TRANSCENDENCE OF INFINITE SERIES OVER LATTICES

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Abstract. In this paper, we study the arithmetic nature of series of the form

$$\sum_{\omega \in \Lambda} \frac{A(\omega)}{B(\omega)},$$

where Λ is a two-dimensional lattice in $\mathbb{C}$, $A(X)$ and $B(X)$ are suitable polynomials over $\mathbb{C}$, with $\deg A \leq \deg B - 3$. In particular, we focus on the cases when the roots of the polynomial $B(X)$ are either algebraic numbers or rational multiples of a non-zero period of Λ.

1. **Introduction**

The study of the arithmetic nature of special values of infinite series originates with the theorem of Euler that for a positive integer $k \geq 1$,

$$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

where $\zeta(s)$ is the Riemann zeta-function and B_n is the n^{th} Bernoulli number defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n,$$

from which we see that the Bernoulli numbers are rational numbers. Using Lindemann's theorem that π is transcendental, one can deduce that the values $\zeta(2k)$ are transcendental. However, the transcendental nature of $\zeta(2k+1)$ remains unknown.

More generally, one may consider the series

$$\sum_{n \in \mathbb{Z}} \frac{A(n)}{B(n)} = \lim_{N \to \infty} \sum_{|n| \leq N} \frac{A(n)}{B(n)}, \tag{1}$$

for suitable polynomials $A(X)$ and $B(X)$ when $A(X)$ and $B(X)$ have algebraic coefficients. The study of these series was initiated by M. Ram Murty and C. Weatherby in [7] and [8]. To deduce the transcendence of sums of the form (1), they applied the well-known cotangent expansion

$$\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{z+n} = \lim_{N \to \infty} \sum_{|n| \leq N} \frac{1}{z+n},$$

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together with the conjectures of Schneider, namely that, for algebraic numbers $\alpha \neq 0, 1$ and β of degree $d \geq 2$, the numbers

$$\alpha^{\beta},\, \alpha^{\beta^2},\, \cdots,\, \alpha^{\beta^{d-1}}$$

are algebraically independent; or more generally that of Gelfond-Schneider, i.e.,

$$\log(\alpha),\, \alpha^{\beta},\, \alpha^{\beta^2},\, \cdots,\, \alpha^{\beta^{d-1}}$$

are algebraically independent (see [7]). A striking example of their work is given by the explicit evaluation,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(An^2 + Bn + C)} = \frac{2\pi}{\sqrt{D}} \left(\frac{e^{2\pi\sqrt{D}/A - 1}}{e^{2\pi\sqrt{D}/A} - 2\cos(\pi B/A) e^{\pi\sqrt{D}/A} + 1} \right),$$

for $A, B, C \in \mathbb{Z}$ and $-D = B^2 - 4AC < 0$. This sum is transcendental by invoking a deep result of Nestrenko [9] on the algebraic independence of π and $e^{\pi\sqrt{D}}$.

Several discoveries in mathematics have been driven by analogy. If one treats the above discussion as a 'one-dimensional' scenario, it is natural to inquire if similar theorems hold in the 'two-dimensional' setup, i.e., the elliptic world. This question was first taken up by M. Ram Murty and Akshaa Vatwani [6] in 2014. They initiated the study of elliptic analogues of the Hurwitz zeta-function and the Dirichlet L-functions. In particular, they investigated the special values of elliptic analogues of Dirichlet L-functions, thus generalizing Hecke's work.

Continuing on the path illumined by analogy, we would like to study the arithmetic nature of series of the form

$$\mathcal{S}(A, B) := \sum_{\omega \in \Lambda} \frac{A(\omega)}{B(\omega)}, \tag{2}$$

where Λ is a two-dimensional lattice in $\mathbb{C}$ and $A(X), B(X) \in \mathbb{C}[X]$ are co-prime polynomials with $\deg A \leq \deg B - 3$ with other suitable conditions imposed to ensure the absolute convergence of the above series. Suppose ω_1, ω_2 are fundamental periods of the lattice Λ, chosen such that $\mathrm{Im}(\omega_2/\omega_1) > 0$. Then, the sum should be interpreted as

$$\lim_{M \to \infty} \lim_{N \to \infty} \sum_{|m| \leq M} \sum_{|n| \leq N} \frac{A(m\,\omega_2 + n\,\omega_1)}{B(m\,\omega_2 + n\,\omega_1)}.$$

Our considerations lead us to elliptic analogues of the cotangent function and its derivatives, which form building blocks of the corresponding sums in the classical case. These are precisely given by the Weierstrass ζ-function, the Weierstrass $\wp$-function and its derivatives. For a lattice Λ, let $\Lambda^* := \Lambda \backslash \{0\}$. Then the Weierstrass ζ-function attached to Λ is given by

$$\zeta(z; \Lambda) := \frac{1}{z} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right].$$

The derivative of the Weierstrass ζ-function gives the Weierstrass $\wp$-function,

$$\wp(z; \Lambda) = -\zeta'(z; \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z - \omega)^2} + \frac{1}{\omega^2} \right],$$

which is an elliptic function with respect to the lattice Λ. Since the above series converges uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$,

$$\wp^{(j)}(z;\Lambda) = (-1)^j (j+1)! \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^{j+2}}.$$

Associated to each lattice Λ, we define the invariants,

$$g_2(\Lambda) = 60 \sum_{\omega\in\Lambda^*} \frac{1}{\omega^4}, \;\; g_3(\Lambda) = 140 \sum_{\omega\in\Lambda^*} \frac{1}{\omega^6}. \tag{3}$$

Since the lattice Λ will usually be fixed in our discussion, we will often drop the reference to it in our notation for the sake of brevity. With the notation in place, we state our main theorems below.

Theorem 1.1. *Let $A(X)$, $B(X) \in \mathbb{C}[X]$ be co-prime polynomials with $\deg A \leq \deg B - 3$. Let the distinct roots of $B(X)$ be $\alpha_1, \cdots, \alpha_r$ with multiplicities $\mu_1, \cdots, \mu_r$. Suppose that the partial fraction decomposition of $A(X)/B(X)$ is*

$$\frac{A(X)}{B(X)} = \sum_{i=1}^{r}\sum_{j=1}^{\mu_r} \frac{\lambda_{i,j}}{(X-\alpha_i)^j}.$$

Let $\mathcal{M} := \max_{1\leq i\leq r} \mu_i$ and define $\lambda_{i,j} = 0$ for $\mu_i < j \leq \mathcal{M}$, $1 \leq i \leq r$. Further suppose that none of the roots α_i of $B(X)$ lie in Λ. Then

$$\mathcal{S}(A,B) = -\sum_{i=1}^{r} \lambda_{i,1}\,\zeta(\alpha_i) + \sum_{i=1}^{r}\sum_{j=2}^{\mathcal{M}} \frac{\lambda_{i,j}}{(j-1)!}\,\wp^{(j-2)}(\alpha_i),$$

where $\zeta(z)$ is the Weierstrass ζ-function and $\wp(z)$ is the Weierstrass $\wp$-function.

In the next two theorems, we will concentrate on determining the arithmetic nature of the sum $\mathcal{S}(A,B)$. When the roots of $B(X)$ are non-integral rational multiples of non-zero periods, we prove the following.

Theorem 1.2. *Let $\Lambda = \omega_1\,\mathbb{Z} \oplus \omega_2\,\mathbb{Z}$ with $Im(\omega_2/\omega_1) > 0$ and $g_2(\Lambda)$, $g_3(\Lambda) \in \overline{\mathbb{Q}}$. Let $A(X) \in \overline{\mathbb{Q}}[X]$ and $B(X) \in \mathbb{C}[X]$ be as in Theorem 1.1. Fix a non-zero period in Λ, say $\omega \in 2^k\Lambda \setminus 2^{k+1}\Lambda$ for some $k \in \mathbb{Z}_{\geq 0}$. Assume that $B(X)$ is monic and that the roots α_i of $B(X)$ are of the form*

$$\alpha_i = \frac{a_i}{b_i}\,\omega \;\; \notin \Lambda,$$

where a_i, $b_i \in \mathbb{Z}$, $a_i \neq 0$, $b_i > 1$ and $\gcd(a_i, b_i) = 1$.

a) If all roots of $B(X)$ are simple, then $\mathcal{S}(A,B)$ is a rational function in ω over $\overline{\mathbb{Q}}$ and hence, is either zero or transcendental.

b) If $B(X)$ has at least one repeated root, $\sum_{i=1}^{r} \lambda_{i,1}\,\alpha_i \neq 0$ and Λ has complex multiplication, then $\mathcal{S}(A,B)$ is a non-zero rational function in ω and the corresponding quasi-period $\eta(\omega)$ over $\overline{\mathbb{Q}}$, and hence is transcendental.

Remark. *Let $A(X)$, $C(X)$, $D(X)$ be non-zero polynomials in $\overline{\mathbb{Q}}[X]$ such that $D(X)$ has rational roots and $B(X)$ is as in Theorem 1.2 a) with Λ having CM. If*

$$\sum_{n\in\mathbb{Z}} \frac{C(n)}{D(n)} \;\; \text{and} \;\; \sum_{\omega\in\Lambda} \frac{A(\omega)}{B(\omega)}$$

are both non-zero, then by [7, Theorem 1] and Theorem 2.8 (see Section 2.3), the above two sums are algebraically independent. Thus, our considerations give rise to 'new' transcendental numbers.

Inspired by the classical case, we study the nature of $\mathcal{S}(A,B)$ when $B(X)$ has algebraic roots. This involves an understanding of the algebraic independence of values of the Weierstrass functions at algebraic arguments. In this context, the following conjectures have been put forth, which are motivated by the nature of values of the exponential function, in particular, the Lindemann-Weierstrass theorem and Schanuel's conjecture (see [5]).

An elliptic analogue of the Lindemann-Weierstrass theorem was conjectured by G. V. Chudnovsky [2] in 1980.

Conjecture 1 (Chudnovsky)**.** *Suppose that $n \geq 1$, the Weierstrass $\wp(z)$ has algebraic invariants (i.e., g_2, $g_3 \in \overline{\mathbb{Q}}$), does not have complex multiplication and $\alpha_1, \cdots, \alpha_n$ are algebraic numbers that are $\mathbb{Q}$-linearly independent. Then the numbers*

$$\wp(\alpha_1), \cdots, \wp(\alpha_n)$$

are algebraically independent over $\mathbb{Q}$.

This conjecture has only been proved when $n = 1$, which is a theorem of Schneider (see Theorem 2.4 in Section 2 of this paper).

In 1983, Wüstholz [13] and P. Philippon [10] proved that the elliptic analogue of the Lindemann-Weierstrass theorem is true in the CM-case. In particular,

Theorem 1.3 (G. Wüstholz, P. Philippon)**.** *Suppose that $n \geq 1$, the Weierstrass $\wp(z)$ has algebraic invariants and complex multiplication, and $\alpha_1, \cdots, \alpha_n$ are linearly independent over the field of complex multiplication. Then the numbers*

$$\wp(\alpha_1), \cdots, \wp(\alpha_n)$$

are algebraically independent over $\mathbb{Q}$.

The algebraic independence of values of the Weierstrass $\wp$-function together with the Weierstrass ζ-function is predicted by an elliptic analogue of the classical Schanuel's conjecture. Following the authors in [11], we call this the *elliptic Schanuel conjecture* and state it below.

Conjecture 2 (Elliptic Schanuel)**.** *Let Λ be a lattice and $\wp$, ζ denote the associated Weierstrass functions. Let K be the field of endomorphisms of Λ (i.e., $K = \mathbb{Q}(\omega_1/\omega_2)$ if Λ has complex multiplication and $K = \mathbb{Q}$ otherwise). Let $x_1, \cdots, x_n \in \mathbb{C}\backslash\Lambda$ such that they are linearly independent over K. Then*

$$\text{tr deg } \mathbb{Q}\Big(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, x_1, \cdots, x_n, \wp(x_1), \cdots, \wp(x_n), \zeta(x_1), \cdots, \zeta(x_n)\Big)$$
$$\geq 2n + \frac{4}{[K:\mathbb{Q}]}.$$

Under the assumption of the above conjectures, we prove the following.

Theorem 1.4. *Let Λ be a lattice with $g_2(\Lambda)$, $g_3(\Lambda) \in \overline{\mathbb{Q}}$. Let $A(X)$, $B(X) \in \overline{\mathbb{Q}}[X]$ be co-prime polynomials as in Theorem 1.1. Suppose that $B(X)$ is monic and that the distinct roots of $B(X)$, namely, $\alpha_1, \alpha_2, \cdots, \alpha_r$ are $\mathbb{Q}$-linearly independent.*

a) If $\lambda_{i,1} \neq 0$ for at least one i, $1 \leq i \leq r$, then Conjecture 2 implies that $\mathcal{S}(A, B)$ is transcendental.

b) If $\lambda_{i,1} = 0$ for all $1 \leq i \leq r$ and Λ does not have complex multiplication, then $\mathcal{S}(A, B)$ is transcendental conditional on Conjecture 1.

c) If $\lambda_{i,1} = 0$ for all $1 \leq i \leq r$ and Λ has complex multiplication, then $\mathcal{S}(A, B)$ is transcendental.

2. Preliminaries

In this section, we recall relevant aspects of Weierstrass functions and prove crucial lemmas. A detailed introduction to elliptic functions can be found in the classic book of Whittaker and Watson [12, Chapter XX]. However, for a brief review, we refer the reader to [5, Chapter 10].

2.1. The Weierstrass $\wp$-function and its derivatives.

Let $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$, with $\mathrm{Im}(\omega_2/\omega_1) > 0$ be a two-dimensional lattice in $\mathbb{C}$. The elements of Λ are called *periods* of the lattice. The numbers ω_1 and ω_2 that generate the lattice Λ are called *fundamental periods.* An *elliptic function* with respect to the lattice Λ is a meromorphic function periodic with periods ω_1, ω_2 and hence any element of Λ. An example of a non-constant elliptic function is provided by the Weierstrass $\wp$-function associated to Λ, defined as

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right], \quad z \in \mathbb{C} \backslash \Lambda,$$

where Λ^* denotes the set of non-zero periods. It can be shown that the above series converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$, thus defining an analytic function on $\mathbb{C}\backslash\Lambda$, with poles of order 2 at every period in Λ. On differentiating the series term-by-term for $z \in \mathbb{C}\backslash\Lambda$, it follows that the derivatives of $\wp$,

$$\wp^{(j)}(z) = (-1)^j (j+1)! \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^{j+2}}, \quad j \geq 1,$$

are elliptic functions on $\mathbb{C}$. Although it may seem that the Weierstrass $\wp$-function is only one specific example of an elliptic function, it turns out that $\wp(z)$ and its derivative $\wp'(z)$ are, in a sense, universal. That is, any elliptic function with respect to Λ is a rational function in $\wp$ and $\wp'$.

The Weierstrass functions $\wp(z)$ and $\wp'(z)$ satisfy a differential equation with coefficients g_2 and g_3, namely,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{4}$$

where the invariants g_2 and g_3 are as in (3). Observe that

$$\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1+\omega_2}{2}\right) = 0,$$

as $\wp'(z)$ is an odd elliptic function. Hence, the numbers

$$\wp\left(\frac{\omega_1}{2}\right), \ \wp\left(\frac{\omega_2}{2}\right) \ \text{and} \ \wp\left(\frac{\omega_1+\omega_2}{2}\right) \tag{5}$$

are roots of the equation $4x^3 - g_2x - g_3 = 0$ and so are algebraic when $g_2, g_3 \in \overline{\mathbb{Q}}$.

From (4), one deduces an *addition law* for the Weierstrass $\wp$-function. Let $z_1, z_2 \in \mathbb{C}\backslash\Lambda$, with $z_1 \pm z_2 \notin \Lambda$. Then

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2. \tag{6}$$

Taking the limit as $z_1 \to z_2$, one obtains the *duplication formula*

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp(z)}\right)^2.$$

Thus, when the invariants g_2, g_3 are algebraic, the addition law together with the observation that the numbers (5) are algebraic implies that

$$\wp\left(\frac{\omega_1}{n}\right) \text{ and } \wp\left(\frac{\omega_2}{n}\right)$$

are algebraic numbers for all non-zero integers $n \in \mathbb{Z}_{>1}$. Furthermore, the addition and duplication formulas give that

$$\wp\left(\frac{m}{n}\,\omega_1\right) \text{ and } \wp\left(\frac{m}{n}\,\omega_2\right),$$

with m, $n \in \mathbb{Z}$, $n > 1$, and $n \nmid m$, are also algebraic.

Since the derivatives of the Weierstrass $\wp$-function will be essential to our understanding of the nature of $\mathcal{S}(A,B)$, we prove the following lemma regarding their representation as a polynomial in $\wp$ and $\wp'$.

Lemma 2.1. *Let $\wp(z)$ be the Weierstrass $\wp$-function associated to a lattice Λ, with g_2, $g_3 \in \overline{\mathbb{Q}}$. Then for every $l \geq 0$, there exist polynomials $F_l(X)$, $G_l(X) \in \overline{\mathbb{Q}}[X]$ such that*

$$\wp^{(l)}(z) = F_l\left(\wp(z)\right)\,\wp'(z) + G_l\left(\wp(z)\right).$$

Moreover, $F_{2l}(X) = 0$, $G_{2l+1}(X) = 0$ and $\deg F_{2l+1}(X) = l$, $\deg G_{2l}(X) = l+1$.

Proof. The claim is a tautology for $l = 0$ with $F_0(X) = 0$, $G_0(X) = X$ and for $l = 1$ with $F_1(X) = X$ and $G_1(X) = 0$. For $l > 1$ the existence of such polynomials will be seen to be a consequence of differentiating the algebraic differential equation,

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\,\wp(z) - g_3$$

to obtain

$$\wp'(z)\,\wp''(z) = \left(6\,\wp^2(z) - \frac{g_2}{2}\right)\wp'(z).$$

In other words,

$$\wp^{(2)}(z) = 6\,\wp^2(z) - \frac{g_2}{2}. \tag{7}$$

So we have expressed $\wp^{(2)}(z)$ as desired, with $G_2(X) = 6\,X^2 - (g_2/2)$ and $F_2(X) = 0$. When we differentiate any expression for $\wp^{(k)}(z)$, $k \geq 2$, expressed as a linear polynomial in $\wp'(z)$ with coefficients themselves polynomials in $\wp(z)$, then (7) allows us to express $\wp^{(k+1)}(z)$ in that form as well, and we have the claim in general by induction.

Indeed the claim on the degrees and vanishing of the polynomials F_l and G_l has been exhibited for $l = 0, 1, 2$ and can be seen for $l > 2$ by considering how the polynomials F_l and G_l are obtained in the induction steps. Suppose that the lemma holds for all $k \leq l$, $1 < l$. If $l = 2m$ for some $m \in \mathbb{N}$. Then

$$\wp^{(2m)}(z) = G_{2m}\left(\wp(z)\right),$$

with $\deg G_{2m} = m + 1$. Differentiating both sides with respect to z gives

$$\wp^{(2m+1)}(z) = G'_{2m}\left(\wp(z)\right)\, \wp'(z),$$

where $G'_{2m}(X)$ is the derivative of the polynomial $G_{2m}(X)$. Therefore, $F_{2m+1}(X) = G'_{2m}(X)$, $\deg F_{2m+1}(X) = m$ and $G_{2m+1} = 0$. Similarly, suppose $l = 2m + 1$ for some $m \in \mathbb{N}$, then

$$\wp^{(2m+1)}(z) = F_{2m+1}\left(\wp(z)\right)\, \wp'(z),$$

where $\deg F_{2m+1} = m$. Differentiating this expression gives

$$\wp^{(2m+2)}(z) = F'_{2m+1}\left(\wp(z)\right) \left(\wp'(z)\right)^2 + F_{2m+1}\left(\wp(z)\right)\, \wp''(z).$$

Hence, using (4) and (7), we obtain

$$\wp^{(2m+2)}(z) = F'_{2m+1}\left(\wp(z)\right) \left(4\wp^3(z) - g_2\wp(z) - g_3\right) + F_{2m+1}\left(\wp(z)\right) \left(6\,\wp^2(z) - \frac{g_2}{2}\right).$$

Thus, $\wp^{(2m+2)}(z) = G_{2m+2}(\wp(z))$, for a polynomial $G_{2m+2}(X) \in \overline{\mathbb{Q}}[X]$. Suppose that

$$F_{2m+1}(X) = \sum_{k=0}^{m} f(k)\, X^k.$$

Then the leading term in $G_{2m+2}(X)$ is evidently

$$\left(m\, f(m) + 6\, f(m)\right) X^{m+2}.$$

Since $f(m) \neq 0$ and $m > 0$, the coefficient of X^{m+2} in $G_{2m+2}(X)$ is not zero. Thus, the degree of $G_{2(m+1)}(X) = m + 2$, as claimed in the lemma. $\square$

2.2. **The Weierstrass zeta-function.** Another related function that makes an appearance in our study is the Weierstrass zeta-function (*not* to be confused with the Riemann zeta-function) defined as

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right),$$

which converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$, and hence defines an analytic function there. It is clear from the definition that $\zeta'(z) = -\wp(z)$, for $z \in \mathbb{C}\backslash\Lambda$. Note that $\zeta(z)$ is *not* doubly periodic. In particular,

$$\zeta(z + \omega) = \zeta(z) + \eta(\omega), \tag{8}$$

for some constant $\eta(\omega)$ independent of z. If ω_1 and ω_2 are the fundamental periods of Λ, then the corresponding $\eta_1 := \eta(\omega_1)$ and $\eta_2 := \eta(w_2)$ are called *quasi-periods* of the Weierstrass ζ-function. One can observe that $\eta(\omega)$ is $\mathbb{Z}$-linear in ω and hence, other quasi-periods of ζ are $\mathbb{Z}$-linear combinations of η_1 and η_2.

Also, note that the Weierstrass ζ-function is an odd function. Thus, evaluating (8) at $z = -\omega_1/2$ and $\omega = \omega_1$ gives values

$$\eta_1 = 2\,\zeta\left(\frac{\omega_1}{2}\right), \quad \eta_2 = 2\,\zeta\left(\frac{\omega_2}{2}\right). \tag{9}$$

The Weierstrass ζ-function also satisfies an addition law, given by

$$\zeta(z_1+z_2)=\zeta(z_1)+\zeta(z_2)+\frac{1}{2}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right), \tag{10}$$

for $z_1,\, z_2 \in \mathbb{C}\backslash\Lambda$ such that $z_1 \pm z_2 \notin \Lambda$. As $z_1 \to z_2$, we obtain a duplication formula analogous to the one for the Weierstrass $\wp$-function. More generally, on repeated application of the addition law, we have for $m \in \mathbb{N}$,

$$\zeta(mz)=m\,\zeta(z)+\frac{1}{2}\sum_{j=2}^{m-1}\mathcal{F}_j(z)+\frac{1}{2}\,\frac{\wp''(z)}{\wp'(z)}, \tag{11}$$

where

$$\mathcal{F}_j(z)=\frac{\wp'(jz)-\wp'(z)}{\wp(jz)-\wp(z)},$$

provided that $jz \notin \Lambda$ for all $0 \le j \le m$.

With the background established so far, we prove the following lemmas that will be particularly useful in the proof of main theorems.

Lemma 2.2. *Let Λ be a lattice with algebraic invariants $g_2,\, g_3$ and fundamental periods $\omega_1,\, \omega_2$. Let $l,\, n \in \mathbb{Z}$ with $n > 1$ and $n \nmid l$. Then there exists an algebraic number $\epsilon_1(l,\, n)$ such that*

$$\zeta\left(\frac{l}{n}\,\omega_1\right)=\frac{2l}{n}\,\zeta\left(\frac{\omega_1}{2}\right)+\epsilon_1(l,\,n).$$

The analogous statement holds when ω_1 is replaced by ω_2. We note that the number $\epsilon_1(l,n)$ depends only on ω_1, Λ, n and $l \pmod n$.

Proof. Note that it suffices to prove the lemma when $1 \le l \le n-1$. Indeed, suppose the lemma holds for $1 \le l \le n-1$. Now assume that $l \ge n$. Then one can write

$$\frac{l}{n}=q+\frac{l'}{n},$$

with $q \in \mathbb{N}$ and $0 < l' \le n-1$. Thus,

$$\zeta\left(\frac{l}{n}\,\omega_1\right)=\zeta\left(q\,\omega_1+\frac{l'}{n}\,\omega_1\right)=\zeta\left(\frac{l'}{n}\,\omega_1\right)+\eta(q\,\omega_1)=\frac{2l'}{n}\,\zeta\left(\frac{\omega_1}{2}\right)+\epsilon_1(l',\,n)+2q\,\zeta\left(\frac{\omega_1}{2}\right),$$

which proves the lemma in this case. Moreover, if $l < 0$, then the lemma follows from the fact that $\zeta(z)$ is an odd function.

As a preliminary step for the case $1 \le l \le n-1$, take $m = n$ and $z = \omega_1/2$ in (11). Since ω_1 is a fundamental period, $(j/2n)\,\omega_1 \notin \Lambda$ for $0 < j \le n$. Therefore,

$$\zeta\left(\frac{\omega_1}{2}\right)=n\,\zeta\left(\frac{\omega_1}{2n}\right)+\frac{1}{2}\sum_{j=2}^{n-1}\mathcal{F}_j\left(\frac{\omega_1}{2n}\right)+\frac{1}{2}\,\frac{\wp''}{\wp'}\left(\frac{\omega_1}{2n}\right).$$

As seen earlier, since g_2 and g_3 are algebraic, the values $\wp(\omega_1/2n)$, $\wp'(\omega_1/2n)$ and $\wp''(\omega_1/2n)$ are all algebraic. Hence, we get

$$\zeta\left(\frac{\omega_1}{2n}\right)=\frac{1}{n}\,\zeta\left(\frac{\omega_1}{2}\right)+\delta_1(l,\,n), \tag{12}$$

for some algebraic number $\delta_1(l,\, n)$, which can be explicitly written down from the expressions above.

Now suppose that $1 \le l \le n-1$. Once again, we aim to utilize (11), but now with $m = 2l$ and $z = \omega_1/2n$ to write

$$\zeta\left(\frac{l}{n}\,\omega_1\right) = 2l\,\zeta\left(\frac{\omega_1}{2n}\right) + \frac{1}{2}\sum_{k=2}^{2l-1}\mathcal{F}_k\left(\frac{\omega_1}{2n}\right) + \frac{1}{2}\,\frac{\wp''}{\wp'}\left(\frac{\omega_1}{2n}\right),$$

which is defined because $k+1 \le (2l-1)+1 = 2l < 2n$ when $l < n$. The lemma is now immediate from (12), and the algebraicity of $\wp(z)$ and its derivatives at $\omega_1/2n$. □

Remark. *The above proof will not go through in the most general case of rational multiples of any non-zero period in* Λ*, since* (11) *will not be valid when one of the intermediate values* $jz \in \Lambda$ *for some* $0 < j \le m$*. In particular, if* $\omega = c\,\omega_1 + d\,\omega_2$ *and* n *are such that* $j\omega/2n \in \Lambda$*, then the above proof will fail. For example, consider* $\omega = 4\omega_1 + 4\omega_2$*,* $n = 6$ *and* $j = 3$*. Then* $j\omega/2n \in \Lambda$ *while* $\omega/n \notin \Lambda$ *and* $j < n$*.*

However, using the above lemma, we can prove an analogous statement for values of $\zeta(z)$ at rational multiples of $\omega \in 2^k\Lambda \setminus 2^{k+1}\Lambda$ with $k \in \mathbb{Z}_{\ge 0}$.

Lemma 2.3. *Let* Λ *be a lattice with algebraic invariants* g_2*,* g_3 *and fundamental periods* ω_1 *and* ω_2*. Let* $\omega = 2^k(c\,\omega_1 + d\,\omega_2) \in 2^k\Lambda \setminus 2^{k+1}\Lambda$ *with* $k \in \mathbb{Z}_{\ge 0}$*. Let* $l, n \in \mathbb{Z}$ *with* $n > 0$ *and* $(l/n)\omega \notin \Lambda$*. Then there exists an algebraic number* $\epsilon(l, n, \omega, \omega_1, \omega_2)$ *such that*

$$\zeta\left(\frac{l}{n}\,\omega\right) = \frac{2^{k+1}\,l}{n}\left(c\,\zeta\left(\frac{\omega_1}{2}\right) + d\,\zeta\left(\frac{\omega_2}{2}\right)\right) + \epsilon(l, n, \omega, \omega_1, \omega_2).$$

Proof. As earlier, one can assume that $0 < l < n$ and further assume that $\gcd(l,n) = 1$. By hypotheses on ω, we have $n \nmid 2^k\,c$ or $n \nmid 2^k\,d$ and c is odd.

We now consider two cases.

a) Suppose $2^k\,c = nq$, $n \nmid 2^k\,d$. Then $n \nmid 2^k\,l\,d$ and

$$\begin{aligned}\zeta\left(\frac{l}{n}\,\omega\right) &= \zeta\left(\frac{l\,n\,q}{n}\,\omega_1 + \frac{2^k\,l\,d}{n}\,\omega_2\right)\\ &= lq\,\eta_1 + \zeta\left(\frac{2^k\,l\,d}{n}\,\omega_2\right), \qquad \text{by quasi-periodicity}\\ &= \frac{2^{k+1}\,l\,c}{n}\,\zeta\left(\frac{\omega_1}{2}\right) + \frac{2^{k+1}\,l\,d}{n}\,\zeta\left(\frac{\omega_2}{2}\right) + \epsilon_2(2^k\,l\,d,\, n), \qquad \text{by Lemma 2.2}\\ &= \frac{2^{k+1}\,l}{n}\left[c\,\zeta\left(\frac{\omega_1}{2}\right) + d\,\zeta\left(\frac{\omega_2}{2}\right)\right] + \epsilon_2(2^k\,l\,d,\, n).\end{aligned}$$

The same proof as above goes through in the case when $n \nmid 2^k\,c$ and $n \mid 2^k\,d$.

b) The only case remaining to consider is when $n \nmid 2^k\,c$ and $n \nmid 2^k\,d$. Here

$$\begin{aligned}\zeta\left(\frac{l}{n}\,\omega\right) &= \zeta\left(\frac{l\,2^k\,c}{n}\,\omega_1 + \frac{l\,2^k\,d}{n}\,\omega_2\right)\\ &= \frac{2^{k+1}\,l\,c}{n}\,\zeta\left(\frac{\omega_1}{2}\right) + \frac{2^{k+1}\,l\,d}{n}\,\zeta\left(\frac{\omega_2}{2}\right) + \epsilon_1(2^k\,l\,c,\, n) + \epsilon_2(2^k\,l\,d,\, n), \qquad \text{by Lemma 2.2}\\ &= \frac{2^{k+1}\,l}{n}\left[c\,\zeta\left(\frac{\omega_1}{2}\right) + d\,\zeta\left(\frac{\omega_2}{2}\right)\right] + \epsilon_1(2^k\,l\,c,\, n) + \epsilon_2(2^k\,l\,d,\, n).\end{aligned}$$

This proves the claim.

□

2.3. **Transcendence of values of elliptic functions.** An excellent reference for the compilation of results regarding algebraic independence of values of elliptic functions is the book "Number Theory IV", by N. I. Fel'dman and Yu. V. Nesterenko [3]. Proofs for some of the results can also be found in [5]. Throughout this subsection, we consider the lattice Λ, with fundamental periods ω_1 and ω_2 such that $\mathrm{Im}(\omega_2/\omega_1) > 0$. Furthermore, we also suppose that *the associated invariants* g_2 *and* g_3 *are algebraic.*

Recall that the ring of endomorphisms (as a $\mathbb{Z}$-module) of the lattice Λ, say $E(\Lambda)$, consists of complex numbers λ such that $\lambda\Lambda \subseteq \Lambda$. It can be shown that $E(\Lambda)$ is either $\mathbb{Z}$ or an order in an imaginary quadratic field. The second case occurs if and only if $\tau := \omega_2/\omega_1$ is a quadratic imaginary irrational number. In this case, the lattice Λ (or the associated Weierstrass $\wp$-function) is said to have *complex multiplication* and the field $K = \mathbb{Q}(\tau)$ will be called the *field of complex multiplication.*

In the 1930s, Theodor Schneider proved several important results regarding the values of $\wp(z)$ and $\zeta(z)$ as well as their periods and quasi-periods. We state a few relevant theorems below (as appeared in [3]).

Theorem 2.4 (Schneider)**.** *Suppose that* $\wp(z)$ *has algebraic invariants* g_2 *and* g_3*. Then for any* $\alpha \in \mathbb{C}\backslash\Lambda$ *algebraic,* $\wp(\alpha)$ *is transcendental.*

Theorem 2.5 (Schneider)**.** *Suppose that the invariants of* $\wp(z)$ *and* $\zeta(z)$*, namely,* $g_2,\, g_3 \in \overline{\mathbb{Q}}$ *and let* $\phi(z) := az + b\,\zeta(z)$ *for* $a,\, b \in \overline{\mathbb{Q}}$ *with* $|a| + |b| > 0$*. If* β *is an algebraic number with* $\beta \notin \Lambda$*, then at least one of the numbers* $\phi(\beta)$ *and* $\wp(\beta)$ *is transcendental.*

The above two theorems together with the Schneider-Lang theorem imply that

Theorem 2.6 (Schneider)**.** *For a lattice* Λ *with algebraic invariants* g_2 *and* g_3*, any non-zero period or quasi-period of* Λ *is transcendental.*

However, more can be said in case Λ has complex multiplication. A crucial lemma of D. Masser [4, Lemma 3.1] in this context is the following.

Lemma 2.7 (Masser)**.** *Let* $\wp(z)$ *be a Weierstrass* $\wp$*-function with algebraic invariants* g_2*,* g_3 *and complex multiplication. Let* ω_1*,* ω_2 *and* η_1*,* η_2 *be certain periods and quasi-periods respectively. Then* ω_2 *and* η_2 *are algebraic over the field* $\mathbb{Q}(\omega_1, \eta_1)$*.*

This result was independently proved by D. Brownawell and K. Kubota in [1, Theorem 8]. As a consequence of Lemma 2.7 and an important theorem of Yu. V. Nesterenko [9], the following result can be obtained when Λ has complex multiplication (see [5, Chapter 17] for details).

Theorem 2.8. *Let* $\wp(z)$ *be a Weierstrass* $\wp$*-function for a lattice with algebraic invariants* g_2*,* g_3 *and complex multiplication by an order of the imaginary quadratic field* K*. Let* ω *be a non-zero period and* η *the corresponding quasi-period. Then for any* $\tau \in K$ *with* $Im(\tau) \neq 0$*, each of the sets*

$$\{\pi,\, \omega,\, e^{2\pi i\tau}\} \text{ and } \{\omega,\, \eta,\, e^{2\pi i\tau}\}$$

is algebraically independent over $\mathbb{Q}$*.*

2.4. **Partial fraction expression of rational functions.** In this section, we prove an intriguing lemma regarding the coefficients of partial fraction expression of rational functions. This lemma will play a vital role in the proof of Theorem 1.1.

Lemma 2.9. *Let $A(X)$, $B(X) \in \mathbb{C}[X]$ be co-prime polynomials with $\deg A \leq \deg B - 3$. Let the distinct roots of $B(X)$ be $\alpha_1, \cdots, \alpha_r$ with multiplicities $\mu_1, \cdots, \mu_r$. Suppose that the partial fraction decomposition of $A(X)/B(X)$ is*

$$\frac{A(X)}{B(X)} = \sum_{i=1}^{r} \sum_{j=1}^{\mu_r} \frac{\lambda_{i,j}}{(X-\alpha_i)^j}.$$

Let $\mathcal{M} := \max_{1\leq i\leq r} \mu_i$ and define $\lambda_{i,j} = 0$ for $\mu_i < j \leq \mathcal{M}$, $1 \leq i \leq r$. Then

$$\text{(a)} \sum_{i=1}^{r} \lambda_{i,1} = 0, \text{ and (b)} \sum_{i=1}^{r} (\lambda_{i,2} + \lambda_{i,1}\,\alpha_i) = 0.$$

Proof. From the partial fraction decomposition, we obtain

$$A(X) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_r} \lambda_{i,j} \frac{B(X)}{(X-\alpha_i)^j}. \tag{13}$$

Since $j \geq 1$, $\deg(B(X)/(X-\alpha_j)^j) \leq \deg B(X) - 1$. However, $\deg A(X) \leq \deg B(X) - 3$. Hence, the coefficients of $X^{\deg B - 1}$ and $X^{\deg B-2}$ must be zero on the right hand side. Since $X^{\deg B-1}$ is the highest degree term in (13), the contribution to this term is solely from the polynomials $B(X)/(X-\alpha_i)$ for $1 \leq i \leq r$. This implies part (a) of the lemma.

Similarly, towards part (b), the coefficient of $X^{\deg B-2}$ is comprised of the coefficient of the highest degree term in the polynomials $B(X)/(X-\alpha_i)^2$ for $1 \leq i \leq r$ as well as the coefficient of the second highest degree term in the polynomials $B(X)/(X-\alpha_i)$. Since the coefficient of the second highest degree term in a polynomial is given by the negative of the sum of its roots, the coefficient of $X^{\deg B-2}$ in $B(X)/(X-\alpha_{i_0})$ equals

$$-\Big[\sum_{i=i}^{r} (\alpha_i \mu_i) - \alpha_{i_0}\Big].$$

Let $S := \sum_{i=1}^{r} \alpha_i \mu_i$. Therefore, the coefficient of $X^{\deg B-2}$ on the right hand side of (13) becomes

$$\sum_{i=1}^{r} \lambda_{i,2} + \sum_{i=1}^{r} \lambda_{i,1}\big(\alpha_i - S\big) = \sum_{i=1}^{r} \lambda_{i,2} + \sum_{i=1}^{r} \lambda_i \alpha_i - S \sum_{i=1}^{r} \lambda_{i,1}.$$

By part (a) of this lemma, the last term in the above expression vanishes and part(b) is proved. □

3. **Proofs of main theorems**

3.1. **Proof of Theorem 1.1.** We first prove an expression for the sums $\mathcal{S}(A, B)$ in terms of Weierstrass functions, i.e., Theorem 1.1.

Proof. Suppose $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$. Let $\sum'$ mean that the term $(m, n) = (0, 0)$ is omitted from the summation below.

$$\sum_{\substack{m,\, n \in \mathbb{Z}, \\ |m| \le M, \\ |n| \le N}}' \frac{A(m\,\omega_1 + n\,\omega_2)}{B(m\,\omega_1 + n\,\omega_2)} = \sum_{i=1}^{r} \sum_{j=1}^{\mathcal{M}} \sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{\lambda_{i,j}}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^j}.$$

The only terms for which convergence needs to be checked are $j = 1, 2$ because

$$\sum_{\omega \in \Lambda^*} 1/|\omega|^{2+\epsilon} < \infty,$$

for any $\epsilon > 0$. For $j = 1$, we add the necessary factors for convergence and subtract the extra terms. This gives

$$\sum_{i=1}^{r} \sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{\lambda_{i,1}}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)}$$

$$= \left(\sum_{i=1}^{r} \lambda_{i,1} \right) \sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{1}{(m\,\omega_1 + n\,\omega_2)} + \left(\sum_{i=1}^{r} \lambda_{i,1} \alpha_i \right) \sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{1}{(m\,\omega_1 + n\,\omega_2)^2}$$

$$- \sum_{i=1}^{r} \lambda_{i,1} \sum_{\substack{|m| \le M, \\ |n| \le N}}' \left\{ \frac{1}{(\alpha_i - (m\,\omega_1 + n\,\omega_2))} + \frac{1}{(m\,\omega_1 + n\,\omega_2)} + \frac{\alpha_i}{(m\,\omega_1 + n\,\omega_2)^2} \right\}.$$

The above expression can be further simplified using Lemma 2.9. The coefficient of the term

$$\sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{1}{(m\,\omega_1 + n\,\omega_2)}$$

vanishes and one can replace $\sum_{i=1}^{r} \lambda_{i,1} \alpha_i = -\sum_{i=1}^{r} \lambda_{i,2}$. Similar computations for $j = 2$ give

$$\sum_{i=1}^{r} \sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{\lambda_{i,2}}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^2}$$

$$= \sum_{i=1}^{r} \lambda_{i,2} \sum_{\substack{|m| \le M, \\ |n| \le N}}' \left\{ \frac{1}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^2} - \frac{1}{(m\,\omega_1 + n\,\omega_2)^2} \right\} + \left(\sum_{i=1}^{r} \lambda_{i,2} \right) \sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{1}{(m\,\omega_1 + n\,\omega_2)^2}.$$

The coefficients corresponding to the sum

$$\sum_{\substack{|m| \le M, \\ |n| \le N}}' \frac{1}{(m\,\omega_1 + n\,\omega_2)^2}$$

in the original terms $j = 1$ and $j = 2$ cancel. Thus, we obtain

$$\sum_{i=1}^{r}\sum_{j=1}^{\mathcal{M}}\sum_{\substack{|m|\le M,\\ |n|\le N}}{}' \frac{\lambda_{i,j}}{((m\,\omega_1+n\,\omega_2)-\alpha_i)^j}$$

$$= -\sum_{i=1}^{r}\lambda_{i,1}\sum_{\substack{|m|\le M,\\ |n|\le N}}{}' \left\{\frac{1}{(\alpha_i-(m\,\omega_1+n\,\omega_2))}+\frac{1}{(m\,\omega_1+n\,\omega_2)}+\frac{\alpha_i}{(m\,\omega_1+n\,\omega_2)^2}\right\}$$

$$+\sum_{i=1}^{r}\lambda_{i,2}\sum_{\substack{|m|\le M,\\ |n|\le N}}{}' \left\{\frac{1}{((m\,\omega_1+n\,\omega_2)-\alpha_i)^2}-\frac{1}{(m\,\omega_1+n\,\omega_2)^2}\right\}$$

$$+\sum_{i=1}^{r}\sum_{j=3}^{\mathcal{M}}\lambda_{i,j}\sum_{\substack{|m|\le M,\\ |n|\le N}}{}' \frac{1}{((m\,\omega_1+n\,\omega_2)-\alpha_i)^j}.$$

Hence, the theorem is proved by taking the limits as $M,\, N \to \infty$ and the term corresponding to $m = n = 0$ is added to both sides. □

Remark. *This theorem shows that $\mathcal{S}(A, B)$ can be expressed as a linear combination of the Weierstrass ζ-function and its derivatives. This is analogous to the evaluation of sums of the form $\sum_{n\in\mathbb{Z}} A(n)/B(n)$ in terms of the cotangent function and its derivatives. In this sense, the Weierstrass zeta-function $\zeta(z)$ can be thought of as an elliptic analogue of the cotangent function, $\pi\cot(\pi z)$.*

3.2. **Roots of $B(X)$: Rational multiples of periods.** Since we know the transcendence of quasi-periods, whereas the Weierstrass $\wp$-functions take algebraic values at non-integral rational multiples of periods, we can deduce the transcendence of the sum $\mathcal{S}(A, B)$ (Theorem 1.2).

Proof of Theorem 1.2. Recall that by Theorem 1.1, we have

$$\mathcal{S}(A,B) = -\sum_{i=1}^{r}\lambda_{i,1}\,\zeta(\alpha_i)+\sum_{i=1}^{r}\sum_{j=2}^{\mathcal{M}}\frac{\lambda_{i,j}}{(j-1)!}\,\wp^{(j-2)}(\alpha_i).$$

If $\alpha_i = (a_i/b_i)\,\omega$ as in the hypothesis of Theorem 1.2, then as seen in Section 2,

$$\wp^{(j-2)}\left(\frac{a_i}{b_i}\,\omega\right)\in\overline{\mathbb{Q}},\quad \text{for all } 1\le i\le r,\ 2\le j\le \mathcal{M}.$$

Moreover, Lemma 2.3 implies that if $\omega = 2^k\,(c\,\omega_1 + d\,\omega_2) \in 2^k\Lambda \setminus 2^{k+1}\Lambda$,

$$\zeta\left(\frac{a_i}{b_i}\,\omega\right) = 2^k\,\frac{a_i}{b_i}\left[2c\,\zeta\left(\frac{\omega_1}{2}\right)+2d\,\zeta\left(\frac{\omega_2}{2}\right)\right]+\epsilon(a_i,\,b_i,\,\omega,\,\omega_1,\,\omega_2),$$

for some algebraic number $\epsilon(a_i, b_i, \omega, \omega_1, \omega_2)$. Using (9), this can be expressed in terms of the fundamental quasi-periods as

$$\zeta\left(\frac{a_i}{b_i}\,\omega\right) = 2^k\,\frac{a_i}{b_i}\left[c\,\eta_1+d\,\eta_2\right]+\epsilon(a_i,\,b_i,\,\omega,\,\omega_1,\,\omega_2).$$

Therefore, we get that

$$\mathcal{S}(A,B) = -2^k \left(\sum_{i=1}^{r} \lambda_{i,1} \frac{a_i}{b_i} \right) \Big[c\,\eta_1 + d\,\eta_2 \Big] + \sum_{i=1}^{r} \lambda_{i,1}\, \epsilon(a_i,\, b_i,\, \omega,\, \omega_1,\, \omega_2)$$
$$+ \sum_{i=1}^{r} \sum_{j=2}^{\mathcal{M}} \frac{\lambda_{i,j}}{(j-1)!}\, \wp^{(j-2)}(\alpha_i). \tag{14}$$

Now suppose that $B(X)$ has simple roots. Then part (b) of Lemma 2.9 implies that

$$\sum_{i=1}^{r} \lambda_{i,1} \frac{a_i}{b_i} = 0,$$

and there are no terms involving the Weierstrass $\wp$-function and its derivatives. Therefore, (14) reduces to

$$\mathcal{S}(A,B) = \sum_{i=1}^{r} \lambda_{i,1}\, \epsilon(a_i,\, b_i, \omega\,, \omega_1,\, \omega_2). \tag{15}$$

We know that for $i \le i_0 \le r$,

$$\lambda_{i_0,1} = \frac{A(\alpha_{i_0})}{B'(\alpha_{i_0})} = \frac{A(\alpha_{i_0})}{\prod\limits_{\substack{1 \le i \le r,\\ i \ne i_0}} (\alpha_{i_0} - \alpha_i)}.$$

Since $\deg(A) \le \deg(B) - 3$ and $\alpha_i = (a_i/b_i)\,\omega$,

$$\lambda_{i_0,1}\, \epsilon(a_{i_0},\, b_{i_0},\, \omega,\, \omega_1,\, \omega_2) = \frac{F_{i_0}(\omega)}{\omega^{r-1}},$$

where $F_{i_0}(X) \in \overline{\mathbb{Q}}[X]$ of degree less than or equal to $\deg B - 3 = r - 3$. Thus, we have expressed $\mathcal{S}(A,B)$ as a rational function with algebraic coefficients evaluated at ω. If $\mathcal{S}(A,B)$ is a non-zero algebraic number, say β, then clearing the denominators in

$$0 \ne \beta = \frac{\left(\sum_{i=1}^{r} F_i(\omega)\right)}{\omega^{r-1}}$$

gives a polynomial relation, $\beta\,\omega^{r-1} - \sum_{i=1}^{r} F_i(\omega) = 0$ of ω over $\overline{\mathbb{Q}}$. This polynomial is non-trivial as $\beta \ne 0$ and $\deg(F_i) \le r - 3$. This implies that ω is algebraic and contradicts Schneider's Theorem 2.6. Thus, $\mathcal{S}(A,B)$ is either zero or transcendental. This proves part a).

Now assume that $B(X)$ has multiple roots. By the hypothesis in part (b), the coefficient of $\eta := 2^k(c\,\eta_1 + d\,\eta_2)$ (which is the quasi-period corresponding to ω) in (14) is not zero. If $\mathcal{S}(A,B)$ is an algebraic number, then the right hand side of (14) is a non-trivial polynomial in η over $\overline{\mathbb{Q}}(\omega)$. However, Theorem 2.8 proves the algebraic independence of ω and η, provided that Λ has complex multiplication. Therefore, $\mathcal{S}(A,B)$ cannot be a non-zero algebraic number, and so is transcendental. □

Remark. *It is possible that the sum $\mathcal{S}(A,B) = 0$ in certain cases. For example, let $A(X) = 1$ and*

$$B(X) = \left(X - \frac{m_1}{2}\,\omega_1\right)\left(X - \frac{m_2}{2}\,\omega_1\right)\left(X - \frac{m_3}{2}\,\omega_1\right),$$

where m_1, m_2 and m_3 are distinct odd positive integers. Then the linearity of the η-function implies that

$$\zeta\left(\frac{m_j}{2}\,\omega_1\right) = m_j\,\zeta\left(\frac{\omega_1}{2}\right), \;\; j = 1,\, 2,\, 3.$$

Therefore, $\epsilon(m_j, 2, \omega_1, \omega_2) = 0$ *for all* $j = 1$, 2 *and* 3. *Thus, equation* (15) *implies that* $\mathcal{S}(A, B) = 0$. *It is interesting to ask if these are the only cases when* $\mathcal{S}(A, B) = 0$. *We relegate this to future research.*

Remark. *If* $B(X)$ *is monic, has multiple roots and the coefficient of* η *is zero, then* (14) *gives*

$$\mathcal{S}(A, B) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \mathcal{A}_{i,j}\, \lambda_{i,j},$$

for some algebraic numbers $\mathcal{A}_{i,j}$. *Since*

$$\lambda_{i,j} = \frac{1}{(\mu_i - j)!} \left[\frac{d^{(\mu_i - j)}}{dX^{(\mu_i - j)}} \left\{ (X - \alpha_i)^{\mu_i} \frac{A(X)}{B(X)} \right\} \right] \Bigg|_{X = \alpha_i},$$

it follows that $\lambda_{i,j}$ *are rational functions in* ω. *If we show that* $\sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \mathcal{A}_{i,j}\, \lambda_{i,j}$ *is a non-trivial rational function, then it follows that* $\mathcal{S}(A, B)$ *is either zero or transcendental. However, this is more subtle and does not follow from mere degree considerations.*

3.3. **Roots of $B(X)$: Algebraic numbers.** Inspired by the conjectures and theorems regarding the transcendental nature of values of the Weierstrass functions at algebraic arguments (Section 2), we focus on the nature of $\mathcal{S}(A, B)$ when $B(X)$ is monic and has algebraic roots in Theorem 1.4.

Proof of Theorem 1.4. We recall that by Theorem 1.1,

$$\mathcal{S}(A, B) = -\sum_{i=1}^{r} \lambda_{i,1}\, \zeta(\alpha_i) + \sum_{i=1}^{r} \sum_{j=2}^{\mathcal{M}} \frac{\lambda_{i,j}}{(j-1)!}\, \wp^{(j-1)}(\alpha_i).$$

Since the roots of $A(X)$, $B(X) \in \overline{\mathbb{Q}}[X]$ are algebraic, the partial fraction coefficients $\lambda_{i,j}$ lie in $\overline{\mathbb{Q}}$. Therefore, $\mathcal{S}(A, B)$ is an algebraic linear combination of $\zeta(\alpha_i)$, $\wp(\alpha_i)$ and special values of the derivatives of $\wp(z)$ at α_i. As seen in Lemma 2.1, the derivatives of $\wp(z)$ can be expressed as polynomials in $\wp$ and $\wp'$ with algebraic coefficients. Thus, $\mathcal{S}(A, B)$ is, in fact, a polynomial with algebraic coefficients evaluated at $\zeta(\alpha_i)$, $\wp(\alpha_i)$ and $\wp'(\alpha_i)$ for $1 \le i \le r$. Suppose $\lambda_{i_0,1} \neq 0$ for some i_0, $1 \le i_0 \le r$, then $\mathcal{S}(A, B)$ is a non-trivial polynomial in $\zeta(\alpha_{i_0})$ with coefficients in

$$\overline{\mathbb{Q}\Big(\{\wp(\alpha_i) \,:\, 1 \le i \le r\} \Big) \cup \overline{\mathbb{Q}}\Big(\{\zeta(\alpha_i) \,:\, 1 \le i \le r,\, i \neq i_0\} \Big)}.$$

If $\mathcal{S}(A, B)$ is algebraic, then one would obtain that $\zeta(\alpha_{i_0})$ is algebraic over the above field, implying that the transcendence degree of

$$\mathbb{Q}\Big(\omega_1, \omega_2, \eta_1, \eta_2, \wp(\alpha_1), \cdots, \wp(\alpha_r), \zeta(\alpha_1), \cdots, \zeta(\alpha_r) \Big)$$

is less than $2r + 4$ if Λ does not have CM, and the transcendence degree of

$$\mathbb{Q}\Big(\omega_1, \eta_1, \wp(\alpha_1), \cdots, \wp(\alpha_r), \zeta(\alpha_1), \cdots, \zeta(\alpha_r) \Big)$$

is less than $2r+2$ if Λ has CM, by Lemma 2.7. This contradicts Conjecture 2, thus proving part (a).

Now suppose that $\lambda_{i,1} = 0$ for all $1 \le i \le r$. Thus,

$$\mathcal{S}(A, B) = \sum_{i=1}^{r} \sum_{j=2}^{\mathcal{M}} \widetilde{\lambda}_{i,j}\, \wp^{(j-2)}(\alpha_i),$$

where $\widetilde{\lambda}_{i,j} := \lambda_{i,j}/(j-1)! \in \overline{\mathbb{Q}}$. Let

$$\mathcal{M}_e := \begin{cases} \dfrac{\mathcal{M}-1}{2}, & \text{if } \mathcal{M} \text{ is odd,} \\ \dfrac{\mathcal{M}}{2}, & \text{if } \mathcal{M} \text{ is even,} \end{cases}$$

and

$$\mathcal{M}_o := \begin{cases} \dfrac{\mathcal{M}-1}{2}, & \text{if } \mathcal{M} \text{ is odd,} \\ \dfrac{\mathcal{M}-2}{2}, & \text{if } \mathcal{M} \text{ is even.} \end{cases}$$

Thus, separating the terms corresponding to j-even and j-odd in the expression for $\mathcal{S}(A,B)$ gives

$$\mathcal{S}(A,B) = \sum_{i=1}^{r}\sum_{u=1}^{\mathcal{M}_o} \widetilde{\lambda}_{i,2u+1}\, \wp^{(2u-1)}(\alpha_i) + \sum_{i=1}^{r}\sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, \wp^{(2v-2)}(\alpha_i).$$

Now using Lemma 2.1, we express the derivatives of the Weierstrass $\wp$-function as polynomials in $\wp$ and $\wp'$ to get

$$\begin{aligned} \mathcal{S}(A,B) &= \sum_{i=1}^{r} \left\{ \left[\sum_{u=1}^{\mathcal{M}_o} \widetilde{\lambda}_{i,2u+1}\, F_{2u-1}\Big(\wp(\alpha_i)\Big) \right] \wp'(\alpha_i) \right\} + \sum_{i=1}^{r}\sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, G_{2v-2}\Big(\wp(\alpha_i)\Big) \\ &= \sum_{i=1}^{r} \Big(\mathcal{C}_i\, \wp'(\alpha_i) \Big) + \mathcal{D}, \end{aligned}$$

where $\mathcal{C}_i$, $\mathcal{D} \in \overline{\mathbb{Q}}(\wp(\alpha_1), \cdots, \wp(\alpha_r))$ are defined as

$$\mathcal{C}_i = \sum_{u=1}^{\mathcal{M}_o} \widetilde{\lambda}_{i,2u+1}\, F_{2u-1}\Big(\wp(\alpha_i)\Big), \;\; \mathcal{D} = \sum_{i=1}^{r}\sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, G_{2v-2}\Big(\wp(\alpha_i)\Big).$$

Suppose that $\mathcal{S}(A,B)$ is algebraic and that $\mathcal{C}_{i_0} \neq 0$ for some i_0 with $1 \leq i_0 \leq r$. This implies that $\wp'(\alpha_i)$ satisfies a linear equation over the field

$$\overline{\mathbb{Q}}\Big(\{\wp(\alpha_i) \,:\, 1 \leq i \leq r\} \Big) \cup \overline{\mathbb{Q}}\Big(\{\wp'(\alpha_i) \,:\, 1 \leq i \leq r,\, i \neq i_0\} \Big).$$

By Conjecture 1, if Λ does not have CM or by Theorem 1.3 if Λ has CM, we know that the numbers $\wp(\alpha_i)$ are algebraically independent for $1 \leq i \leq r$. Thus, the above conclusion implies that $\wp'(\alpha_{i_0})$ satisfies a non-trivial linear relation over the field $\overline{\mathbb{Q}}(\wp(\alpha_{i_0}))$. However, we know by (4) and Theorem 2.4 that $\wp'(\alpha_{i_0})$ is quadratic over $\overline{\mathbb{Q}}(\wp(\alpha_{i_0}))$. Hence, the theorem is proved in this case.

Now assume that $\mathcal{C}_i = 0$ for all $1 \leq i \leq r$. We will show that this can only happen if the polynomial $A(X)$ is the identically zero polynomial to begin with. Let

$$G_{2l}(X) := \sum_{m=0}^{l+1} g_l(m)\, X^m,$$

where $G_{2l}(X)$ is the polynomial that we encountered in Lemma 2.1. Therefore, we have

$$\begin{aligned}\mathcal{S}(A,B) &= \sum_{i=1}^{r}\sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, G_{2v-2}\Big(\wp(\alpha_i)\Big)\\ &= \sum_{i=1}^{r}\sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v} \sum_{m=0}^{v} g_{v-1}(m)\left(\wp(\alpha_i)\right)^m\\ &= \sum_{i=1}^{r}\sum_{m=0}^{\mathcal{M}_e} \delta_{m,i}\left(\wp(\alpha_i)\right)^m,\end{aligned}$$

where $\delta_{m,i} = \sum_{v=m}^{\mathcal{M}_e} g_{v-1}(m)\,\widetilde{\lambda}_{i,2v}$ and the last step is obtained by interchanging the order of summation. Therefore, if

$$P_i(X) := \sum_{m=0}^{\mathcal{M}_e} \delta_{m,i}\, X^m \;\in \overline{\mathbb{Q}}[X],$$

then we get

$$\mathcal{S}(A,B) = \sum_{i=1}^{r} P_i\Big(\wp(\alpha_i)\Big).$$

If $P_i(X)$ is a non-constant polynomial for even a single i, $1 \le i \le r$, then by Conjecture 1 in the non-CM case and by Theorem 1.3 in the CM case, we deduce that $\mathcal{S}(A,B)$ must be transcendental.

Thus, we can assume that $\deg P_i(X) = 0$ for all $1 \le i \le r$. Therefore, the coefficients $\delta_{m,i} = 0$ for $1 \le i \le r$, $1 \le m \le \mathcal{M}_e$, that is,

$$\sum_{v=0}^{\mathcal{M}_e} g_{v-1}(m)\,\widetilde{\lambda}_{i,2v} = 0, \tag{16}$$

since Lemma 2.1 proves that $g_{v-1}(m) = 0$ for $m > v$. The above relation can be interpreted as a matrix equation as follows. Let $\mathcal{G}$ be the $\mathcal{M}_e \times \mathcal{M}_e$ matrix whose (m,v)-th entry is $g_{v-1}(m)$ for $1 \le v$, $m \le \mathcal{M}_e$. Let $\mathcal{N}$ be the $\mathcal{M}_e \times r$ matrix whose (v,i)-th entry is $\widetilde{\lambda}_{i,2v}$, for $1 \le i \le r$, $1 \le v \le \mathcal{M}_e$. Hence, equation (16) is equivalent to the matrix identity

$$\mathcal{G}\,\mathcal{N} = \begin{bmatrix} 0\\ \vdots\\ 0\end{bmatrix},$$

where the right hand side is the 0-matrix. Observe that $\mathcal{G}(m,v) = 0$ for all $m > v$, i.e., the matrix $\mathcal{G}$ is an upper triangular matrix. Thus,

$$\det(\mathcal{G}) = \prod_{m=1}^{\mathcal{M}_e} g_{m-1}(m).$$

By Lemma 2.1, $\deg G_{2m-2}(X) = m$, which implies that $g_{m-1}(m) \neq 0$ for all $m \in \mathbb{N}$. Hence, $\mathcal{G}$ is an invertible matrix and $\mathcal{N}$ is the zero matrix. This concludes the proof of parts (b) and (c). □

Owing to the involved nature of the addition formula of $\zeta(z)$ and $\wp(z)$, it is not a priori clear if $\mathcal{S}(A,B)$ is transcendental or even non-zero, in the case when $\alpha_1, \cdots, \alpha_r$ are not $\mathbb{Q}$-linearly independent. We relegate this to future study.

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