(s,f) = \sum_{d\mid N}\left(1-\frac{d^k}{d^s}\right) L(s,g_d).$$

We prove this statement below assuming the Chowla-Milnor conjecture.

First we show that the functions g_d are linearly independent over $\mathbb{Q}$, i.e. if

$$\sum_{\substack{d\mid N,\\ d>1}}\left(1-\frac{d^k}{d^s}\right) L(s,g_d) = 0, \tag{13}$$

for $g_d \in F_D(N/d)$, then $g_d \equiv 0$. Indeed, writing the function g_d in terms of its character decomposition

$$\sum_{\chi \bmod N/d} c_{\chi,(N/d)}\, \chi$$

and rearranging the sum over the primitive characters mod d for each proper divisor d of N we obtain the following :

$$0 = \sum_{D|N} \left(1 - \frac{D^k}{D^s}\right) \sum_{d|(N/D)} \sum_{\substack{\chi \bmod (N/D) \\ \mathrm{cond}(\chi)=d}} c_{\chi,(N/D)}\, L(s,\chi)$$

$$= \sum_{D|N} \sum_{\substack{\chi \bmod D \\ \chi \text{ primitive}}} \sum_{d|(N/D)} \left(c_{\chi,(N/d)} \left(1 - \frac{d^k}{d^s}\right) \prod_{\substack{p|(N/D) \\ p\nmid d}} \left(1 - \frac{\chi(p)}{p^s}\right)\right) L(s,\chi).$$

By the linear independence of $L(s,\chi)$ over the ring of Dirichlet polynomials (see [14]), we get that for every divisor d of N, and for every character χ mod D with $D \not\equiv 2 \bmod 4$,

$$\sum_{d|N/D} \left(c_{\chi,(N/d)} \left(1 - \frac{d^k}{d^s}\right) \prod_{\substack{p|(N/D) \\ p\nmid d}} \left(1 - \frac{\chi(p)}{p^s}\right)\right) = 0. \tag{14}$$

For brevity, henceforth we write N/D as q and remove the subscript χ while denoting $c_{\chi,(N/d)}$. Expanding (14), for each divisor d of q, and setting $\alpha(d) = \chi(d)\mu(d)$ we get :

$$\sum_{d|q} c_d \sum_{\substack{e|q \\ (e,d)=1}} \frac{\alpha(e)}{e^s} = \sum_{d|q} \frac{c_d\, d^k}{d^s} \sum_{\substack{e|q \\ (e,d)=1}} \frac{\alpha(e)}{e^s}$$

$$\implies \sum_{d|q} \frac{\alpha(d)}{d^s} \sum_{\substack{e|q \\ (e,d)=1}} c_e = \sum_{d|q} \frac{1}{d^s} \sum_{\substack{e|d \\ (d/e,e)=1}} e^k\, c_e\, \alpha\left(\frac{d}{e}\right).$$

By equating the coefficients of d^s, we have for every $d \mid q$,

$$\alpha(d) \sum_{\substack{e|q \\ (e,d)=1}} c_e = \sum_{\substack{e|d \\ (d/e,e)=1}} e^k\, c_e\, \alpha\left(\frac{d}{e}\right). \tag{15}$$

We shall first prove that if $\alpha(d) = 0$, then $c_d = 0$. We write $d = d_f d_s$, where d_s is the largest divisor of d for which $\alpha(d_s) \neq 0$. Here d_f consists of prime factors p of q such that either $\alpha(p) = 0$ or $p^2 \mid q$. Note that for $d_f \neq 1$ and $d_s = 1$, we obtain $c_d = 0$ from (15) as for any $e \mid d$ with $e \neq 1$ and $(e, d/e) = 1$, $\alpha(e) = 0$. If $d_s > 1$, the elements $e \mid d$ satisfying $(d/e, e) = 1$ and $\alpha(d/e) \neq 0$ are the ones of the form $e = d_f b$ with $b \mid d_s$. By proceeding via induction in (15) on the number of prime factors of d_s, we obtain $c_{d_f b} = 0$ for $b \mid d_s$. Thus, for any $d \mid q$ with $\alpha(d) = 0$, we have $c_d = 0$ and it remains to consider the case when $\alpha(d) \neq 0$. This also implies that d is a squarefree divisor of q.

In what follows whenever we mention a divisor d of q, we also assume that d satisfies $\alpha(d) \neq 0$. In (15), write $\alpha(d) = \alpha(de)\,\overline{\alpha(e)}$ for divisor d of q to get

$$\sum_{e|q/d} c_e = \sum_{e|d} c_e\, e^k\, \overline{\alpha(e)} = \sum_{e|d} c_e\, \beta(e), \tag{16}$$

where $\beta(e) = e^k\, \overline{\alpha}(e)$. Since α is multiplicative, so is β.

Now define two functions arithmetic $\mathcal{A}$ and $\mathcal{B}$ for suitable j, d, q by

$$\mathcal{A}_j(d;q) := \sum_{e|d} c_{je}; \qquad \mathcal{B}_j(d;q) := \sum_{e|d} \beta(e)\, c_{je}$$

Therefore for each divisor d of q, (16) can be written as follows :

$$\mathcal{A}_1(d;q) = \mathcal{B}_1(q/d;q).$$

For a prime p, setting $q = pq'$ and $d = pd'$ (whenever $p \mid d$), the above equation can be expressed as

$$\mathcal{A}_1\left(\frac{q'}{d'};q'\right) = \mathcal{B}_1(d';q') + \beta(p)\,\mathcal{B}_p(d';q') \tag{17}$$

$$\mathcal{A}_p\left(\frac{q'}{d'};q'\right) + \mathcal{A}_1\left(\frac{q'}{d'};q'\right) = \mathcal{B}_1(d';q'). \tag{18}$$

To see this, note that the coefficients of c_e appearing in $\mathcal{A}_j(d;q)$, $\mathcal{B}_j(d;q)$ are independent of q, and that we have the following property for $\mathcal{A}_1(d;q)$ and $\mathcal{B}_1(d;q)$ when $d = pd'$.

$$\mathcal{A}_1(d;q) = \mathcal{A}_1(d',q') + \mathcal{A}_p(d',q'), \qquad \mathcal{B}_1(d;q) = \mathcal{B}_1(d',q') + \beta(p)\,\mathcal{B}_p(d',q').$$

Comparing (17) and (18) we immediately obtain that

$$\mathcal{A}_p\left(\frac{q'}{d'};q'\right) = -\beta(p)\,\mathcal{B}_p(d';q') \tag{19}$$

for each divisor d' of q' with $\alpha(d') \neq 0$.

We show that this system of equations does not have any non-trivial solutions. For every divisor d' of q', we can rearrange the above equation as

$$c_{pd'} = \sum_{e|d'} \mu(e)\,\mathcal{A}_p\left(\frac{d'}{e};q'\right) = -\beta(p)\sum_{e|d'}\mu(e)\,\mathcal{B}_p(d';q'). \tag{20}$$

Writing X as the column vector $[c_{pd'}]_{d'|q'}$, we have

$$X = -\beta(p)\mathscr{A}X$$

where $\mathscr{A}$ is a matrix with entries in algebraic *integers*. If X is a non-zero vector then $-1/\beta(p)$ is an eigenvalue for $\mathscr{A}$. However, the eigenvalues of the matrix $\mathscr{A}$ are algebraic integers. Hence, X is the zero vector and $c_d = 0$ whenever $p \mid d$.

Recall that we had set $d = pd'$ and $q = pq'$. For each divisor d' of q', we have $\mathcal{A}_j(d;q) = \mathcal{A}_j(d';q')$ and $\mathcal{B}_j(d;q) = \mathcal{B}_j(d';q')$. Thus, it suffices to consider the system of equations

$$\mathcal{A}_1(d';q') = \mathcal{B}_1(q'/d';q').$$

Proceeding inductively on the number of divisors of q', we obtain the result that $c_d = 0$ for all divisors d of q.

Therefore, we have proved that (13) has only trivial solutions. From here, we see that if

$$V_{span} := \mathbb{Q}\left\langle \left(1 - \frac{d^k}{d^s}\right) L(s,g_d) \middle|\ d \mid N,\ g_d \in F_D\left(\frac{N}{d}\right)\right\rangle,$$

then $\dim_{\mathbb{Q}} V_{span} = N - \varphi(N)$. Since $V_{span} \subseteq \mathcal{O}_k(N)$ and $\dim_{\mathbb{Q}} \mathcal{O}_k(N) = N - \varphi(N)$ from (10), Theorem 1.2 is proved. ∎

4. Induction of arithmetical functions

Towards the proof of Theorem 1.3, we define induction of arithmetical functions akin to that of Dirichlet characters. Given a character $\chi \bmod q$, its lift, $\chi_N \bmod N$ is given by $\chi_N = \chi\,\chi_{0,N}$, where $\chi_{0,N}$ is the principal character modulo N. The values $L(k,\chi)$ and $L(k,\chi_N)$ differ by at most an Euler factor. Rephrasing this observation, we can say that given a character $\chi \bmod q$ and a natural number N such that $q \mid N$, we have another arithmetic function Ψ of period N taking values in $\mathbb{Q}(\chi)$ such that $L(k,\chi) = L(k,\Psi)$. We generalize this observation for any function $f \in F_D(q;K)$.

If $f \in F_D(q;K)$, then we can write f uniquely as

$$f = \sum_{\chi \bmod q} c_\chi(f)\,\chi, \qquad \text{where} \qquad c_\chi(f) := \frac{1}{\varphi(q)} \sum_{a=1}^{q} f(a)\,\overline{\chi}(a).$$

Using this expression for the function f, we define a 'lift' of f mod q to a function mod N preserving the value at k as follows.

Definition 4.1. *Let k, q, N be positive integers greater than 1, such that $q \mid N$. Let $f \in F_D(q;K)$ have the character decomposition, $f = \sum_{\chi \bmod q} c_\chi(f)\,\chi$. We define*

$$Ind_q^N(f) := \sum_{\chi \bmod q} \frac{c_\chi(f)}{\prod_{p|N}(1-\chi(p)p^{-k})}\ \chi_N,$$

where χ_N denotes the character mod N induced from $\chi \bmod q$.

From the above definition, we note that

$$L\left(s, \mathrm{Ind}_q^N(f)\right) = \sum_{\chi \bmod q} \left[\frac{c_\chi(f)}{\prod_{p|N}(1-\chi(p)p^{-k})} \left(\prod_{p|N}\left(1 - \frac{\chi(p)}{p^s}\right)\right) L(s,\chi)\right],$$

because $\chi_N = \chi\,\chi_{0,N}$. Therefore,

$$L\left(k, \mathrm{Ind}_q^N(f)\right) = \sum_{\chi \bmod q} c_\chi(f)\,L(k,\chi) = L(k,f).$$

Clearly, $\mathrm{Ind}_q^N(f) \in F_D(N;K(\chi))$, where $K(\chi)$ is the field obtained by adjoining the character values $\chi(n)$ to K. However, one can further prove the following.

Lemma 4.2. *Let $f \in F_D(q;K)$ have the character decomposition $f = \sum_{\chi \bmod q} c_\chi(f)\,\chi$. Then, for any prime p, the arithmetic function*

$$g = \sum_{\chi \bmod q} \frac{c_\chi(f)\,\chi}{(1-\chi(p)p^{-k})} \in F_D(q;K).$$

Proof. If $p \mid q$, then we have $\chi(p) = 0$, and hence $g = f$. Hence, we consider the case when $p \nmid q$ and we should prove that the function g is K-valued, as we know that g is supported on co-prime residue classes modulo q. We first note that if $f \in F_D(q;K)$, then, for any r co-prime to q, the function $\sigma_r(f)(n) = f(r^{-1}n) \in F_D(q,K)$ has the character decomposition

$$\sigma_r(f) = \sum_{\chi \bmod q} c_\chi(f)\,\overline{\chi}(r)\,\chi \ \in F_D(q,K). \tag{21}$$

Let l be an exponent of p in $(\mathbb{Z}/q\mathbb{Z})^*$, i.e. $p^l \equiv 1 \bmod q$. Then note that

$$\left(1 - \frac{\chi(p)}{p^k}\right)\left(\sum_{i=0}^{l-1} \frac{\chi(p^i)}{p^{ik}}\right) = C^{-1} \implies \left(1 - \frac{\chi(p)}{p^k}\right)^{-1} = C\left(\sum_{i=0}^{l-1} \frac{\chi(p^i)}{p^{ik}}\right)$$

where $C := \left(1 - \frac{1}{p^{lk}}\right)^{-1}$. Substituting the above expression in g gives

$$g = C \sum_{\chi \bmod q} c_\chi(f) \left(\sum_{i=0}^{l-1} \frac{\chi(p^i)}{p^{ik}}\right) \chi = C \sum_{i=0}^{l-1} \frac{1}{p^{ik}} \sum_{\chi \bmod q} c_\chi(f)\, \chi(p^i)\, \chi. \tag{22}$$

From (21), we conclude that g is K valued.

□

Now by applying Lemma 4.2, and iterating (22) consecutively, we obtain that the arithmetic function $\mathrm{Ind}_q^N(f)$ is also K-valued. Thus, we have proved the following proposition.

Proposition 4.3. *The operator Ind_q^N defined in 4.1 is an injective operator,*

$$Ind_q^N : F_D(q;K) \to F_D(N;K)$$

such that $L(k, Ind_q^N(f)) = L(k, f)$.

The injectivity of the Ind operator is clear as $\mathrm{Ind}_q^N(f) = 0$ implies that $c_\chi(f) = 0$, that is $f \equiv 0$ by the orthogonality of characters mod N.

With the above setup in place, Theorem 1.3 can be easily proved.

Proof of Theorem 1.3. Assume that the Chowla-Milnor Conjecture 2 is true modulo N. This is equivalent to stating that the map $F_D(N) \mapsto \mathbb{C}$, sending $f \to L(k, f)$ is injective. Let d be a divisor of N. By Proposition 4.3, we see that the map $F_D(N/d) \mapsto F_D(N)$ sending $f \to \mathrm{Ind}_{N/d}^N(f)$ is also injective. Hence we conclude that the composite map $F_D(N/d) \to \mathbb{C}$,

$$f \mapsto \mathrm{Ind}_{N/d}^N(f) \mapsto L(k, \mathrm{Ind}_{N/d}^N(f))$$

is injective. Since $L(k, \mathrm{Ind}_{N/d}^N(f)) = L(k, f)$, the Chowla-Milnor conjecture is true modulo N/d.

□

4.1. **The functions $\widetilde{f_1^{(k)}}$ and $\mathrm{Ind}_q^N(f)$.**

Before concluding our discussion, we underline the inherent connection between the function $\widetilde{f_1^{(k)}}$ from Section 3 and the Ind operator defined above. To do so, we first prove the following key lemma.

Lemma 4.4. *Let $f \in F_D(q;K)$ and $q \mid N$. For $(n, N) = 1$, we have*

$$Ind_q^N(f)(n) = \sum_{m \in \mathcal{M}(N)} \frac{f(mn)}{m^k}$$

Proof. Since $|\chi(p)p^{-k}| \le 1$, we note that

$$\left(1 - \frac{\chi(p)}{p^k}\right)^{-1} = \sum_{n \in \mathcal{M}(p)} \frac{\chi(n)}{n^k},$$

which gives

$$\prod_{p \mid N} \left(1 - \frac{\chi(p)}{p^k}\right)^{-1} = \sum_{n \in \mathcal{M}(N)} \frac{\chi(n)}{n^k}.$$

Therefore, for $(n, N) = 1$, we have

$$\mathrm{Ind}_q^N(f)(n) = \sum_{\chi \bmod q} \sum_{m \in \mathcal{M}(N)} \frac{c_\chi(f)\chi(m)}{m^k} \chi_N(n) = \sum_{m \in \mathcal{M}(N)} \frac{f(mn)}{m^k}.$$

In the last step we used the character decomposition of f to evaluate $f(mn)$. This proves the lemma. ◻

For a divisor d of N, define the map $\mathcal{B}_d : F(N;K) \to F_D(N/d;K)$ by

$$\mathcal{B}_d(f)(n) := \begin{cases} f(dn) & \text{if } (n, N/d) = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

so that given any arithmetic function $f \in F(N;K)$, $L(s,f)$ naturally decomposes as

$$L(s,f) = \sum_{d|N} \frac{1}{d^s} L(s, \mathcal{B}_d(f)) \qquad \text{with} \qquad \mathcal{B}_d(f) \in F_D(N/d;K).$$

Thus, $B_d(f)$ can be viewed as the building block of f modulo N/d.

The following proposition establishes the relation between the function $\widetilde{f_1^{(k)}}$ and the Ind operator.

Proposition 4.5. *Let $f \in F(N;K)$. With notation as before,*

$$L\left(s, \widetilde{f_1^{(k)}}\right) = \sum_{d|N} \frac{1}{d^k} L\left(s, Ind_{N/d}^N \mathcal{B}_d(f)\right).$$

Proof. Now let $g \in F_D(N;K)$ be defined such that

$$L(s,g) := \sum_{d|N} \frac{1}{d^k} L\left(s, \mathrm{Ind}_{N/d}^N \mathcal{B}_d(f)\right).$$

On expanding this function by Lemma 4.4, whenever $(n,N) = 1$, we have

$$\begin{aligned} g(n) &= \sum_{d|N} \frac{1}{d^k} \sum_{m \in \mathcal{M}(N/d)} \frac{\mathcal{B}_d(f)(mn)}{m^k} \\ &= \sum_{d|N} \frac{1}{d^k} \sum_{m \in \mathcal{M}(N/d)} \chi_{0,N/d}(mn) \frac{f(dmn)}{m^k} = \sum_{m \in \mathcal{M}(N)} \frac{f(mn)}{m^k}. \end{aligned}$$

In the last step, we note that given $m \in \mathcal{M}(N)$ there is exactly one d such that $(N/d, mn/d) = 1$. Therefore $\chi_{0,N/d_1}(mn/d_1) = 0$ unless $d_1 = d$. Hence $g = \widetilde{f_1^{(k)}}$. ◻

An alternate proof of an analog of Theorem 3.7 for $k > 1$ can be obtained from the above observations. Note that $L(k,f) = L(k, \widetilde{f_1^{(k)}})$. In order to show that

$$f - \widetilde{f_1^{(k)}} \in K \left\langle \mathrm{Ann}_p(F(N/p)) \ : \ p \mid N \right\rangle,$$

it suffices to prove

$$\mathcal{B}_d(f) - \mathrm{Ind}_{N/d}^N \mathcal{B}_d(f) \in K \left\langle \mathrm{Ann}_p(F(N/p)) \ : \ p \mid N \right\rangle.$$

To do so, we consider the Dirichlet series associated to $\mathcal{B}_d(f)$ and $\mathrm{Ind}_{N/d}^N \mathcal{B}_d(f)$. We use the character decomposition of these functions and proceed along the same lines as in the proof of Lemma 4.2.

5. Explicit computations

In Theorem 1.2, we showed that if $L(k, f) = 0$, then there exist unique functions $g_d \in F_D(N/d)$ such that

$$L(s, f) = \sum_{d|N} \left(1 - \frac{d^k}{d^s}\right) L(s, g_d).$$

A natural question is whether the functions $g_d \in F_D(N/d)$ can be explicitly given in terms of the function f. This is a difficult problem in general. However, we address the cases when N is a product of two or three primes in this section. These computations highlight the intricacies of the desired endeavour.

5.1. **Example $N = pq$.** Suppose that $f \in \mathcal{O}_k(N)$. By Theorem 1.2, we know that there exists unique $g_d \in F_D(N/d)$ (for every proper $d \mid N$) such that

$$L(s, f) = \left(1 - \frac{p^k}{p^s}\right) L(s, g_p) + \left(1 - \frac{q^k}{q^s}\right) L(s, g_q) + \left(1 - \frac{N^k}{N^s}\right) L(s, g_N).$$

By Lemma 3.2, we have

$$L(s, f) = -\left(1 - \frac{p^k}{p^s}\right) L(s, \widetilde{f_p^{(k)}}) - \left(1 - \frac{q^k}{q^s}\right) L(s, \widetilde{f_q^{(k)}}) + \prod_{\substack{t|N \\ t \text{ prime}}} \left(1 - \frac{t^k}{t^s}\right) L\left(s, \widetilde{f_{pq}^{(k)}}\right). \quad (24)$$

For a Dirichlet character χ of period co-prime to pq, we write

$$\prod_{\substack{t|N \\ t \text{ prime}}} \left(1 - \frac{t^k}{t^s}\right)$$
$$= c_{p,\chi} \left(1 - \frac{p^k}{p^s}\right)\left(1 - \frac{\chi(q)}{q^s}\right) + c_{q,\chi} \left(1 - \frac{q^k}{q^s}\right)\left(1 - \frac{\chi(p)}{p^s}\right) + c_{pq,\chi} \left(1 - \frac{(pq)^k}{(pq)^s}\right). \quad (25)$$

If we solve this equation for $c_{d,\chi}$ and substitute in (24) (with χ being the trivial character), we get the desired functions g_d for $d \mid pq$, $d \neq 1$.

Continuing (25), by equating the coefficients of d^{-s} for $d \mid N$, we get the following system of equations :

$$c_{p,\chi} + c_{q,\chi} + c_{pq,\chi} = 1, \qquad p^k c_{p,\chi} + \chi(p) c_{q,\chi} = p^k, \qquad \chi(q) c_{p,\chi} + q^k c_{q,\chi} = q^k,$$

$$p^k \chi(q) c_{p,\chi} + q^k \chi(p) c_{q,\chi} + (pq)^k c_{pq,\chi} = (pq)^k.$$

The solution for this system of equations is obtained as

$$c_{p,\chi} = \frac{1 - \frac{\chi(p)}{p^k}}{1 - \frac{\chi(pq)}{(pq)^k}}, \qquad c_{q,\chi} = \frac{1 - \frac{\chi(q)}{q^k}}{1 - \frac{\chi(pq)}{(pq)^k}}, \qquad c_{pq,\chi} = -\frac{\prod_{t|pq}\left(1 - \frac{\chi(t)}{t^k}\right)}{1 - \frac{\chi(pq)}{(pq)^k}}. \quad (26)$$

Substituting these values in (24), we get:

$$L(s,g_p) = -L(s,\widetilde{f_p^{(k)}}) + f(N)\frac{1-p^{-k}}{(1-N^{-k})}\,L(s,\chi_{0,q}),$$
$$L(s,g_q) = -L(s,\widetilde{f_q^{(k)}}) + f(N)\frac{1-q^{-k}}{(1-N^{-k})}\,L(s,\chi_{0,p}),$$
$$L(s,g_{pq}) = -\frac{f(N)}{(1-N^{-k})}\,\zeta(s).$$

5.2. **Example $N = pqr$.** By symmetry, it suffices to compute g_p, g_{pq}, g_{pqr}. Henceforth, whenever we refer to $\prod_{t|d}$ for a divisor d of N, the product should be taken over all the primes t dividing d.

Proceeding as in the earlier example, by Lemma 3.2, it is enough to decompose $\prod_{t|p_ip_j}(1-t^{k-s})$ and $\prod_{t|N}(1-t^{k-s})$ in a 'suitable' manner consisting of Euler factors arising from the Dirichlet series associated to $L(s,\chi)$. We first compute g_{pq}.

Let $h \in F_D(r)$. Then we have

$$\prod_{t|pq}\left(1-\frac{t^k}{t^s}\right) L(s,h) = \sum_{\chi \bmod r} c_\chi(h) \prod_{t|pq}\left(1-\frac{t^k}{t^s}\right) L(s,\chi)$$
$$= \sum_{t|pq}\left(1-\frac{t^k}{t^s}\right) \sum_{\chi \bmod r} c_\chi(h)\, c_{t,\chi} \left(1-\frac{\chi(pq/t)}{(pq/t)^s}\right) L(s,\chi) + \left(1-\frac{(pq)^k}{(pq)^s}\right) \sum_{\chi \bmod r} c_\chi(h)\, c_{pq,\chi}\, L(s,\chi) \tag{27}$$
$$= \sum_{t|pq}\left(1-\frac{t^k}{t^s}\right) L(s,h_t) + \left(1-\frac{(pq)^k}{(pq)^s}\right) L(s,h_{pq})$$
$$= \sum_{d|pq}\left(1-\frac{d^k}{d^s}\right) L(s,h_d). \tag{28}$$

In the above t runs over the primes dividing pq. We also note that $h_d \in F_D(N/d)$ and we get (27) by (25). From Lemma 3.2, we need to consider the above equation for $h = \widetilde{f_{pq}^{(k)}}$ in order to compute h_d.

Now, decompose the function h as follows. For $(n,r)=1$ we have

$$\begin{aligned}\widetilde{f_{pq}^{(k)}}(n) &= \sum_{m\in\mathcal{M}(N)} \frac{f(p_1p_2mn)}{(p_1p_2m)^k}\\ &= \sum_{m\in M(pq)} \frac{f(p_1p_2mn)}{(p_1p_2m)^k} + \sum_{\substack{m\in\mathcal{M}(N)\\ r\mid m}} \frac{f(p_1p_2mn)}{(p_1p_2m)^k}\\ &= \sum_{m\in M(pq)} \frac{f(p_1p_2mn)}{(p_1p_2m)^k} + \frac{f(N)}{N^k}\sum_{m\in\mathcal{M}(N)}\frac{1}{m^k}\\ &= \sum_{m\in M(pq)} \frac{f(p_1p_2mn)}{(p_1p_2m)^k} + \frac{f(N)}{N^k}\prod_{t\mid N}\left(1-\frac{1}{t^k}\right)^{-1}.\end{aligned}$$

Rewriting the first summand in terms of $\mathcal{B}_{pq}(f)$ (see 23), and by arguments similar to Lemma 4.4, we have

$$L(s,h) = \frac{1}{(pq)^k}\sum_{\chi \bmod r}\frac{c_\chi(\mathcal{B}_{pq}(f))}{\prod_{t\mid pq}(1-\chi(t)t^{-k})}L(s,\chi) + \frac{f(N)}{N^k}\prod_{t\mid N}\left(1-\frac{1}{t^k}\right)^{-1}L(s,\chi_{0,r})$$

By substituting values $c_{\chi,p}$ and $c_{\chi,pq}$ as mentioned in (26), we have

$$\begin{aligned}L(s,h_p) = \frac{1}{(pq)^k}&\sum_{\chi \bmod r}\frac{c_\chi(\mathcal{B}_{pq}(f))}{(1-\chi(q)q^{-k})(1-\chi(pq)(pq)^{-k})}L(s,\chi_{N/p})\\ &+\frac{f(N)}{N^k}\prod_{\substack{t\mid N/p\\ t \text{ prime}}}\left(1-\frac{1}{t^k}\right)^{-1}\left(1-\frac{1}{(pq)^k}\right)^{-1}L(s,\chi_{0,qr})\end{aligned} \tag{29}$$

$$\begin{aligned}L(s,h_{pq}) = \frac{1}{(pq)^k}&\sum_{\chi \bmod r}\frac{c_\chi(\mathcal{B}_{pq}(f))}{(1-\chi(pq)(pq)^{-k})}L(s,\chi_{N/p})\\ &+\frac{f(N)}{N^k}\left(1-\frac{1}{r^k}\right)^{-1}\left(1-\frac{1}{pq^k}\right)^{-1}L(s,\chi_{0,r}).\end{aligned} \tag{30}$$

We do a similar analysis for the term $\prod_{t\mid N}(1-t^{k-s})L(s,\widetilde{f_q^{(k)}})$. Since

$$\widetilde{f_q^{(k)}} = f(N)N^{-k}\prod_{t\mid N}(1-t^{-k})^{-1},$$

we require decomposition akin to (25) along the following lines:

$$\begin{aligned}\prod_{t\mid N}\left(1-\frac{1}{t^k}\right) = \sum_{p_i\mid N}c_{p_i}\left(1-\frac{p_i^k}{p_i^s}\right)&\prod_{t\mid N/p_i}\left(1-\frac{1}{t^s}\right)\\ &+\sum_{p_i<p_j}c_{p_ip_j}\left(1-\frac{(p_ip_j)^k}{(p_ip_j)^s}\right)\prod_{t\mid N/p_ip_j}\left(1-\frac{1}{t^s}\right)+c_N\left(1-\frac{N^k}{N^s}\right).\end{aligned} \tag{31}$$

We now expand (31) and equate the coefficients of Dirichlet polynomials as done in Example 5.1. For each proper divisor D of N, we get the following equation :

$$-\sum_{d|D} \mu(d)\, c_d\, d^k + \sum_{d|N/D} c_d = D^k.$$

Solving the above system of equations (using SAGE), we find that

$$c_D = -\mu(D)\, \frac{\prod_{p|D}\left(1-p^{-k}\right)}{1-N^{-k}}.$$

We therefore obtain :

$$\begin{aligned}
\widetilde{f_q^{(k)}}(1) \prod_{t|N}\left(1-\frac{t^k}{t^s}\right) \zeta(s) = & \frac{\widetilde{f_q^{(k)}}(1)}{1-N^{-k}} \Bigg(\sum_{p_i|N} \left(1-\frac{1}{p_i^k}\right)\left(1-\frac{p_i^k}{p_i^s}\right) L(s,\chi_{0,N/p_i}) \\
& - \sum_{p_i<p_j} \prod_{t|p_ip_j} \left(1-\frac{1}{t^k}\right)\left(1-\frac{(p_ip_j)^k}{(p_ip_j)^s}\right) L(s,\chi_{0,N/p_ip_j}) \\
& + \prod_{t|N}\left(1-\frac{1}{t^k}\right)\left(1-\frac{N^k}{N^s}\right) \zeta(s). \qquad (32)
\end{aligned}$$

On substituting the values (32), (29), (30) in Lemma 3.2, we obtain the following evaluations:

$$\begin{aligned}
L(s,g_p) &= -L(s,\widetilde{f_p^{(k)}}) + L(s,h_p) - \frac{f(N)}{N^k}\left(\prod_{t|qr}\left(1-\frac{1}{t^k}\right)\left(1-\frac{1}{N^k}\right)\right)^{-1} L(s,\chi_{0,qr}) \\
L(s,g_{pq}) &= L(s,h_{pq}) - \frac{f(N)}{N^k}\left(\left(1-\frac{1}{r^k}\right)\left(1-\frac{1}{N^k}\right)\right)^{-1} L(s,\chi_{0,r}) \\
L(s,g_N) &= -\frac{f(N)}{(1-N^{-k})}\,\zeta(s).
\end{aligned}$$

6. **Arithmetic nature of $L(k,\chi)$**

Analogous to the odd zeta-values, it is expected that the special values of the Dirichlet L-functions $\{L(k,\chi)\ :\ \chi \bmod N,\ \chi(-1)\neq(-1)^k\}$ are transcendental and algebraically independent (see [5, Section 4]). However, the polylog conjecture is not sufficient to conclude this. In this section, we consider instead the effect of the Polylog conjecture (Conjecture 1) on the *algebraicity* of $L(k,\chi)$.

If $L(k,\chi)\in\overline{\mathbb{Q}}$, it is a natural question to investigate the Galois structure imparted to the set $\mathcal{S}_k := \{\chi \bmod N\ :\ L(k,\chi)\in\overline{\mathbb{Q}},\ N\geq 2,\ \text{squarefree}\}$. We deduce this in Theorem 1.4. Throughout this section, we assume the Polylog conjecture.

We begin with a lemma about the $\overline{\mathbb{Q}}$ vector space of tuples of algebraic numbers. Let $\alpha_1,\ldots,\alpha_n$ be distinct algebraic numbers with $|\alpha_i|\leq 1$ and let the evaluation map $\Phi:\overline{\mathbb{Q}}^n\to\mathbb{C}$ be given by

$$\Phi(a_1,\ldots a_n) = \sum_{i=1}^{n} a_i\,\mathrm{Li}_k(\alpha_i)\,. \qquad (33)$$

Let $V_{\overline{\alpha}} = \{(a_1,\ldots,a_n)\in\overline{\mathbb{Q}}^n \mid \Phi(a_1,\ldots,a_n)\in\overline{\mathbb{Q}}\}$. Note that $V_{\overline{\alpha}}$ is a $\overline{\mathbb{Q}}$-vector space.

Lemma 6.1. *Suppose that $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are such that $Li_k(\alpha_1)$, $Li_k(\alpha_2)$, $\cdots$, $Li_k(\alpha_n)$ are $\mathbb{Q}$-linearly independent. Then, conditional on the Polylog conjecture, there exists a $C \in \overline{\mathbb{Q}}^*$ such that for any $(a_1, \dots a_n)$, $(b_1, \dots b_n) \in V_{\overline{\alpha}} \setminus \{0\}$, $a_i = Cb_i$.*

Proof. Let $V = V_{\overline{\alpha}}$. First, note that $\dim_{\overline{\mathbb{Q}}} V/\ker\Phi \leq 1$. Indeed, if $V = \{(0, \dots, 0)\}$, there is nothing to prove. If not, then the map $\Phi|_V : V \mapsto \overline{\mathbb{Q}}$ is surjective and hence, $V/\ker(\Phi) \cong \overline{\mathbb{Q}}$. Since $\mathrm{Li}_k(\alpha_1), \dots, \mathrm{Li}_k(\alpha_n)$ be linearly independent over $\mathbb{Q}$, by Polylog conjecture, $\mathrm{Li}_k(\alpha_1), \dots, \mathrm{Li}_k(\alpha_n)$ are linearly independent over $\overline{\mathbb{Q}}$, that is, $\ker\Phi = \{0\}$. Thus $\dim_{\overline{\mathbb{Q}}} V \leq 1$ and we have the lemma. □

Using equation (2), we know that

$$L(k, \chi) = \sum_{a=1}^{N-1} \widehat{\chi}(a) \,\mathrm{Li}_k(e^{2\pi i a/N}),$$

with $\widehat{\chi}(a) \in \mathbb{Q}(\chi)(e^{2\pi i/N})$ and $\mathbb{Q}(\chi) = \mathbb{Q}(\{\chi(n) : 1 \leq n \leq N\})$. However, if N is squarefree, one can further prove the following.

Proposition 6.2. *Let N be squarefree and χ be a character $\mathrm{mod} N$. Then*

$$L(k, \chi) = \widehat{\chi(1)} \sum_{a=1}^{N-1} c_{a,\chi} \, Li_k(e^{2\pi i a/N}),$$

where $c_{a,\chi} \in \mathbb{Q}(\chi)$ and $\widehat{\chi(1)} = N^{-1} \sum_{a=1}^{N} \chi(a)\, e^{-2\pi i a/N}$.

Proof. When $(a, N) = 1$, it is easy to see that

$$\widehat{\chi(a)} = \frac{1}{N} \sum_{b=1}^{N} \chi(b)\, e^{-2\pi i ab/N} = \overline{\chi(a)}\, \widehat{\chi(1)}.$$

Now suppose $(a, N) = d > 1$. Write $a = a_1 d$. Therefore we have

$$\begin{aligned}
\widehat{\chi(a)} = \frac{1}{N} \sum_{\substack{b=1 \\ (b,N)=1}}^{N} \chi(b)\, e^{-\frac{2\pi i a_1 b}{(N/d)}} &= \frac{\mu^2(d)}{N} \sum_{\substack{b=1 \\ (b,N)=1}}^{N} \chi(b)\, e^{-\frac{2\pi i a_1 b}{(N/d)}} \\
&= \frac{\mu(d)}{N} \sum_{\substack{b=1 \\ (b,N)=1}}^{N} \chi(b)\, e^{-\frac{2\pi i a_1 b}{(N/d)}} \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e^{-2\pi i bc/d}, \qquad (34)
\end{aligned}$$

using the fact that $\mu^2(d) = 1$ since d is squarefree and that $\mu(d) = \sum\limits_{\substack{c=1 \\ (c,d)=1}}^{d} \zeta_d^{-bc}$. Let c_1 and c_2 be positive integers such that

$$\frac{1}{N} = c_1 \frac{1}{N/d} + c_2 \frac{1}{d},$$

and S_a be the set

$$S_a := \left\{ m \in (\mathbb{Z}/N\mathbb{Z})^* \mid m \equiv a_1 c_1^{-1} \bmod N/d, m \equiv c c_2^{-1} \bmod d, \text{ as } c \text{ varies over } (\mathbb{Z}/d\mathbb{Z})^* \right\}$$

Note that for $m \in S_a$, we have

$$e^{2\pi i m/N} = e^{-\frac{2\pi i a_1}{(N/d)}}\, e^{2\pi i c/d},$$

for some c co-prime to d. Hence we can write (34) as

$$\widehat{\chi(a)} = \frac{\mu(d)}{N} \sum_{m \in S_a} \sum_{b=1}^{N} \chi(b)\, e^{-2\pi i m b/N} = \widehat{\chi(1)}\mu(d) \sum_{m \in S_a} \overline{\chi(m)} = \widehat{\chi(1)}\, c_{a,\chi},$$

where $c_{a,\chi} \in \mathbb{Q}(\chi)$, proving the proposition. □

With these preliminaries in place, we prove the theorem regarding the Galois structure of $\mathcal{S}_k$ below. Recall the disjointness hypothesis (say ($\mathcal{H}_{dis}$)) that we assume in the statement of Theorem 1.4: there exists a squarefree integer N and two distinct characters χ and Ψ mod N such that

$$\chi, \Psi \in \mathcal{S}_k \text{ and } \mathbb{Q}(\chi) \cap \mathbb{Q}(\Psi) = \mathbb{Q}. \tag{$\mathcal{H}_{dis}$}$$

Proof of Theorem 1.4. Let N be as in $(\mathcal{H}_{dis})$ and $l_1, \dots l_t$ be a maximal $\mathbb{Q}$-linearly independent subset of $\{\mathrm{Li}_k(e^{2\pi i a/N}) \,|\, 1 \le a < N\}$. We write,

$$\mathrm{Li}_k(e^{2\pi i j/N}) = \sum_{i=1}^{t} a_{ij}\, l_i \text{ for } 1 \le j < N.$$

Re-writing $L(k,\chi)$ from Proposition 6.2 as a $\overline{\mathbb{Q}}$-linear combination of l_i for $1 \le i \le t$, we get :

$$L(k,\chi) = \widehat{\chi(1)} \sum_{j=1}^{N-1} c_{j,\chi} \sum_{i=1}^{t} a_{ij}\, l_i = \widehat{\chi(1)} \sum_{i=1}^{t} \left(\sum_{j=1}^{N-1} a_{ij}\, c_{j,\chi} \right) l_i.$$

Then by Lemma 6.1, for any distinct characters, periodic modulo the same modulus, we have

$$\frac{\widehat{\chi(1)}(\sum_{j=1}^{N-1} a_{mj}\, c_{j,\chi})}{\widehat{\Psi(1)}(\sum_{j=1}^{N-1} a_{mj}\, c_{j,\Psi})} = \frac{\widehat{\chi(1)}(\sum_{j=1}^{N-1} a_{nj}\, c_{j,\chi})}{\widehat{\Psi(1)}(\sum_{j=1}^{N-1} a_{nj}\, c_{j,\Psi})},$$

for any two m, n with $1 \le m, n \le t$ and $m \ne n$. In the above we are considering only those natural numbers m and n for which $\sum_{j=1}^{N-1} a_{mj} c_{j,\chi} \ne 0$. Since N is squarefree, $\widehat{\chi}(1)$, $\widehat{\Psi}(1)$ are non-zero and we have the following:

$$\frac{(\sum_{j=1}^{N-1} a_{mj}\, c_{j,\chi})}{(\sum_{j=1}^{N-1} a_{nj}\, c_{j,\chi})} = \frac{(\sum_{j=1}^{N-1} a_{mj}\, c_{j,\Psi})}{(\sum_{j=1}^{N-1} a_{nj}\, c_{j,\Psi})} =: C_{mn}. \tag{35}$$

Since the corresponding fields are disjoint, $C_{mn} \in \mathbb{Q}$. Also, note that C_{mn} depends only on the period and is independent of the characters χ, Ψ. Thus, we have

$$L(k,\chi) = \frac{\widehat{\chi(1)}}{N} \left(\sum_{j=1}^{N-1} a_{1j}\, c_{j,\chi} \right) \left(\sum_{i=1}^{t} C_{i1}\, l_i \right) \in \overline{\mathbb{Q}} \implies \sum_{i=1}^{t} C_{i1}\, l_i \; \in \overline{\mathbb{Q}}. \tag{36}$$

Noting that (35) is valid if we replace χ by its conjugate (say χ^σ), and therefore by (36), we get

$$L(k,\chi^\sigma) = \frac{\widehat{\chi^\sigma(1)}}{N} \left(\sum_{j=1}^{N-1} a_{1j}\, c_{j,\chi^\sigma} \right) \left(\sum_{i=1}^{t} C_{i1}\, l_i \right) \in \overline{\mathbb{Q}}.$$

Thus, under ($\mathcal{H}_{dis}$), we have that

$$L(k,\chi) \in \overline{\mathbb{Q}} \implies L(k,\chi^\sigma) \in \overline{\mathbb{Q}}$$

for all characters χ^σ conjugate to χ.

We still have to prove the result for a character $\eta \in \mathcal{S}_k$ of a period different from N. Let $\eta \in \mathcal{S}_k$ be of period M and set $Q := \operatorname{lcm}(N, M)$. Proceeding as above: replace N by Q, χ and Ψ mod N by χ_Q and Ψ_Q mod Q respectively. Then we see that (35) also holds for characters χ_Q, Ψ_Q and η_Q. Since we know that $C_{mn} \in \mathbb{Q}$ (by comparing χ_Q mod Q and Ψ_Q mod Q in (35)), we obtain

$$L(k, \eta_Q) \in \overline{\mathbb{Q}} \implies L(k, \eta_Q^\sigma) \in \overline{\mathbb{Q}}.$$

As $L(k, \eta)$ and $L(k, \eta_Q)$ differ by an algebraic number, the result is also holds for $L(k, \eta)$. □

Remark 6.3. *We make the following observations about the above result.*

1. *In general, it is not true that when χ and k are of the same parity, then $L(k, \chi^\sigma)/\pi^k = \sigma(L(k, \chi)/\pi^k)$. For instance, this can be seen by taking $k = 1$ and χ to be a primitive odd character of conductor p^2. We know that*
$$p^2 \, L(1, \chi) = -i \, \pi \, B_{1,\chi} \, \tau(\chi),$$
where $B_{1,\chi} \in \overline{\mathbb{Q}}$ denotes the generalized Bernoulli number. When we apply the automorphism $\sigma_j : e^{2\pi i/p^2} \to e^{2\pi i j/p^2}$, we obtain $\sigma(\tau(\chi)) = \chi^\sigma(j)\tau(\chi^\sigma)$, which gives
$$\sigma\left(\frac{L(1, \chi)}{\pi}\right) = \chi^\sigma(j) \left(\frac{L(1, \chi^\sigma)}{\pi}\right).$$
2. *If one restricts to quadratic characters, then proceeding along the lines of* [9, Proposition 5] *implies that given a number field K, there exist at most $[K : \mathbb{Q}] + 1$ quadratic characters χ (of opposite parity), such that $L(k, \chi) \in K$, conditional on the Chowla-Milnor conjecture.*

7. Concluding Remarks

Characterizing rational valued periodic arithmetical functions f such that $L(k, f) = 0$ is not only an interesting question in its own right, but also subsumes the investigation of possible linear relations among the special values of Dirichlet L-functions, the polylogarithm functions and the Hurwitz zeta-functions respectively. The tools developed in Sections 3 and 4 provide new framework for approaching these questions and advancing our understanding.

For instance, we have seen that

$$L(k, f) = \frac{1}{N^k} \sum_{a=1}^{N} f(a) \, \zeta\left(k, \frac{a}{N}\right),$$

which relates the vanishing of $L(k, f)$ to linear relations among the numbers

$$\left\{\zeta\left(k, \frac{a}{N}\right) \mid 1 \leq a \leq N\right\}.$$

This problem was addressed by Milnor [12] using the theory of distributions (see also [23, Chapter 12]). Fix any complex number s. A function f on the interval $(0, 1)$ is said to be a *Kubert function* if f satisfies *the Kubert relations or distribution relations*, namely: for every natural number m and $x \in (0, 1)$,

$$f(x) = m^{s-1} \sum_{l=0}^{m-1} f\left(\frac{x + l}{m}\right). \qquad (*_s)$$

Examples of such functions include the polylogarithm functions $l_s(x) := \mathrm{Li}_s(e^{2\pi i x})$, that satisfy $(*_s)$ when $s \neq -1, -2, \cdots$ and the Hurwitz zeta functions $\zeta(s, x)$ with s fixed, that satisfy $(*_{1-s})$ whenever $s \neq 0, 1, 2, \cdots$. A Kubert function is said to be *universal* if all $\mathbb{Q}$-linear relations among the values of f are generated by Kubert relations. The function $\cot(\pi x)$ is an example of a universal function (see [12, Theorem 3]).

In [12, Section 6], Milnor conjectures that if $k > 1$ is an integer, then the function $\zeta(k, x)$ is universal. In the same discussion, he alludes to the equivalence between the universality of $\zeta(k, x)$ as defined above and the statement that for every integer $N \geq 2$, the numbers

$$\left\{\zeta\left(k, \frac{a}{N}\right) \mid 1 \leq a \leq N,\, (a, N) = 1\right\}$$

are $\mathbb{Q}$-linearly independent, which we refer to as the Chowla-Milnor conjecture. However, it is not made immediately clear why these two claims are equivalent.

The method adopted in Section 3 can be viewed as an analytic perspective towards the algebraic setup described above. Indeed, if $g_b(m) := 1$ whenever $m \equiv b \bmod N/d$ and 0 otherwise, then unraveling the identity

$$L(s, f) = \left(1 - \frac{d^k}{d^s}\right) L(s, g_b) \qquad \Longrightarrow \qquad L(k, f) = 0$$

in terms of the Hurwitz zeta-function leads to the distribution relation

$$\zeta\left(k, \frac{b}{N/d}\right) = d^{-k} \sum_{l=0}^{d-1} \zeta\left(k, \frac{\frac{b}{N/d} + l}{d}\right).$$

In other words, Theorem 3.3 gives an explicit proof that all possible $\mathbb{Q}$-linear relations among the Hurwitz zeta-values are generated by the distribution relations. In fact, in Theorem 1.2, we go one step further and prove a stronger statement, which is that the relations mod N arising from distributions modulo distinct divisors of N are linearly independent. Thus, our approach provides an 'analytic' perspective to an 'algebraic' problem.

In the same spirit, Girstmair [8] utilized the theory of character coordinates and relative traces of cyclotomic numbers to obtain an explicit formula for the coefficients a_j in expressions such as

$$\cot\left(\frac{\pi}{d}\right) = \sum_{\substack{1 \leq j \leq N/2, \\ (j,N)=1}} a_j \cot\left(\frac{j\pi}{N}\right),$$

where $N \geq 3$ and $d \geq 2$ is a divisor of N. Thus, the values of trigonometric functions at a 'lower level' are written as a linear combination of the values at a 'higher level'.

This can also be achieved via the induction operator Ind_q^N defined in Section 4. Indeed, let

$$f_{a,q} = \begin{cases} 1 & n \equiv a \mod q \\ (-1)^k & n \equiv -a \mod q \\ 0 & \text{otherwise.} \end{cases}$$

As derived in [18], note that

$$L(k, f_{a,q}) = \frac{(-1)^k}{(k-1)!q^k} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot(\pi x)) \bigg|_{x=a/q}.$$

Similarly, we have the evaluation

$$L(k, \mathrm{Ind}_q^N f_{a,q}) = \frac{(-1)^k}{(k-1)!N^k} \sum_{\substack{n=1 \\ (n,N)=1}}^{N} c_{a,q,n,k} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot(\pi x)) \bigg|_{x=n/N},$$

where $c_{a,q,n,k}$ are given by Lemma (4.4). By definition of the Ind operator, we have that $L(k, f_{a,q}) = L(k, \mathrm{Ind}_q^N f_{a,q})$. Hence, comparing the above two equations immediately leads to the desired expression.

Thus, the methods introduced in the paper have wider applicability and potential for further development. We relegate this to future research.

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Department of Mathematics and Statistics, 48 University Avenue, Jeffery Hall, Kingston, ON K7L 3N6, Canada
Email address: `atb4@queensu.ca`

Chennai Mathematical Institute, H-1 SIPCOT IT Park, Siruseri Kelambakkam, Tamil Nadu 603 103, India.
Email address: siddhi@cmi.ac.in