CONVOLUTION OF VALUES OF THE LERCH ZETA-FUNCTION

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Abstract. We investigate generalizations arising from the identity

$$\zeta_2(n-1,1) = \frac{n-1}{2}\zeta(n) - \frac{1}{2}\sum_{j=2}^{n-2}\zeta(j)\,\zeta(n-j),$$

where $\zeta_2(k,1)$ denotes a double zeta value at $(k,1)$, or an Euler-Zagier sum. In particular, we prove analogues of the above identity for Lerch zeta-functions and Dirichlet L-functions. Such an attempt has met with limited success in the past. We highlight that this study naturally leads one into the realm of *multiple* L-values and multiple L^*-values.

1. Introduction

In the early 18th century, Euler extensively studied infinite series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^k},$$

for any positive integer $k>1$. After Riemann's introduction of the zeta-function,

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \Re(s)>1,$$

we recognize the series studied by Euler as special values of $\zeta(s)$ at positive integers. In particular, Euler's resolution of the Basel problem leads to

$$\zeta(2k)\in\pi^{2k}\,\mathbb{Q}^*,$$

for any positive integer $k\geq 1$. Thus, the values $\zeta(2k)$ are all transcendental, thanks to Lindemann's theorem that π is transcendental. However, the arithmetic nature of $\zeta(2k+1)$ for an integer $k\geq 1$ remains shrouded in mystery.

Recently, significant progress was made in this direction when Apéry [2] proved that $\zeta(3)$ is irrational, T. Rivoal [20] proved that infinitely many of $\zeta(2k+1)$ are irrational and W. Zudilin [24] showed that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational. The transcendence of the values $\zeta(2k+1)$ is not known, although they are expected to be so. Moreover, it is widely believed that

$$\pi,\ \zeta(3),\ \zeta(5),\ \zeta(7),\ \cdots$$

are algebraically independent.

In an attempt to understand the nature of the special values of the Riemann zeta-function, it seems fruitful to adopt a larger perspective. The values then seem intimately connected with special values

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of the multi-zeta functions. A multi-zeta value (MZV) of depth r and weight w is defined as the nested sum,

$$\zeta_r(k_1, k_2, \cdots, k_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{{n_1}^{k_1}\, {n_2}^{k_2}\, \cdots\, {n_r}^{k_r}},$$

where k_i are positive integers, $k_1 \geq 2$ and $k_1 + k_2 + \cdots + k_r = w$. These values not only appear in several areas of mathematics but also in quantum physics. MZVs have been the focus of intense research in recent times. They satisfy a wide variety of relations. Recently, F. Brown [7] proved a remarkable theorem which states that all multiple zeta-values of weight n are $\mathbb{Q}$-linear combinations of

$$\Big\{\zeta(a_1, \cdots, a_r) \,:\, a_i \in \{2,\, 3\} \text{ for } 1 \leq i \leq r,\ a_1 + \cdots + a_r = n\Big\}.$$

The MZVs are also intricately related to the values of the Riemann zeta-function itself. Perhaps the most striking example of such a relation is that

$$\zeta_2(2,1) = \zeta(3),$$

or more generally,

$$\zeta_2(n-1, 1) = \frac{n-1}{2}\zeta(n) - \frac{1}{2}\sum_{j=2}^{n-2} \zeta(j)\, \zeta(n-j), \tag{1}$$

for a positive integer $n \geq 3$, which was certainly known to Euler. Thus, it is expected that the study of MZVs will shed light upon the arithmetic nature of $\zeta(2k+1)$.

These convolution sum identities suggest that there must exist similar identities for other L-functions such as Dirichlet L-functions. Yet, to our knowledge, no one has derived such analogues. The closest we come to such an attempt revolves around a celebrated theorem of Ramanujan: let α, $\beta > 0$ with $\alpha\beta = \pi^2$, and let k be any non-zero integer. Then

$$\begin{aligned}\alpha^{-k}\left\{\frac{1}{2}\,\zeta(2k+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}(e^{2\alpha n} - 1)}\right\} =& (-\beta)^{-k}\left\{\frac{1}{2}\,\zeta(2k+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}\,(e^{2\beta n} - 1)}\right\} \\ &- 2^{2k}\sum_{j=0}^{k+1} (-1)^j\, \frac{B_{2j}}{(2j)!}\, \frac{B_{2k+2-2j}}{(2k+2-2j)!}\, \alpha^{k+1-j}\, \beta^j,\end{aligned}$$

where B_n denotes the n-th Bernoulli number (see [3]). The last term on the right hand side of the above identity can be viewed as a convolution sum of zeta values since

$$\zeta(2k) = \frac{(2\pi i)^{2k}\, B_{2k}}{2\,(2k)!}.$$

Attempts to generalize this identity to Dirichlet L-functions have met with limited success. For example, S. Chowla [8] derived an analog of this identity if $\zeta(s)$ is replaced by $L(s, \chi_4)$ where χ_4 is the non-trivial Dirichlet character modulo 4 (see [3, pg. 277]). It is the purpose of this note to initiate a systematic study of such convolution identities. As Dirichlet L-functions are linear combinations of Hurwitz zeta-functions, it seems appropriate to derive convolution sum identities for them, and more generally for the Lerch zeta-functions.

The Hurwitz zeta-function was isolated for independent study by A. Hurwitz [13] in 1882. For $0 < x \leq 1$, the Hurwitz zeta-function is defined as the series

$$\zeta(s; x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \Re(s) > 1.$$

In 1887, Lerch studied an exponential twist of the Hurwitz zeta-function. For $|z| \leq 1$, $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$, the Lerch zeta-function is defined as

$$\Phi(z;\alpha;s) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s},$$

which converges for $\Re(s) > 1$ if $z = 1$ and $\Re(s) > 0$ otherwise. The Riemann and Hurwitz zeta-functions are special cases of the Lerch zeta-function.

Moreover, this function generalizes another special function that makes an appearance in the theory of special values of zeta-functions, namely, the polylogarithm. For $|z| \leq 1$, the s-th polylogarithm is defined as

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\,\Phi(z,1,s).$$

This series converges for $s > 1$ when $z = 1$ and $s > 0$ when $|z| \leq 1$ and $z \neq 1$.

Fix a positive integer $q \geq 3$. A Dirichlet character χ modulo q is a group homomorphism, $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$, extended as a completely multiplicative, periodic function on the integers. The L-function associated to χ is defined as

$$L(s;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges absolutely for $\Re(s) > 1$. It can be shown that (see [16, Section 5] for details) $L(k;\chi) \in \pi^k\,\mathbb{Q}^*$ when k and χ have the same parity, i.e., both are either odd or even. However, when k and χ have the opposite parity, the nature of the values $L(k;\chi)$ is unknown. Since the function χ is periodic, for $\Re(s) > 1$,

$$L(s;\chi) = \frac{1}{q^s}\sum_{a=1}^{q} \chi(a)\,\zeta\left(s;\frac{a}{q}\right).$$

Thus, the Hurwitz zeta-functions are building blocks of the Dirichlet L-series.

That the above functions are inter-related is immediate from the following observations.

$$\zeta\left(k;\frac{1}{2}\right) = \left(2^k - 1\right)\zeta(k), \quad \mathrm{Li}_k(-1) = -\Phi(-1;\,1;\,k) = \left(1 - \frac{2}{2^k}\right)\zeta(k)$$
$$\Phi\left(-1;\,\frac{1}{2};\,k\right) = 2^k\,L(k;\,\chi_4), \tag{2}$$

where χ_4 is the non-trivial character modulo 4.

Multi-variable analogs of the above zeta-functions have been studied by various authors. The theory of meromorphic continuation of multiple Hurwitz zeta-function was studied by Akiyama and Ishikawa [1] and by the first author and Kaneenika Sinha [18]. Around the same time, the multiple Hurwitz zeta-functions were also studied in [15]. The *multiple Hurwitz zeta-function*,

$$\widetilde{\zeta}(s_1,\cdots,s_r\,;\,x_1,\cdots,x_r) := \sum_{n_1>n_2>\cdots>n_r\geq 1} \frac{1}{(n_1+x_1)^{s_1}\cdots(n_r+x_r)^{s_r}},$$

converges when $x_i \in (0,\infty)$ and

$$\Re(s_1) > 1,\ \Re(s_1+s_2) > 2,\ \cdots,\Re(s_1+\cdots+s_r) > r. \tag{3}$$

The analytic continuation of these multiple Hurwitz zeta-functions was the center of interest in [18]. However, the arithmetic nature of special values of the multiple Hurwitz zeta-functions have not been studied previously in full generality. In the special case that $x_i = 1/2$, the multiple Hurwitz zeta-values are called multiple t-values. A detailed study of these special values in the spirit of multiple zeta-values has been carried out by M. E. Hoffman in [12], who conjectured that the dimension of the $\mathbb{Q}$-vector space generated by the weight k multiple t-values is the k^{th} Fibonacci number. A basis for this vector space was conjectured by B. Saha in [22].

In order to ensure elegance of our formulas, we modify the above definition slightly to include the indices equal to 0 and ensure that $x_i \notin \{0, -1, -2, \cdots\}$. Thus, throughout this paper, we will consider the multiple Hurwitz zeta-function to be

$$\zeta(s_1, \cdots, s_r\,;\, x_1, \cdots, x_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 0} \frac{1}{(n_1 + x_1)^{s_1} \cdots (n_r + x_r)^{s_r}}. \tag{4}$$

Adopting this convention implies that

$$\zeta(s_1, \cdots, s_r\,;\, 1, \cdots, 1) = \zeta_r(s_1, \cdots, s_r),$$

the usual multi-zeta function. Note that

$$\zeta(s_1, \cdots, s_r\,;\, x_1, \cdots, x_r) = \widetilde{\zeta}(s_1, \cdots, s_r\,;\, x_1, \cdots, x_r) + \frac{1}{x_r^{s_r}}\, \widetilde{\zeta}(s_1, \cdots, s_{r-1}\,;\, x_1, \cdots, x_{r-1}).$$

In the same vein, the meromorphic continuation of *multiple Lerch zeta-function* was studied by S. Gun and B. Saha in [11]. We will define a multiple Lerch-zeta function as

$$\Phi(z_1, \cdots, z_r; \alpha_1, \cdots, \alpha_r; s_1, \cdots, s_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 0} \frac{z_1^{n_1} \cdots z_r^{n_r}}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}}, \tag{5}$$

for $\alpha_i \in (0, \infty)$, $s_1, \cdots s_r$ satisfying (3) and z_i such that $\prod_{i=1}^{j} |z_i| \leq 1$ for all $1 \leq j \leq r$ (for a proof, see [21, Section 2.2]). Note that we include the term corresponding to $n_r = 0$ to ensure clean identities, so our definition includes more terms than that used in [11]. It is then easy to see that $\Phi(1, \cdots, 1; \alpha_1, \cdots, \alpha_r; s_1, \cdots, s_r) = \zeta(s_1, \cdots, s_r; \alpha_1, \cdots, \alpha_r)$ and the multiple polylogarithms (see [23]),

$$\mathrm{Li}_{s_1, \cdots, s_r}(z_1, \cdots, z_r) = \sum_{n_1 > \cdots > n_r \geq 1} \frac{{z_1}^{n_1} \cdots {z_r}^{n_r}}{{n_1}^{s_1} \cdots {n_r}^{s_r}} = z_1 \cdots z_r\, \Phi(z_1, \cdots, z_r; 1, \cdots, 1; s_1, \cdots, s_r).$$

When $z_i = \pm 1$, the corresponding multiple polylogarithms are called *alternating Euler-Zagier sums*, which have been extensively studied in the literature (for example, see [5] and [6]). In order to maintain consistency of notation for depth 2 sums, we use the following convention.

$$\zeta_2(\overline{r}, s) := \sum_{m=2}^{\infty} \frac{(-1)^m}{m^r} \sum_{n=1}^{m-1} \frac{1}{n^s}, \quad \zeta_2(\overline{r}, \overline{s}) := \sum_{m=2}^{\infty} \frac{(-1)^m}{m^r} \sum_{n=1}^{m-1} \frac{(-1)^n}{n^s}. \tag{6}$$

Multiple Dirichlet L-functions were considered by Akiyama and Ishikawa [1] and also appear in the work of Goncharov [10]. Let χ_1, χ_2, $\cdots$, χ_r be primitive Dirichlet characters of the same modulus q. Then the associated multiple Dirichlet L-function is defined as

$$L(s_1, \cdots, s_r\,;\, \chi_1, \cdots, \chi_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{\chi_1(n_1) \cdots \chi_r(n_r)}{{n_1}^{s_1} \cdots {n_r}^{s_r}}.$$

The convolution of values of Dirichlet L-functions is considerably more involved. The multiple L-functions that appear are more general than the multiple Dirichlet L-functions above, namely, if f_1, f_2, $\cdots$, f_r be functions on the integers, that are periodic modulo the same modulus, q and satisfy $\sum_{a=1}^{q} f_j(a) = 0$ for all $1 \leq j \leq r$, then define the multiple L-function,

$$L(s_1, \cdots, s_r\,;\, f_1, \cdots, f_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{f_1(n_1) \cdots f_r(n_r)}{{n_1}^{s_1} \cdots {n_r}^{s_r}}, \tag{7}$$

which converges for $s_1, \cdots, s_r$ satisfying (3).

Another multiple Dirichlet series allied to (7) are *quasi-multiple L-functions*, where the strict inequality in (7) is replaced by a possible equality. Analogously, one can also define the *quasi-multiple Hurwitz zeta-functions*. They are a special case of a general multiple zeta-function introduced by Matsumoto [15] and are discussed in [18, pg. 13]. In particular, the quasi-multiple L-functions that appear in our work will be

$$L^*(s_1, \cdots, s_r\,;\, x_1, \cdots, x_r) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{f_1(n_1)\,\cdots\, f_r(n_r)}{{n_1}^{s_1}\,\cdots\,{n_r}^{s_r}}, \tag{8}$$

for $x_i \in \mathbb{R}\backslash\{0, -1, -2, \cdots\}$, $1 \leq i \leq r$ and s_i satisfying (3). These can be related to the multiple L-functions via a simple inclusion-exclusion principle.

The identities we obtain naturally also include the digamma function $\psi(x)$, which is defined as the logarithmic derivative of the gamma function. Owing to the infinite product of $\Gamma(z)$, one obtains a series expansion for $\psi(z)$, namely

$$\psi(z) := -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right), \;\; z \neq 0,\, -1,\, -2, \cdots.$$

Here γ denotes the Euler-Mascheroni constant. Thus, $\psi(1) = -\gamma$.

We first prove convolution sum identities for Lerch zeta-functions where the argument $z \neq 1$. From these identities, we derive the analogous expressions for Hurwitz zeta-functions by careful analysis of the effect of taking limit as $z \to 1^-$. Thus, our main theorem is

Theorem 1.1. *Let $k \geq 3$ be a positive integer, $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$ and $z_1, z_2 \in \mathbb{C}$ with $0 < |z_1|, |z_2| \leq 1$, $z_1 \neq 1$ and $z_2 \neq 1$. Then*

$$\begin{aligned}
&\sum_{j=1}^{k-1} \Phi(z_1; \alpha; j)\, \Phi(z_2; \alpha; k-j) \\
&= (k-1)\, \Phi(z_1\, z_2; \alpha; k) - \Big(\log(1-z_1) + \log(1-z_2)\Big)\, \Phi(z_1\, z_2; \alpha; k-1) \\
&\quad - z_2^{-1}\, \Phi(z_1\, z_2,\; {z_2}^{-1}\,;\, \alpha, 1\,;\, k-1, 1) - z_1^{-1}\, \Phi(z_1\, z_2,\; {z_1}^{-1}\,;\, \alpha, 1\,;\, k-1, 1),
\end{aligned}$$

where the last two terms are multiple Lerch zeta-functions as defined in (5).

As an easy corollary of this theorem using (2), we deduce the following identity for values of $L(s;\, \chi_4)$, where χ_4 denotes the non-trivial Dirichlet character modulo 4.

Corollary 1.1. *Let $k \geq 3$ be a positive integer and χ_4 denote the non-trivial Dirichlet character modulo 4. Then,*

$$\begin{aligned}
\sum_{j=1}^{k-1} L(j;\, \chi_4)\, L(k-j;\, \chi_4) = (k-1)\, \left(1 - \frac{1}{2^k}\right) \zeta(k) - \log 2\, \left(1 - \frac{1}{2^{k-1}}\right) \zeta(k-1) \\
+ 2\, \Phi\left(1; -1;\, \frac{1}{2}, 1;\, k-1, 1\right)
\end{aligned}$$

Now, we fix z_2 and take the limit as $z_1 \to 1^-$ in Theorem 1.1. This gives the following theorem for values of the Lerch and the Hurwitz zeta-function.

Theorem 1.2. *Let $k \geq 3$ be a positive integer, $0 < |z| \leq 1$ and $z \neq 1$ and $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$. Then*

$$\sum_{j=2}^{k-1} \zeta(j;\alpha)\,\Phi(z;\,\alpha;\,k-j) = (k-1)\,\Phi(z;\,\alpha;\,k) + \Big(\psi(\alpha)+\gamma-\log(1-z)\Big)\Phi(z;\,\alpha;\,k-1) - z^{-1}\,\Phi(z,z^{-1};\,\alpha,1;\,k-1,1) - \Phi(z,1;\,\alpha,1;\,k-1,1)$$

where the last two terms are multiple Lerch zeta-functions as defined in (5).

Similarly, on taking the limit as $z \to 1^-$ in the above theorem, we get

Corollary 1.2. *Let $k \geq 4$ be a positive integer and $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$. Then*

$$\sum_{j=2}^{k-2} \zeta(j;\alpha)\,\zeta(k-j;\alpha) = (k-1)\,\zeta(k;\alpha) + 2\Big(\psi(\alpha)+\gamma\Big)\zeta(k-1;\alpha) - 2\,\zeta(k-1,1;\,\alpha,1).$$

In particular, when $k = 3$ and $\alpha = 1$, the above corollary implies $\zeta(3) = \zeta_2(2,1)$. Moreover, we can take $z = -1$ and $\alpha = 1$ in Theorem 1.2 to obtain

Corollary 1.3. *For any integer $k \geq 4$,*

$$\sum_{j=2}^{k-2}\left(1-\frac{2}{2^{k-j}}\right)\zeta(j)\,\zeta(k-j) = (k-1)\,\zeta(k) - \left(1-\frac{2}{2^{k-1}}\right)(\log 2)\,\zeta(k-1) + \zeta_2(\overline{k-1},\bar{1}) - \zeta_2(\overline{k-1},1),$$

where the last terms are alternating Euler-Zagier sums, defined in (6).

This identity has been discussed in detail in [5, Section 4, (15)].

In [17], the first author emphasized that $\zeta(2) = \pi^2/6$ itself implies the more general fact that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, simply because of the neat identity

$$\left(k+\frac{1}{2}\right)\zeta(2k) = \sum_{j=1}^{k-1}\zeta(2j)\,\zeta(2k-2j). \tag{9}$$

This is another relation among the zeta-values that Euler was familiar with. It is natural to inquire if convolutions of values of the Lerch zeta-functions at *even* positive integers lead to new identities, different from the ones described previously. Towards this question, we prove the following.

Theorem 1.3. *Let $k \geq 2$ be a positive integer and complex numbers z_1 and z_2 such that $0 < |z_1| = |z_2| \leq 1$ and $z_1,\, z_2 \neq 1$. Then*

$$\begin{aligned}
&\sum_{j=1}^{k-1}\Phi(z_1;\,\alpha;2j)\,\Phi(z_2;\,\alpha;2k-2j)\\
&= \left(k-\frac{1}{2}\right)\Phi(z_1z_2;\,\alpha;\,2k) - \frac{1}{2}\Phi(z_1z_2;\,\alpha;\,2k-1)\Big(\log(1-z_1)+\log(1-z_2)\Big)\\
&- \frac{1}{2}\Phi(z_1^{-1}z_2;\,\alpha;\,2k-1)\,\Phi(z_1;\,2\alpha;\,1) - \frac{1}{2}\Phi(z_1z_2^{-1};\,\alpha;\,2k-1)\,\Phi(z_2;\,2\alpha;\,1)\\
&- \frac{z_2^{-1}}{2}\Phi(z_1z_2,z_2^{-1};\,\alpha,1;\,2k-1,1) + \frac{1}{2}\Phi(z_1z_2^{-1},z_2;\,\alpha,2\alpha;\,2k-1,1)\\
&- \frac{z_1^{-1}}{2}\Phi(z_1z_2,z_1^{-1};\,\alpha,1;\,2k-1,1) + \frac{1}{2}\Phi(z_1^{-1}z_2,z_1;\,\alpha,2\alpha;\,2k-1,1).
\end{aligned}$$

Taking $z_1 = z_2 = -1$ and $\alpha = 1/2$ in the above theorem, we deduce that

Corollary 1.4. *Let χ_4 denote the non-trivial character modulo 4 and $k \geq 2$ be an integer. Then*

$$\begin{aligned}
&\sum_{j=1}^{k-1} L(2j;\, \chi_4)\, L(2k-2j;\, \chi_4) \\
&= \left(1 - \frac{1}{2^{2k}}\right) \left(k - \frac{1}{2}\right) \zeta(2k) - \left(1 - \frac{1}{2^{2k-1}}\right) (\log 2)\, \zeta(2k-1) + \frac{1}{2^{2k-1}}\, \Phi\left(1, -1;\, \frac{1}{2}, 1;\, 2k-1, 1\right).
\end{aligned}$$

On the other hand, considering the equation in Theorem 1.3 at $z = z_1 = z_2$, $|z| < 1$ and taking the limit as $z \to 1^-$, we deduce the following identity for the values of Hurwitz zeta-functions.

Corollary 1.5. *Let $k \geq 2$ be an integer and $\alpha \in \mathbb{C} \backslash \{0, -1, -2, \cdots\}$. Then*

$$\begin{aligned}
\sum_{j=1}^{k-1} \zeta(2j;\alpha)\, \zeta(2k-2j;\alpha) = \left(k - \frac{1}{2}\right) \zeta(2k;\alpha) + \Big(\psi(2\alpha) + \gamma\Big)\, \zeta(2k-1;\alpha) \\
- \zeta(2k-1, 1\,;\, \alpha, 1) + \zeta(2k-1, 1\,;\, \alpha, 2\alpha),
\end{aligned}$$

where the last two terms are multiple Hurwitz zeta-functions as in (4).

For Dirichlet L-functions associated to *primitive* Dirichlet characters, we have the following theorem.

Theorem 1.4. *Let χ_1, χ_2 be primitive Dirichlet characters modulo $q \geq 3$ and $k \geq 3$ be an integer. For a primitive Dirichlet character χ mod q, define two allied periodic functions mod q by*

$$T_{q,a,\chi}(n) := \chi(n)\, \zeta_q^{an}, \ \textit{and}\ T_{q,a}(n) := \zeta_q^{an},$$

for any $a \in \mathbb{Z}$. Also, let $\tau(\chi) = \sum_{a=1}^{q} \chi(a)\zeta_q^a$ be the Gauss sum associated to χ. Then,

$$\begin{aligned}
&\sum_{j=1}^{k-1} L(j;\chi_1)\, L(k-j\,;\chi_2) \\
&= (k-1)L(k\,;\, \chi_1\chi_2) - \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \Big(\overline{\chi_2}(a)\, \log(1-\zeta_q^a)\, L(k-1; T_{q,a,\chi_1})\Big) \\
&- \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \Big(\overline{\chi_1}(a)\, \log(1-\zeta_q^a)\, L(k-1; T_{q,a,\chi_2})\Big) - \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \overline{\chi_2}(a)\, \zeta_q^{-a}\, L^*(k-1, 1\,;\, T_{q,a,\chi_1}, T_{q,-a}) \\
&\qquad - \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \overline{\chi_1}(a)\, \zeta_q^{-a}\, L^*(k-1, 1\,;\, T_{q,a\chi_2}, T_{q,-a})
\end{aligned}$$

where the last terms involve multiple L^*-function as defined in (8).

This is a generalization of Corollary 1.1 and gives an idea of the various combinations of special values involved. It is not difficult to see that for $r, s \in \mathbb{N}$ with $1 < r$ and $1 \leq s$, and a primitive character χ mod q,

$$\begin{aligned}
L^*(r, s\,;\, T_{q,a,\chi}, T_{q,-a}) &= \sum_{m=1}^{\infty} \frac{\chi(m)\, \zeta_q^{am}}{m^r} \cdot \frac{\zeta_q^{-am}}{m^s} + \sum_{m=1}^{\infty} \frac{\chi(m)\, \zeta_q^{am}}{m^r} \sum_{j=1}^{m-1} \frac{\zeta_q^{aj}}{j^s} \\
&= L(r+s, \chi) + L(r, s\,;\, T_{q,a,\chi}, T_{q,a}).
\end{aligned}$$

Using this in the above theorem, together with the fact that for a primitive Dirichlet character χ mod q,

$$\sum_{a=1}^{q} \chi(a)\, \zeta_q^{-a} = \chi(-1)\tau(\chi),$$

simplifies the identity as follows:

$$\begin{aligned}
&\sum_{j=1}^{k-1} L(j\,;\,\chi_1)\, L(k-j\,;\,\chi_2) \\
&= (k-1)L(k\,;\,\chi_1\chi_2) - \chi_2(-1)\, L(k\,;\,\chi_1) - \chi_1(-1)\, L(k\,;\,\chi_2) \\
&- \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \left(\overline{\chi_2}(a)\, \log(1-\zeta_q^a)\, L(k-1\,;\,T_{q,a,\chi_1}) \right) - \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \left(\overline{\chi_1}(a)\, \log(1-\zeta_q^a)\, L(k-1\,;\,T_{q,a,\chi_2}) \right) \\
&- \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \overline{\chi_2}(a)\, \zeta_q^{-a}\, L(k-1, 1\,;\,T_{q,a,\chi_1}, T_{q,-a}) - \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \overline{\chi_1}(a)\, \zeta_q^{-a}\, L(k-1, 1\,;\,T_{q,a\chi_2}, T_{q,-a}).
\end{aligned}$$

Remark. *It is evident from the above theorem that in order to study the special values of Dirichlet L-function, one must investigate the allied functions*

$$\mathrm{Li}_k(z;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)\, z^n}{n^k}, \quad |z| \leq 1,$$

for a Dirichlet character χ modulo q. By the duality between Dirichlet characters and arithmetic progressions, these sums will be naturally related to the function,

$$\sum_{\substack{n=1,\\ n\equiv a \bmod q}}^{\infty} \frac{z^n}{n^k},$$

which is essentially the Lerch zeta-function $\Phi(z;\, a/q;\, k)$.

2. **Proof of main theorems**

The method of summation in evaluating the sums that arise in our theorems is based on the same general principle, which we outline below. Fix a positive integer $r \geq 1$ and a positive integer $k \geq 4$. For complex numbers z_1, z_2 with $|z_i| \leq 1$ and $z_i \neq 1$, $i = 1, 2$, $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$, let the r-level convolution be defined as

$$\mathcal{C}_r(z_1, z_2\,;\,\alpha) := \sum_{j=1}^{k-1} \Phi(z_1;\, \alpha;\, rj)\ \Phi(z_2\,;\,\alpha\,;\,r(k-j)).$$

Then we expand the right hand side as

$$\begin{aligned}
\mathcal{C}_r(z_1, z_2\,;\,\alpha) &= \sum_{j=1}^{k-1} \sum_{n,m=0}^{\infty} \frac{z_1^m}{(m+\alpha)^{rj}} \cdot \frac{z_2^n}{(n+\alpha)^{r(k-j)}} \\
&= (k-1) \sum_{n=0}^{\infty} \frac{(z_1\, z_2)^n}{(n+\alpha)^{rk}} + \sum_{\substack{n,m=0,\\ n\neq m}}^{\infty} z_1^m\, z_2^n \sum_{j=1}^{k-1} \frac{1}{(m+\alpha)^{rj}} \cdot \frac{1}{(n+\alpha)^{r(k-j)}} \\
&= (k-1)\Phi(z_1\, z_2;\, \alpha;\, rk) + \sum_{\substack{n,m=0,\\ n\neq m}}^{\infty} \frac{z_1^m\, z_2^n}{(n+\alpha)^{rk}} \sum_{j=1}^{k-1} \left(\frac{n+\alpha}{m+\alpha} \right)^{rj}.
\end{aligned}$$

Now, the inner sum can be evaluated as a geometric series,

$$\frac{1}{(n+\alpha)^{rk}} \sum_{j=1}^{k-1} \left(\frac{n+\alpha}{m+\alpha} \right)^{rj} = \frac{1}{(m+\alpha)^{r(k-1)}(n+\alpha)^{r(k-1)}} \left(\frac{(n+\alpha)^{r(k-1)} - (m+\alpha)^{r(k-1)}}{(n+\alpha)^r - (m+\alpha)^r} \right).$$

Thus we get

$$\mathcal{C}_r(z_1, z_2\,; \alpha) = (k-1)\Phi(z_1\,z_2;\ \alpha;\ rk) + \sum_{m=0}^{\infty} \frac{z_1^m}{(m+\alpha)^{r(k-1)}} \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z_2^n}{(n+\alpha)^r - (m+\alpha)^r}$$
$$+ \sum_{n=0}^{\infty} \frac{z_2^n}{(n+\alpha)^{r(k-1)}} \sum_{\substack{m=0,\\ m\neq n}}^{\infty} \frac{z_1^m}{(m+\alpha)^r - (n+\alpha)^r}. \tag{10}$$

Therefore, the above computations naturally lead one into the study of the auxiliary sums

$$\mathcal{S}_{r,m}(z,\alpha) := \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z^n}{(n+\alpha)^r - (m+\alpha)^r}, \tag{11}$$

where $z \in \mathbb{C}$ with $|z| \leq 1$, $z \neq 1$ and $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$. Our focus will mostly be on the cases $r = 1$ and $r = 2$. We will also later indicate the difficulties in obtaining neat formulas for $r \geq 3$ using the above method.

2.1. **Evaluation of auxiliary sum.** For a non-negative integer m, let H_m denote the m^{th} harmonic number, that is,

$$H_m := \sum_{j=1}^{m} \frac{1}{j},$$

if m is a strictly positive integer and $H_0 := 0$. It is not difficult to see that

$$H_N = \log N + \gamma + O\left(\frac{1}{N}\right). \tag{12}$$

Analogous to the harmonic numbers, we introduce the *generalized harmonic numbers*, defined as

$$H_k(z,\alpha) := \begin{cases} \sum_{j=0}^{k} \frac{z^j}{(j+\alpha)}, & \text{if } k \geq 0,\\ 0 & \text{otherwise,} \end{cases}$$

for $|z| \leq 1$ and $\alpha \in \mathbb{C}\backslash\{0,\,-1,\,-2,\,\cdots\}$. Let $H_k(\alpha) := H_k(1,\alpha)$, so that $H_m = H_{m-1}(1)$. The asymptotic behaviour of these numbers is evident from the following lemma.

Lemma 2.1. *Let $|z| \leq 1$, $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$ and k be a non-negative integer. Then,*

$$H_k(\alpha) = \log k - \psi(\alpha) + O\left(\frac{1}{k}\right),$$

as $k \to \infty$. If $z \neq 1$, then

$$\lim_{k\to\infty} H_k(z,\alpha) = \Phi(z;\alpha;1).$$

Proof. When $z \neq 1$, the series

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)}$$

can be shown to converge using Abel's theorem. When $z = 1$, the asymptotics follow from (12) and the series representation of the digamma function,

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty}\left(\frac{1}{n+\alpha} - \frac{1}{n}\right) = -\gamma - \psi(\alpha),$$

for $\alpha \neq 0,\,-1,\,-2,\,\cdots$. ∎

Also note that for $z \neq 1$ and $0 < |z| \leq 1$,

$$\Phi(z;1;1) = z^{-1}\, \log(1-z).$$

With this background, the auxiliary sum in the case $r = 1$ can be expressed as follows.

Lemma 2.2. *Let $z \in \mathbb{C}$ with $|z| \leq 1$, $z \neq 1$ and m be a non-negative integer. Then*

$$\mathcal{S}_m(z) := \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z^n}{n-m} = -z^m\, \log(1-z) - z^{m-1}\, H_{m-1}(z^{-1}, 1),$$

where the last term involves a generalized harmonic number.

Proof. Separating the sum into two parts gives

$$\begin{aligned}\mathcal{S}_m(z) = \sum_{m<n} \frac{z^n}{n-m} + \sum_{0\leq n<m} \frac{z^n}{n-m} &= \sum_{j=1}^{\infty} \frac{z^{j+m}}{j} - \sum_{j=1}^{m} \frac{z^{m-j}}{j} \\ &= -z^m\, \log(1-z) - \sum_{j=0}^{m-1} \frac{z^{(m-j-1)}}{j+1} \\ &= -z^m\, \log(1-z) - z^{m-1}\, H_{m-1}(z^{-1}, 1).\end{aligned}$$

□

In the case $r = 2$, we have

Lemma 2.3. *Let $z \in \mathbb{C}$ with $|z| \leq 1$, $z \neq 1$ and m be a non-negative integer. Then*

$$\begin{aligned}\mathcal{S}_{2,m}(z,\alpha) &:= \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z^n}{(n+\alpha)^2 - (m+\alpha)^2} \\ &= \frac{z^m}{4(m+\alpha)^2} - \frac{1}{2(m+\alpha)}\left\{z^m\, \log(1-z) + z^{m-1}\, H_{m-1}(z^{-1}, 1)\right\} \\ &- \frac{1}{2(m+\alpha)}\left\{z^{-m}\Phi(z;2\alpha;1) - z^{-m} H_{m-1}(z, 2\alpha)\right\}\end{aligned}$$

Proof. By partial fractions, we know that

$$\frac{1}{(n+\alpha)^2 - (m+\alpha)^2} = \frac{1}{2(m+\alpha)}\left(\frac{1}{n-m} - \frac{1}{n+m+2\alpha}\right).$$

The required sum can then be re-written as

$$\mathcal{S}_{2,m}(z,\alpha) = \frac{1}{2(m+\alpha)}\, \mathcal{S}_m(z) + \frac{z^m}{4(m+\alpha)^2} - \frac{1}{2(m+\alpha)} \sum_{n=0}^{\infty} \frac{z^n}{n+m+2\alpha}.$$

The last sum can be determined as follows

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{z^n}{n+m+2\alpha} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{z^n}{n+m+2\alpha} &= \lim_{N\to\infty} \sum_{j=m}^{N+m} \frac{z^{j-m}}{j+2\alpha} \\ &= z^{-m} \lim_{N\to\infty} \left(\sum_{j=0}^{N+m} \frac{z^j}{j+2\alpha} - \sum_{j=0}^{m-1} \frac{z^j}{j+2\alpha}\right) \\ &= z^{-m} \lim_{N\to\infty} \Big(H_{N+m}(z, 2\alpha) - H_{m-1}(z, 2\alpha)\Big) \\ &= z^{-m}\, \Phi(z;\, 2\alpha;\, 1) - z^{-m}\, H_{m-1}(z, 2\alpha).\end{aligned}$$

The evaluation of $\mathcal{S}_{2,m}(z,\alpha)$ is now evident from Lemma 2.2. □

Remark. *Using partial fractions, it is possible to obtain that for* $|z|\le 1$ *and* $z\neq 1$,

$$\mathcal{S}_{r,m}(z,\alpha)=\frac{1}{r(m+\alpha)^{r-1}}\sum_{k=1}^{r}\zeta_r^k\sum_{\substack{n=0,\\ n\neq m}}^{\infty}\frac{z^n}{(n+\alpha)-\zeta_r^k\,(m+\alpha)},$$

where ζ_r *denotes a primitive* r*-th root of unity. However, for* $r\ge 3$*, since the roots of unity are complex, the evaluation of inner sums is not immediate. Moreover, when* $r=2^s$*, the sums arising above have the special form*

$$\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^{2^t}+(m+\alpha)^{2^t}},\qquad 0\le t\le s-1.$$

When $t=1$, $\alpha=1$ *and* $z=1$*, the resulting sum can be evaluated using* [19, Theorem 2]*. This highlights the importance of the study of the series*

$$\sum_{n=0}^{\infty}\frac{A(n)}{B(n)}\,z^n,$$

where $A(X)$ *and* $B(X)$ *are suitable polynomials with rational coefficients and* $|z|\le 1$.

2.2. **Proof of Theorem 1.1.** Let $r=1$ in (10). Then, we have

$$\begin{aligned}\mathcal{C}_1(z_1,z_2\,;\alpha)&=(k-1)\Phi(z_1,z_2;\,\alpha;\,k)\\&\quad+\sum_{m=0}^{\infty}\left(\frac{z_1^m}{(m+\alpha)^{k-1}}\cdot\mathcal{S}_m(z_2)\right)+\sum_{n=0}^{\infty}\left(\frac{z_2^n}{(n+\alpha)^{k-1}}\cdot\mathcal{S}_n(z_1)\right).\end{aligned}$$

The above two sums can be simplified using the expressions for $\mathcal{S}_m(z)$ obtained in Lemma 2.2. For instance,

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\frac{z_1^m}{(m+\alpha)^{k-1}}\cdot\mathcal{S}_m(z_2)\right)&=-\log(1-z_2)\sum_{m=0}^{\infty}\frac{(z_1\,z_2)^m}{(m+\alpha)^{k-1}}-z_2^{-1}\sum_{m=0}^{\infty}\frac{(z_1\,z_2)^m}{(m+\alpha)^{k-1}}H_{m-1}(z_2^{-1},1)\\&=-\log(1-z_2)\,\Phi(z_1z_2;\,\alpha;\,k-1)-z_2^{-1}\,\Phi(z_1z_2,z_2^{-1};\,\alpha,1;\,k-1,1),\end{aligned}$$

where the last term is a multiple Lerch zeta-function as defined in (5). The remaining sum can also be evaluated similarly. This proves Theorem 1.1.

2.3. **Proof of Theorem 1.2.** The idea of the proof is that for a fixed integer $k>1$, $\alpha\in\mathbb{C}\backslash\{0,-1,-2,\cdots\}$, the function

$$\Phi(z;\,\alpha;\,k):=\sum_{j=0}^{\infty}\frac{z^j}{(j+\alpha)^k}$$

is a continuous function of z on the disk $\{z\in\mathbb{C}\,:\,|z|\le 1\}$. However, when $k=1$, the limit $\lim_{z\to 1^-}\Phi(z;\alpha;1)$ does not exist because of the pole of the Hurwitz zeta-function at $s=1$. Therefore, we re-write the identity obtained in Theorem 1.1 as follows.

$$\begin{aligned}
&\sum_{j=2}^{k-1} \Phi(z_1;\alpha;j)\,\Phi(z_2;\alpha;k-j)\\
&= (k-1)\,\Phi(z_1\,z_2;\alpha;k) - \Big(\log(1-z_2)\Big)\,\Phi(z_1\,z_2;\alpha;k-1)\\
&\quad - z_2^{-1}\,\Phi(z_1\,z_2,\ {z_2}^{-1}\,;\,\alpha,1\,;\,k-1,1) - z_1^{-1}\,\Phi(z_1\,z_2,\ {z_1}^{-1}\,;\,\alpha,1\,;\,k-1,1)\\
&\quad - \Big\{\log(1-z_1)\,\Phi(z_1z_2;\ \alpha;\ k-1) + \Phi(z_1;\ \alpha;\ 1)\,\Phi(z_2;\ \alpha;\ k-1)\Big\}
\end{aligned}$$

For a fixed $z_2 \neq 1$, we would like to consider the limit $z_1 \to 1^-$. That is, we let $z_1 \in \mathbb{R}$ with $0 < z_1 < 1$ and then take the limit as $z_1 \to 1$. For all the terms in the above identity except the ones in curly brackets, the limit as $z_1 \to 1^-$ exists. Hence, we concentrate on just those two terms. Observe that

$$\begin{aligned}
&\lim_{z_1\to 1^-} \log(1-z_1)\,\Phi(z_1z_2;\ \alpha;\ k-1) + \Phi(z_1;\ \alpha;\ 1)\,\Phi(z_2;\ \alpha;\ k-1)\\
&= \lim_{z_1\to 1^-}\lim_{N\to\infty}\left[\Phi(z_2;\ \alpha;\ k-1)\left(\sum_{j=0}^{N}\frac{z_1^j}{j+\alpha}\right) - \Phi(z_1z_2;\ \alpha;\ k-1)\left(\sum_{j=1}^{N}\frac{z_1^j}{j}\right)\right].
\end{aligned}$$

Now, note that for a fixed z_2, $\Phi(z\,z_2;\ \alpha;\ k-1)$ is a continuous function of z. Thus, we have that the limit equals

$$\Phi(z_2;\ \alpha;\ k-1)\lim_{z_1\to 1^-}\lim_{N\to\infty}\left[\left(\sum_{j=0}^{N}\frac{z_1^j}{j+\alpha} - \sum_{j=1}^{N}\frac{z_1^j}{j}\right)\right] = \Phi(z_2;\ \alpha;\ k-1)\lim_{z_1\to 1^-}\lim_{N\to\infty}\left[\frac{1}{\alpha} + \sum_{j=1}^{N}\frac{\alpha\, z_1^j}{j(j+\alpha)}\right].$$

Since $\sum_{j=1}^{\infty} 1/(j\,(j+\alpha)) < \infty$, one can interchange the limits thanks to the dominated convergence theorem, to get that the above limit is in fact

$$\Phi(z_2;\ \alpha;\ k-1)\left[\frac{1}{\alpha} + \lim_{N\to\infty}\sum_{j=1}^{N}\left(\frac{1}{j+\alpha} - \frac{1}{j}\right)\right] = -\Big(\psi(\alpha)+\gamma\Big)\Phi(z_2;\ \alpha;\ k-1).$$

This implies Theorem 1.2.

2.4. **Proof of Theorem 1.3.** We take $r = 2$ in (10). Therefore, we have

$$\begin{aligned}
\mathcal{C}_2(z_1,z_2;\ \alpha) = (k-1)\,\Phi(z_1z_2;\ \alpha;\ 2k) &+ \sum_{m=0}^{\infty}\frac{z_1^m}{(m+\alpha)^{2(k-1)}}\,\mathcal{S}_{2,m}(z_2,\alpha)\\
&+ \sum_{n=0}^{\infty}\frac{z_2^n}{(n+\alpha)^{2(k-1)}}\,\mathcal{S}_{2,n}(z_1,\alpha).
\end{aligned}$$

Using the evaluation of $\mathcal{S}_{2,m}(z,\alpha)$ from Lemma 2.3, we get

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{z_1^m}{(m+\alpha)^{2(k-1)}}\,\mathcal{S}_{2,m}(z_2,\alpha)\\
&= \frac{1}{4}\Phi(z_1z_2;\ \alpha;2k) - \frac{1}{2}\log(1-z_2)\Phi(z_1z_2;\ \alpha;2k-1) - \frac{1}{2}\Phi(z_2;\ 2\alpha;\ 1)\,\Phi(z_1z_2^{-1};\ \alpha;\ 2k-1)\\
&- \frac{z_2^{-1}}{2}\Phi(z_1z_2,z_2^{-1};\ \alpha,1;\ 2k-1,1) + \frac{1}{2}\Phi(z_1z_2^{-1},\ z_2;\ \alpha,2\alpha;\ 2k-1,1),
\end{aligned}$$

where the last two terms are multiple Lerch zeta-functions. The theorem now follows since the remaining sum can be computed by symmetry.

2.5. **Dirichlet L-functions: Proof of Theorem 1.4.** Recall that for a primitive Dirichlet character χ,

$$\sum_{a=1}^{q} \overline{\chi}(a)\, \zeta_q^{an} = \chi(n)\, \tau(\overline{\chi}), \tag{13}$$

where ζ_q is a primitive q-th root of unity and $\tau(\chi) = \sum_{a=1}^{q} \chi(a)\zeta_q^a$ is the Gauss sum associated to χ. Since χ is primitive, $\tau(\chi) \neq 0$. Thus, we have the following lemma.

Lemma 2.4. *Let χ be a primitive Dirichlet character mod q and m be a fixed positive integer. Then,*

$$\sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{\chi(n)}{n-m} = -\frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \left(\overline{\chi}(a)\, \zeta_q^{am} \log(1-\zeta_q^a) \right) - \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \left(\overline{\chi}(a)\, \zeta_q^{a(m-1)}\, H_{m-1}(\zeta_q^{-am}, 1) \right).$$

Proof. Substituting the value of $\chi(n)$ from (13), we have

$$\begin{aligned}
\sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{\chi(n)}{n-m} &= \frac{1}{\tau(\overline{\chi})} \sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{1}{n-m} \sum_{a=1}^{q} \overline{\chi}(a)\zeta_q^{an} \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{\zeta_q^{an}}{n-m} \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \left(\mathcal{S}_m(\zeta_q^a) + \frac{1}{m} \right) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\, \mathcal{S}_m(\zeta_q^a) + \frac{1}{m\, \tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\, \mathcal{S}_m(\zeta_q^a).
\end{aligned}$$

The value of $\mathcal{S}_m(\zeta_q^a)$ can be calculated from Lemma 2.2. This proves the lemma. □

Applying the above lemma, one can prove Theorem 1.4 as follows. For simplicity of notation, let

$$C_k(\chi_1, \chi_2) := \sum_{j=1}^{k-1} L(j; \chi_1)\, L(k-j; \chi_2).$$

Using the definition of the Dirichlet L-functions, we have

$$\begin{aligned}
C_k(\chi_1, \chi_2) &= \sum_{m,n=1}^{\infty} \sum_{j=1}^{k-1} \frac{\chi_1(m)}{m^j} \cdot \frac{\chi_2(n)}{n^{k-j}} \\
&= (k-1) \sum_{m=1}^{\infty} \frac{(\chi_1\,\chi_2)(m)}{m^k} + \sum_{\substack{m,n=1,\\ m\neq n}}^{\infty} \sum_{j=1}^{k-1} \frac{\chi_1(m)}{m^j} \cdot \frac{\chi_2(n)}{n^{k-j}} \\
&= (k-1)L(k\,;\, \chi_1\chi_2) + \sum_{\substack{m,n=1,\\ m\neq 1}}^{\infty} \frac{\chi_1(m)\,\chi_2(n)}{n^k} \sum_{j=1}^{k-1} \left(\frac{n}{m}\right)^j.
\end{aligned}$$

Since $m \neq n$ in the second sum, the inner sum can be simplified as a geometric sum,

$$\frac{1}{n^k}\sum_{j=1}^{k-1}\left(\frac{n}{m}\right)^j = \frac{1}{(n-m)}\left(\frac{1}{m^{k-1}} - \frac{1}{n^{k-1}}\right).$$

Therefore, the convolution sum becomes

$$C_k(\chi_1,\chi_2) = (k-1)L(k\,;\,\chi_1\chi_2) + \sum_{m=1}^{\infty}\frac{\chi_1(m)}{m^{k-1}}\sum_{\substack{n=1,\\ n\neq m}}^{\infty}\frac{\chi_2(n)}{n-m} + \sum_{n=1}^{\infty}\frac{\chi_2(n)}{n^{k-1}}\sum_{\substack{m=1,\\ m\neq n}}^{\infty}\frac{\chi_1(m)}{m-n}.$$

The inner sums were computed in Lemma 2.4. For any Dirichlet character χ mod q and $1 \leq a < q$, let $T_{q,a}(m) := \zeta_q^{am}$ and $T_{q,a,\chi}(m) := \chi(m)\,\zeta_q^{am}$. Thus, $T_{q,a}$ and $T_{q,a,\chi}$ define periodic functions on the integers, periodic modulo q. With this notation, the convolution becomes,

$$\begin{aligned}
&C_k(\chi_1,\chi_2)\\
&= (k-1)L(k\,;\,\chi_1\chi_2) - \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\overline{\chi_2}(a)\,\log(1-\zeta_q^a)\sum_{m=1}^{\infty}\frac{T_{q,a,\chi_1}(m)}{m^{k-1}}\\
&- \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\overline{\chi_1}(a)\,\log(1-\zeta_q^a)\sum_{n=1}^{\infty}\frac{T_{q,a,\chi_2}(n)}{n^{k-1}} - \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\overline{\chi_2}(a)\zeta_q^{-a}\sum_{m=1}^{\infty}\frac{T_{q,a,\chi_1}(m)}{m^{k-1}}\sum_{j=1}^{m}\frac{\zeta_q^{-aj}}{j}\\
&\qquad - \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\overline{\chi_1}(a)\zeta_q^{-a}\sum_{n=1}^{\infty}\frac{T_{q,a,\chi_2}(n)}{n^{k-1}}\sum_{j=1}^{n}\frac{\zeta_q^{-aj}}{j}\\
&= (k-1)L(k\,;\,\chi_1\chi_2) - \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\left(\overline{\chi_2}(a)\,\log(1-\zeta_q^a)\,L(k-1;T_{q,a,\chi_1})\right)\\
&- \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\left(\overline{\chi_1}(a)\,\log(1-\zeta_q^a)\,L(k-1;T_{q,a,\chi_2})\right) - \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\overline{\chi_2}(a)\,\zeta_q^{-a}\,L^*(k-1,1\,;\,T_{q,a,\chi_1},T_{q,-a})\\
&\qquad - \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\overline{\chi_1}(a)\,\zeta_q^{-a}\,L^*(k-1,1\,;\,T_{q,a\chi_2},T_{q,-a})
\end{aligned}$$

where

$$T_{q,a,\chi}(n) = \chi(n)\,\zeta_q^{an} \qquad \text{and} \qquad T_{q,-a}(n) = \zeta_q^{-an}.$$

This proves Theorem 1.4.

Remark. *It is clear from the above proof that in order to understand r-level convolution of values of Dirichlet L-functions, one needs to understand sums of the form*

$$\sum_{\substack{n=1,\\ n\neq m}}^{\infty}\frac{z^n}{n^r - m^r},$$

for a fixed positive integer m and $|z| \leq 1$. These sums are interesting in their own right and we relegate their investigation to future research.

3. Concluding Remarks

The theorems included here are only the opening themes of a larger symphony of ideas. It is now clear that to understand the nature of $\zeta(2k+1)$, it is necessary to study the multi-zeta values. Our paper shows that a similar approach is needed to understand $L(k;\chi)$ when k and χ have opposite parity.

In Theorems 1.1, 1.2 and 1.3, one can consider the more general case when the corresponding Lerch and Hurwitz zeta-functions have different parameters. For example, one can compute the convolution of values of $\Phi(z_1;\alpha_1;s)$ and values of $\Phi(z_2;\alpha_2;s)$ with $\alpha_1 \neq \alpha_2$. The method outlined in this paper would also go through in these cases. However, the identities in these scenarios are not as elegant as the ones mentioned here.

Let G denote the Catalan's constant, that is,

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = L(2;\chi_4) = 4\,\Phi\left(-1;\frac{1}{2};2\right).$$

Then $k = 3$ and $k = 2$ cases of Corollaries 1.1 and 1.4 furnish interesting relations among G, $L(1,\chi_4)$, π^2, $\zeta(3)$ and values of multiple zeta-functions.

A curious observation emerges from the identity stated in Corollary 1.2. For $k = 3$, the left-hand side of the formula in Corollary 1.2 is empty and hence, zero. Substituting $\alpha = 1/2$ and simplifying the right-hand side leads to the identity

$$\zeta(3) = \frac{6}{7}\,(\log 2)\,\zeta(2) + \frac{4}{7}\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}.$$

Furthermore, taking $k = 3$ in Corollary 1.3, we also get

$$\zeta(3) = \frac{1}{4}\,(\log 2)\,\zeta(2) + \frac{1}{2}\zeta(\overline{2},1) - \frac{1}{2}\zeta(\overline{2},\overline{1}).$$

This is interesting since (it seems) Euler conjectured that

$$\zeta(3) = \alpha\,\pi^2\,\log 2 \,+\, \beta\,(\log 2)^2$$

for certain rational numbers α and β (see for example, [9, pg. 60]). This observation leads us to inquire whether

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2} \text{ or } \zeta(\overline{2},1) - \zeta(\overline{2},\overline{1})$$

can be explicitly evaluated in terms of $\pi^2\,\log 2$ and $(\log 2)^2$. Perhaps not. To date, no one has disproved Euler's conjecture.

In this vein, we would like to highlight a conjecture by D. Bailey, J. Borwein and R. Girgensohn [4, Section 7, pg. 27] based on numerical evidence. To each (alternating) Euler-Zagier sum, $\Phi(\epsilon_1,\cdots,\epsilon_r;\,1,\cdots,1;\,k_1,\cdots k_r)$, $\epsilon_j \in \{\pm 1\}$, one can associate the weight $w = k_1 + \cdots + k_r$. Moreover, the weight of the product $\Phi(\epsilon_1,\cdots,\epsilon_r;\,1,\cdots,1;\,k_1,\cdots k_r)\cdot\Phi(\delta_1,\cdots,\delta_r;\,1,\cdots,1;\,m_1,\cdots m_s)$ is given by the sum $k_1+\cdots+k_r+m_1+\cdots+m_s$. Then, the conjecture of Bailey, Borwein and Girgensohn can be stated as follows.

Conjecture 1 (Bailey, J. Borwein, Girgensohn)**.** *Alternating Euler-Zagier sums of different weights are $\mathbb{Q}$-linearly independent.*

Now, $\zeta(3)$ and $\pi^2\log 2$ have weight 3 each. However, $(\log 2)^2 = \Phi(-1;1;1)^2 = 2\zeta(\overline{1},1)$ (see [5, pg. 291]) and hence, has weight 2. Therefore, Conjecture 1 would imply that $\zeta(\overline{2},1) - \zeta(\overline{2},\overline{1})$ is a rational multiple of $\pi^2\log(2)$. This is not expected (see [5, pg. 291]) and thus, Euler's conjecture seems to be false.

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