LINEAR INDEPENDENCE OF VALUES OF THE q-EXPONENTIAL AND RELATED FUNCTIONS

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Abstract. In this paper, we establish the linear independence of values of the q-analogue of the exponential function, $E_q(x)$ and its derivatives at specified algebraic arguments, when q is a Pisot-Vijayraghavan number. We also deduce similar results for cognate functions, such as the Tschakaloff function and certain generalized q-series.

1. **Introduction**

For any complex number q with $|q| > 1$, the q-analogue of the exponential function is defined by the absolutely convergent series

$$E_q(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{[n]_q!},$$

where $[n]_q = (q^n - 1)$ and $[n]_q! = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$. Similarly we have the q-analogue of the logarithm, which is given by

$$L_q(x) := \sum_{n=1}^{\infty} \frac{x^n}{[n]_q}, \qquad \text{for } |x| < |q|.$$

The analogy between the classical functions and their q-analogues is driven by the limit

$$\lim_{q \to 1^+} \frac{q^n - 1}{q - 1} = n.$$

Unlike the classical exponential and logarithm functions, their q-counterparts are related by the following differential relation:

$$L_q(x) = x \frac{E_q'(-x)}{E_q(-x)}$$

for $|x| < |q|$. For more details, we refer the reader to [7, Section 6]. These functions appear in various contexts in combinatorics and number theory, and are studied as interesting functions in their own right.

The value at $x = 1$ of the q-logarithm function is of particular importance, as $L_q(1) = \zeta_q(1)$, where

$$\zeta_q(s) := \sum_{n=1}^{\infty} \frac{n^{s-1}}{[n]_q},$$

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is the q-analogue of the Riemann zeta-function as considered in [6]. The value $\zeta_q(1)$,

$$\zeta_q(1) = \sum_{n=1}^{\infty} \frac{1}{q^n - 1},$$

is often referred to in the literature as the q-harmonic series.

In this paper, we examine the arithmetic nature and linear independence properties of certain special values of the above mentioned functions. Recall that a real algebraic integer ω is said to be *Pisot-Vijayraghavan number* (abbreviated as PV number) if $\omega > 1$, and for all other Galois conjugates $\omega^{(j)}$ of ω, we have $|\omega^{(j)}| < 1$. Immediate examples of PV numbers are positive integers greater than 1. A non-trivial example is obtained by considering β, the real root of $x^4 - x^3 - 2x^2 + 1$ with $\beta > 1$. Then it can be checked that β is a PV number. In fact, Pisot [8] showed that in every real algebraic number field, there exist PV numbers that generate the field. These numbers make a fundamental appearance in Diophantine approximation and have been extensively studied in the literature.

Fix an algebraic integer $q \neq 0$ and let $n_q = [\mathbb{Q}(q) : \mathbb{Q}]$. Let $\sigma_1, \sigma_2, \cdots, \sigma_{n_q}$ denote the embeddings of $\mathbb{Q}(q)$ into $\mathbb{C}$, with σ_1 being identity. Let $\mathcal{O}_q$ be the ring of integers of $\mathbb{Q}(q)$. For any algebraic number $\alpha \in \mathbb{Q}(q)$, the q-relative height of α, $H_q(\alpha)$, is defined as

$$H_q(\alpha) := \prod_{l=1}^{n_q} \max\{1, |\sigma_l(\alpha)|\}.$$

Thus, if q is a PV number, then $H_q(q) = q$.

Our first theorem concerns the linear independence of values of derivatives of a certain generalized q-exponential function. Let $P(X) \in \mathbb{Z}[X]$ be a non-constant polynomial such that $P(q^t) \neq 0$ for all $t \in \mathbb{N}$. Then, the generalized q-exponential function with respect to P given by

$$E_{q,P}(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{t=1}^{n} P(q^t)}.$$

If $P(X) = X - 1$, $E_{q,P}(x) = E_q(x)$, the q-exponential function. Note that $E_{q,P}(x)$ is a basic hypergeometric series, as defined in [5].

With these notations, the first result of this paper is stated below.

Theorem 1.1. *Let q be such that $\pm q$ is a PV number. Let $P(X) = L_D X^D + \cdots + c_d X^d \in \mathbb{Z}[X]$ be a non-constant polynomial with $P(q^t) \neq 0$ for $t \geq 1$, $d \leq D$ and $L_D\, c_d \neq 0$. Let $\alpha_1, \ldots, \alpha_m$ be non-zero algebraic integers in $\mathbb{Q}(q)$ satisfying*

$$|c_d|^{n_q - 1} \max\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_m|\} \prod_{l=2}^{n_q} \max\{1, |\sigma_l(\alpha_1)|, |\sigma_l(\alpha_2)|, \cdots, |\sigma_l(\alpha_m)|\} < |q|^D. \quad (1)$$

Suppose that $\alpha_{k_1}/\alpha_{k_2}$ is not a root of unity for $1 \leq k_1, k_2 \leq m$ and $k_1 \neq k_2$. Then the numbers in the set

$$S := \left\{ E_{q,P}^{(j)}(\alpha_k) \,:\, 1 \leq k \leq m,\, 0 \leq j \leq M \right\} \cup \{1\}$$

are linearly independent over the field $\mathbb{Q}(q)$.

The following is an immediate corollary of this theorem.

Corollary 1.1. *Let q be such that $\pm q$ is a PV number. Let $\alpha_1, \ldots, \alpha_m$ be non-zero algebraic integers in $\mathbb{Q}(q)$ satisfying*

$$\max\{|\alpha_1|,\, |\alpha_2|,\, \cdots,\, |\alpha_m|\} \prod_{l=2}^{n_q} \max\{1,\, |\sigma_l(\alpha_1)|\,,\, |\sigma_l(\alpha_2)|\,,\, \cdots,\, |\sigma_l(\alpha_m)|\} < |q|.$$

Suppose that $\alpha_{k_1}/\alpha_{k_2}$ is not a root of unity for $1 \le k_1,\, k_2 \le m$ and $k_1 \ne k_2$. Then the numbers in the set

$$S := \left\{ E_q^{(j)}(\alpha_k) \,:\, 1 \le k \le m,\, 0 \le j \le M \right\} \cup \{\,1\,\}$$

are linearly independent over the field $\mathbb{Q}(q)$.

In particular, this implies the following about the special functions discussed earlier.

Corollary 1.2. *Let q be such that $\pm q$ is a PV number and $\alpha \in \mathcal{O}_q$ satisfies*

$$0 < \Big(\min\{1,\, |\alpha|\} \Big) H_q(\alpha) < |q|.$$

Then the values $E_q(\alpha)$, $L_q(\alpha) \notin \mathbb{Q}(q)$. In particular, $\zeta_q(1)$ is irrational when $\pm q$ is a PV number.

The irrationality and linear independence of the values of the q-logarithm function have been extensively studied by various authors. We refer to [10] for a comprehensive history of the problem and investigation of the values of a generalization of the q-logarithm function. The irrationality of $\zeta_q(1)$ when q is an integer was first obtained by Erdős [4]. More recently, Y. Tachiya [9, Theorem 2] proved that $\zeta_q(1) \notin \mathbb{Q}(q)$ when q is a PV number, which is also implied by Corollary 1.2 above.

A special function that is closely related to the study of the q-exponential function is the Tschakaloff function, given by

$$T_q(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{q^{\frac{n(n+1)}{2}}}.$$

In our notation, $T_q(x) = E_{q,I}(x)$, where $I(x) = x$. Thus, Theorem 1.1 implies the following.

Corollary 1.3. *Let q be such that $\pm q$ is a PV number. Suppose that $\alpha \in \mathcal{O}_q$ satisfies*

$$0 < \Big(\min\{1,\, |\alpha|\} \Big) H_q(\alpha) < |q|.$$

Then the numbers

$$1,\, T_q(\alpha),\, T_q^{(1)}(\alpha),\, \cdots,\, T_q^{(m)}(\alpha)$$

are linearly independent over $\mathbb{Q}(q)$.

It was brought to our notice by the referee that Theorem 1.1 follows from Corollaries 5.1 and 5.2 in [1], which require a much weaker condition on the α_k's than in Theorem 1.1. The authors, M. Amou, T. Matala-Aho and K. Väänänen, prove a general result regarding linear independence of values of solutions to q-difference equations in [1]. As such, the techniques necessary to prove the result in [1] are involved, whereas the proof of Theorem 1.1 provided in this paper follows from relatively elementary considerations.

The statements so far were concerned with the independence of values of a single function and its derivatives at several arguments. We now address the question of independence of different cognate functions at the same argument. First, for any $M \in \mathbb{N}$ and for any q with $|q| > 1$, we define an arithmetic progression analogue of the $E_q(x)$ as

$$E_{q,M}(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{[Mn]_q!},$$

which is an entire function. Clearly $E_{q,1}(x) = E_q(x)$ and

$$E_{q,M}(x^M) = 1 + \sum_{\substack{n=1,\\ n \equiv 0 \bmod M}}^{\infty} \frac{x^n}{[n]_q!}.$$

Note that $E_{q,M}$ is not a basic hypergeometric function.

For these special functions, we prove the following theorem.

Theorem 1.2. *Let q be such that $\pm q$ is a PV number and $a_1 < \cdots < a_k$ be distinct positive integers. Let $\alpha \in \mathcal{O}_q$ be such that $1 \le |\alpha|$ and*

$$H_q(\alpha) < |q|^{a_1}. \tag{2}$$

Then the numbers

$$1, E_{q,a_1}(\alpha), \ldots, E_{q,a_k}(\alpha) \tag{3}$$

are linearly independent over the field $\mathbb{Q}(q)$.

The approach in this paper is an adaptation of the proof of Theorem 1.1 in [7], which is a slight modification of the argument by Duverney [3]. In essence, it is similar to Fourier's proof of irrationality of the number e. The proof of Theorem 1.1 relies on a Diophantine lemma, which is a consequence of the Skolem-Mahler-Lech theorem. The proof of Theorem 1.2 is completed using a recursive elimination argument.

2. **Proof of the theorems**

An important ingredient in the proofs is the following particular case of the Skolem-Mahler-Lech theorem [2, Theorem 4.3, page 124].

Theorem 2.1. *Let $\alpha_1, \ldots, \alpha_k$ be non-zero algebraic numbers such that α_i/α_j is not a root of unity for $1 \le i < j \le k$. Let $P_1(x), \ldots, P_k(x)$ be non-zero polynomials with algebraic coefficients. Then, there are only finitely many natural numbers n satisfying*

$$P_1(n)\,\alpha_1^n + \cdots + P_k(n)\,\alpha_k^n = 0.$$

This is immediate from the Skolem-Mahler-Lech theorem since the sequence $P_1(n)\,\alpha_1^n + \cdots + P_k(n)\alpha_k^n$ is a non-degenerate recurrence sequence if α_i/α_j is not a root of unity.

2.1. **Proof of Theorem 1.1.** To begin with, let $f_j(x) := x^j E_{q,P}^{(j)}(x)$ for each $0 \le j \le M$. Observe that the result follows if we show that 1 and the values $f_j(\alpha_k)$ are $\mathbb{Q}(q)$-linearly independent for $0 \le j \le M$ and $1 \le k \le m$. Indeed, suppose that ξ_0 and $\xi_{j,k}$ are algebraic numbers in $\mathbb{Q}(q)$ for $1 \le k \le m$ and $0 \le j \le M$, not all zero, such that

$$\xi_0 + \sum_{j=0}^{M} \sum_{k=1}^{m} \xi_{j,k}\, E_{q,P}^{(j)}(\alpha_k) = 0.$$

Then we obtain the non-trivial linear relation

$$\xi_0 + \sum_{j=0}^{M} \sum_{k=1}^{m} \frac{\xi_{j,k}}{\alpha_k^j}\, f_j(\alpha_k) = 0,$$

which again has coefficients in $\mathbb{Q}(q)$. Thus it suffices to establish the linear independence of $f_j(\alpha_k)$'s over $\mathbb{Q}(q)$.

Let $r_0(X) = 1$. For $1 \le j \le M$, let $r_j(X) := X(X-1)\cdots(X-j+1)$. Then

$$f_j(x) = \sum_{n=j}^{\infty} \frac{r_j(n)\, x^n}{\prod_{t=1}^{n} P(q^t)} = \sum_{n=1}^{\infty} \frac{r_j(n)\, x^n}{\prod_{t=1}^{n} P(q^t)},$$

as $r_j(n) = 0$ for $0 \le n \le j-1$. Now suppose λ_0 and $\lambda_{j,k} \in \mathbb{Q}(q)$ are such that

$$\lambda_0 + \sum_{j=0}^{M} \sum_{k=1}^{m} \lambda_{j,k}\, f_j(\alpha_k) = 0.$$

Without loss of generality, we can assume that λ_0 and $\lambda_{j,k}$'s are algebraic integers. For $1 \le k \le m$ let $A_k(X) := \sum_{j=0}^{M} \lambda_{j,k}\, r_j(X)$. Then using the definition of $E_{q,P}(x)$, we get

$$\widetilde{\lambda_0} + \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{m} A_k(n)\, \alpha_k^n}{\prod_{t=1}^{n} P(q^t)} = 0,$$

where $\widetilde{\lambda_0} = \lambda_0 + \sum_{k=1}^{m} \lambda_{0,k}$.

Let N be a sufficiently large positive integer. We truncate the above infinite sum at N and clear denominators to obtain

$$\widetilde{\lambda_0}\left(\prod_{t=1}^{N} P(q^t)\right) + \sum_{n=1}^{N}\left(\sum_{k=1}^{m} A_k(n)\, \alpha_k^n\right) \prod_{t=n+1}^{N} P(q^t) = -\left(\prod_{t=1}^{N} P(q^t)\right) \sum_{n=N+1}^{\infty} \frac{\sum_{k=1}^{m} A_k(n)\, \alpha_k^n}{\prod_{t=1}^{n} P(q^t)}. \quad (4)$$

Denote the left hand side of the above equation as X_N, namely,

$$X_N := \widetilde{\lambda_0}\left(\prod_{t=1}^{N} P(q^t)\right) + \sum_{n=1}^{N}\left(\sum_{k=1}^{m} A_k(n)\, \alpha_k^n\right) \prod_{t=n+1}^{N} P(q^t). \quad (5)$$

Then $X_N \in \mathcal{O}_q$. Moreover, the right hand side of (4) can be written as

$$\left(\prod_{t=1}^{N} P(q^t)\right) \sum_{n=N+1}^{\infty} \frac{\sum_{k=1}^{m} A_k(n)\, \alpha_k^n}{\prod_{t=1}^{n} P(q^t)} = \frac{\sum_{k=1}^{m} A_k(N+1)\alpha_k^{N+1}}{P(q^{N+1})} + \frac{1}{P(q^{N+1})} \sum_{n=2}^{\infty} \frac{\sum_{k=1}^{m} A_k(N+n)\, \alpha_k^{N+n}}{\prod_{t=N+2}^{N+n} P(q^t)}. \quad (6)$$

For simplicity of notation, let

$$\boldsymbol{\alpha} := \max\{|\alpha_1|, |\alpha_2|, \cdots, |\alpha_m|\}$$

Using triangle inequality and the fact that each $A_k(X)$ is a polynomial of degree M, we get that for all $\nu > 0$,

$$\left|\sum_{k=1}^{m} A_k(\nu)\, \alpha_k^{\nu}\right| \leq \boldsymbol{\alpha}^{\nu} \sum_{k=1}^{m} |A_k(\nu)| \ll \nu^M \boldsymbol{\alpha}^{\nu}.$$

Also since $|P(q^t)| \sim |q|^{tD}$ for t sufficiently large, the second term in (6) can be seen to be bounded by

$$\left|\frac{1}{P(q^{N+1})}\sum_{n=2}^{\infty}\frac{\sum_{k=1}^{m} A_k(N+n)\,\alpha_k^{N+n}}{\prod_{t=N+2}^{N+n} P(q^t)}\right| \ll \frac{\boldsymbol{\alpha}^{N+1}}{|P(q^{N+1})|}\sum_{n=2}^{\infty}(n+N)^M \cdot \left(\frac{\boldsymbol{\alpha}}{|q|^{DN}}\right)^{n-1} \cdot |q|^{\frac{-D\left(n^2+n-2\right)}{2}}.$$

The above infinite series converges absolutely as $|q| > 1$ and the terms decay exponentially. Applying these bounds to the expression in (6) gives

$$|X_N| \ll \frac{\boldsymbol{\alpha}^{N+1}}{|P(q^{N+1})|}\, N^M, \tag{7}$$

where the implied constant depends on q, α_k's and the coefficients $\lambda_{j,k}$'s.

We now estimate the size of conjugates of X_N. Since $\pm q$ is a PV number, $|\sigma_l(q)| < 1$ for $2 \leq l \leq n_q$. From the expression for X_N in (5), we have for all $n \geq 0$,

$$\sigma_l(X_N) = \sigma_l(\widetilde{\lambda_0})\left(\prod_{t=1}^{N} P\left(\sigma_l(q)^t\right)\right) + \sum_{n=1}^{N}\left(\sum_{k=1}^{m}\sigma_l(A_k(n))\,\sigma_l(\alpha_k)^n\right)\prod_{t=n+1}^{N} P\left(\sigma_l(q)^t\right).$$

Observe that

$$\left|\prod_{t=n+1}^{N} P\left(\sigma_l(q^t)\right)\right| = \left|c_d\left(\prod_{t=n+1}^{N}\sigma_l(q^t)\right)^d\right|^{N-n} \prod_{t=n+1}^{N}\left|1 + \cdots + \frac{L_D}{c_d}\left(\sigma_l(q^t)\right)^{D-d}\right|.$$

Since $|\sigma_l(q)| < 1$ for all $2 \leq l \leq n_q$, the series

$$\sum_{t=1}^{\infty}\left(\sigma_l\left(q^t\right)\right)^s$$

is absolutely convergent for all $1 \leq s \leq D-d$. Thus, the infinite product

$$\prod_{t=1}^{\infty}\left|1 + \cdots + \frac{L_D}{c_d}\left(\sigma_l(q^t)\right)^{D-d}\right|$$

is convergent and we have

$$\left|\prod_{t=n+1}^{N} P\left(\sigma_l(q)^t\right)\right| \ll |c_d|^{N-n} \prod_{t=n+1}^{N}\left|\left(\sigma_l(q^t)\right)^{d(N-n)}\right| \ll |c_d|^{N-n},$$

as $|\sigma_l(q)| < 1$ for all $2 \leq l \leq n_q$. By these observations, we get that

$$|\sigma_l(X_N)| \ll |c_d|^N\left(1 + \sum_{n=1}^{N}|c_d|^{-n}\left(\sum_{k=1}^{m}|\sigma_l(A_k(n))|\,|\sigma_l(\alpha_k)|^n\right)\right).$$

Note that $c_d \in \mathbb{Z}$, Hence, $|c_d| \geq 1$. Now, $\sigma_l\left(A_k(n)\right) = \sum_{j=0}^{M} \sigma_l\left(\lambda_{j,k}\right) r_j(n)$, which is again a polynomial of degree M in n. Putting these bounds together, we deduce that

$$|\sigma_l(X_N)| \ll N^{M+2} \left|c_d\right|^N \left(\max\left\{1, \left|\sigma_l(\alpha_1)\right|, \cdots, \left|\sigma_l(\alpha_m)\right|\right\} \right)^N. \tag{8}$$

As before, the implied constant above only depends on q, α_k's and the $\lambda_{j,k}$'s.

Multiplying the absolute values of all the conjugates of X_N and the corresponding bounds in (7) and (8), we derive that

$$\begin{aligned}
&\prod_{l=1}^{n_q} |\sigma_l(X_N)| \\
&\ll \frac{N^{n_q(M+2)-2} \left|c_d\right|^{(n_q-1)N} \boldsymbol{\alpha}^N}{|P(q^{N+1})|} \left(\prod_{l=2}^{n_q} \max\left\{1, \left|\sigma(\alpha_1)\right|, \cdots, \left|\sigma(\alpha_m)\right|\right\} \right)^N \\
&\ll N^{n_q(M+2)-2} \left(\frac{\boldsymbol{\alpha} \left|c_d\right|^{(n_q-1)} \prod_{l=2}^{n_q} \max\left\{1, \left|\sigma(\alpha_1)\right|, \cdots, \left|\sigma(\alpha_m)\right|\right\}}{|q|^D} \right)^N
\end{aligned}$$

By the hypothesis (1), the above bound tends to zero as $N \to \infty$. In particular, this implies that

$$\left| \prod_{l=1}^{n_q} \sigma_l(X_N) \right| < 1$$

for all N sufficiently large. However, the left hand side above is a power of the norm of an algebraic integer. Note here that $\mathbb{Q}(X_N)$ may be a strict sub-field of $\mathbb{Q}(q)$. Thus, $\prod_{l=1}^{n_q} \sigma_l(X_N)$ must be a rational integer for all $N > 0$. This is only possible if $X_N = 0$ for *all* N sufficiently large.

Therefore, there exists a natural number N_0 such that for all $N \geq N_0$,

$$\frac{X_N}{\prod_{t=1}^{N} P(q^t)} = \widetilde{\lambda_0} + \sum_{n=1}^{N} \sum_{k=1}^{m} A_k(n)\, \alpha_k^n = 0.$$

Thus, considering the expression,

$$\frac{X_{N+1}}{\prod_{t=1}^{N+1} P(q^t)} - \frac{X_N}{\prod_{t=1}^{N} P(q^t)},$$

which equals zero for $N > N_0$, we obtain that

$$A_1(N)\, \alpha_1^N + \cdots + A_m(N)\, \alpha_m^N = 0,$$

for all $N > N_0$. As $\alpha_{k_1}/\alpha_{k_2}$ is not a root of unity, Theorem 2.1 applied to the above relation immediately gives that $A_k(N) = 0$ for all $1 \leq k \leq m$ and $N > N_0$. Thus, the polynomials $A_k(X)$ are identically zero. Recall that

$$A_k(X) = \sum_{j=0}^{M} \lambda_{j,k}\, r_j(X),$$

and $\deg r_j(X) = j$. Since $r_j(X)$ have distinct degrees, $A_k(X)$ is identically zero if and only if $\lambda_{j,k} = 0$ for all $0 \leq j \leq M$ and $1 \leq k \leq m$. This completes the proof of the theorem. □

2.2. **Proof of Theorem 1.2.** We begin along the same lines as the argument in the proof of Theorem 1.1.

Suppose that the numbers in (3) are linearly dependent over $\mathbb{Q}(q)$. Then, there exist algebraic integers $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathcal{O}_q$ not all zero such that

$$\lambda_0 + \lambda_1 E_{q,a_1}(\alpha) + \cdots + \lambda_k E_{q,a_k}(\alpha) = 0.$$

Without loss of generality, we can assume that $\lambda_1 \neq 0$. For otherwise, we can change notation to replace a_j by a_1 for the smallest $j \leq k$ for which $\lambda_j \neq 0$ and work out the argument below.

Using the definition of the q-exponential function, we get

$$\widetilde{\lambda_0} + \sum_{n=1}^{\infty} \frac{\lambda_1\, \alpha^n}{[a_1 n]_{q!}} + \cdots + \sum_{n=1}^{\infty} \frac{\lambda_k\, \alpha^n}{[a_k n]_q!} = 0, \tag{9}$$

where $\widetilde{\lambda_0} = \lambda_0 + \lambda_1 + \cdots + \lambda_k$. Set $d = \operatorname{lcm}\{a_1, \ldots, a_k\}$ and $d_i = d/a_i$. Choose a large positive integer N which is our parameter and set $N_i = N d_i$ for $i = 1, 2, \ldots, k$. With these choices of N_i, we have

$$a_1 N_1 = a_2 N_2 = \cdots = a_k N_k = dN.$$

Furthermore, for all $i = 1, 2, 3, \ldots, k$,

$$\frac{[dN]_q!}{[a_i(N_i+1)]_q!} = \frac{[a_i N_i]_q!}{[a_i(N_i+1)]_q!} = \frac{(q^{a_i N_i}-1)\cdots(q-1)}{(q^{a_i N_i + a_i}-1)\cdots(q-1)} = \frac{1}{(q^{Nd+a_i}-1)\cdots(q^{Nd+1}-1)}. \tag{10}$$

Now truncate the infinite sums in (9) at N_i's and multiply by $[dN]_q!$ to get

$$\begin{aligned}[dN]_q!\Big(\widetilde{\lambda_0} + \sum_{n=1}^{N_1} \frac{\lambda_1\, \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=1}^{N_k} \frac{\lambda_k\, \alpha^n}{[a_k n]_q!}\Big) \\ = -[dN]_q! \left(\sum_{n=N_1+1}^{\infty} \frac{\lambda_1\, \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=N_k+1}^{\infty} \frac{\lambda_k\, \alpha^n}{[a_k n]_q!} \right).\end{aligned} \tag{11}$$

Let X_N denote the left hand side of the above equation, i.e.,

$$X_N := [dN]_q! \left(\widetilde{\lambda_0} + \sum_{n=1}^{N_1} \frac{\lambda_1\, \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=1}^{N_k} \frac{\lambda_k\, \alpha^n}{[a_k n]_q!} \right). \tag{12}$$

Since $[dN]_q! = [a_i N_i]_q!$ for $1 \leq i \leq k$, X_N is an algebraic integer in $\mathcal{O}_q$. We now estimate the right hand side of (11). Note that by an argument similar to the one in Theorem 1.1 and using (10), we can deduce

$$\left| [dN]_q! \sum_{n=N_j+1}^{\infty} \frac{\alpha^n}{[a_j n]_q!} \right| \ll \frac{|\alpha|^{N_j}}{|q|^{a_j dN}} \ll \left(\left| \frac{\alpha^{d_j}}{q^{a_j d}} \right| \right)^N,$$

since $N_j = N d_j$. Since $a_1 < a_2 < \cdots < a_k$, $d_1 > d_2 > \cdots > d_k$ and $|\alpha| \geq 1$, we derive that

$$|X_N| \ll \left(\left| \frac{\alpha^{d_1}}{q^{a_1 d}} \right| \right)^N. \tag{13}$$

By the same argument as in the proof of Theorem 1.1, we obtain the following estimate for the conjugates of X_N:

$$|\sigma_l(X_N)| \ll N_1 \left(\max\{1, |\sigma_l(\alpha)|\}\right)^{N_1}. \tag{14}$$

As before, the implied constant above only depends on q, a_i's and the $\lambda_{j,k}$'s. Multiplying the bounds (13) and (14) on the absolute values of all the conjugates of X_N and noting that $|\alpha| \geq 1$, we derive that

$$\prod_{l=1}^{n_q} |\sigma_l(X_N)| \ll N_1^{n_q-1} \left(\frac{|\alpha| \prod_{l=2}^{n_q} \max\{1, |\sigma_l(\alpha)|\}}{|q|^{a_1^2}} \right)^{d_1 N}.$$

By (2), the right hand side above tends to zero as $N \to \infty$. However, the left hand side is a rational integer since it is a power of the norm of an algebraic integer. Therefore, there exists a natural number N_0 such that for *all* $N > N_0$, $X_N = 0$, which in turn implies that $X_N = X_{N+1} = 0$. Consequently,

$$\widetilde{\lambda_0} + \sum_{n=1}^{Nd_1} \frac{\lambda_1 \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=1}^{Nd_k} \frac{\lambda_k \alpha^n}{[a_k n]_q!} = 0$$

and

$$\widetilde{\lambda_0} + \sum_{n=1}^{Nd_1+d_1} \frac{\lambda_1 \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=1}^{Nd_k+d_k} \frac{\lambda_k \alpha^n}{[a_k n]_q!} = 0$$

for all $N > N_0$. By subtracting these two relations, we get that

$$\lambda_1 \sum_{n=Nd_1+1}^{Nd_1+d_1} \frac{\alpha^n}{[a_1 n]_q!} + \cdots + \lambda_k \sum_{n=Nd_k+1}^{Nd_k+d_k} \frac{\alpha^n}{[a_k n]_q!} = 0 \tag{15}$$

for all $N > N_0$. Note that for $1 \leq j \leq k$, we have $Nd + a_j \leq a_j n \leq Nd + d$ in the above sums. Therefore, the term

$$\frac{\alpha^{Nd_k+1}}{[Nd + a_1]_q!}$$

divides each term in the above relation. Factoring this out, we obtain

$$\begin{aligned}
&\lambda_1 \left(\alpha^{N(d_1-d_k)} + \frac{\alpha^{N(d_1-d_k)+1}}{(q^{Nd+2a_1}-1)\cdots(q^{Nd+a_1+1}-1)} + \cdots + \frac{\alpha^{N(d_1-d_k)+d_1-1}}{(q^{Nd+d}-1)\cdots(q^{Nd+a_1+1}-1)} \right) + \\
&\lambda_2 \left(\frac{\alpha^{N(d_2-d_k)}}{(q^{Nd+a_2}-1)\cdots(q^{Nd+a_1+1}-1)} + \cdots + \frac{\alpha^{N(d_2-d_k)+d_2-1}}{(q^{Nd+d}-1)\cdots(q^{Nd+a_1+1}-1)} \right) \\
&+ \\
&\vdots \\
&+ \\
&\lambda_k \left(\frac{1}{(q^{Nd+a_k}-1)\cdots(q^{Nd+a_1+1}-1)} + \cdots + \frac{\alpha^{d_k-1}}{(q^{Nd+d}-1)\cdots(q^{Nd+a_1+1}-1)} \right) = 0.
\end{aligned} \tag{16}$$

Now, for $1 \leq j \leq k$ and $0 \leq l \leq d_j - 1$, the absolute value of a general term is of the form

$$\left| \frac{\alpha^{N(d_j-d_k)+l}}{(q^{Nd+(l+1)a_j}-1)\cdots(q^{Nd+a_1+1}-1)} \right| \ll \left| \frac{\alpha^{d_1-d_k}}{q^{\delta d}} \right|^N$$

except for $j = 1$ and $l = 0$, with $\delta = \min\{a_1, a_2 - a_1\}$. Since $1 \leq \delta$, this implies that each term in the above relation is $\ll |\alpha^{d_1-d_k}/q^d|^N$. By (2), this quotient is less than 1 as $1 \leq |\alpha| < |q|^{a_1}$. Hence, taking the limit as $N \to \infty$, all terms in (16) tend to zero except the very first one, that is,

$\alpha^{N(d_1-d_k)}$. This implies that $\lambda_1 = 0$, which is a contradiction. This completes the proof.

□

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