THE OKADA SPACE AND VANISHING OF $L(1, f)$

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Abstract. Fix a positive integer $N \geq 2$. Following Chowla, we associate to each function $f : \mathbb{Z} \to \mathbb{C}$ with period N, the L-series $L(s, f) := \sum_{n \geq 1} f(n)/n^s$. Using a characterization derived by Okada for the vanishing of $L(1, f)$, we construct an explicit basis for the $\mathbb{Q}$-vector space,

$$\mathscr{O}(N) = \{f \bmod N \,:\, f(n) \in \mathbb{Q},\ L(1, f) = 0\}.$$

We analyze the structure of this space and use the explicit basis to extend earlier works of Baker-Birch-Wirsing and Murty-Saradha. The arithmetical nature of Euler's constant γ emerges as a central question in these extensions.

1. Introduction

For a positive integer $N \geq 2$, let $f : \mathbb{Z} \to \mathbb{C}$ be periodic with period N. The L-series associated to f is given by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$. If $\zeta(s, x)$ denotes the Hurwitz zeta function

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \qquad 0 < x \leq 1,$$

then

$$L(s, f) = \frac{1}{N^s} \sum_{a=1}^{N} f(a)\, \zeta(s, a/N).$$

The Hurwitz zeta function extends analytically to the entire complex plane except for $s = 1$ where it has a simple pole with residue 1. As a result, $L(s, f)$ also extends analytically to the entire complex plane except for a simple pole at $s = 1$ with residue

$$\frac{1}{N} \sum_{a=1}^{N} f(a).$$

Thus, $L(1, f)$ exists if and only if

$$\sum_{a=1}^{N} f(a) = 0. \tag{1}$$

Date: April 7, 2021.

2020 *Mathematics Subject Classification.* 11J72, 11J86.

Key words and phrases. Okada's criterion, vanishing of $L(1, f)$, values of the digamma function.

Research of the author was partially supported by an NSERC Discovery grant. Research of the second author was supported by an S. Chowla fellowship.

If $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, then it is known that

$$\zeta(s,x) = \frac{1}{s-1} - \psi(x) + O(s-1)$$

as $s \to 1^+$, and we find

$$L(1,f) = -\frac{1}{N}\sum_{a=1}^{N} f(a)\psi\left(\frac{a}{N}\right) \tag{2}$$

provided (1) holds. Moreover, if $\widehat{f}(m) := N^{-1} \sum\limits_{a=1}^{N} f(a)e^{-2\pi i a m/N}$, then it is not difficult to see by Parseval's theorem that

$$\sum_{a=1}^{N} f(a)\,\psi\left(\frac{a}{N}\right) = N\sum_{a=1}^{N-1}\left(\widehat{f}(a)\log\left(1-e^{2\pi i a/N}\right)\right) + \left(\sum_{a=1}^{N} f(a)\right)\left(\frac{1}{N}\sum_{a=1}^{N}\psi\left(\frac{a}{N}\right)\right).$$

Therefore, together with (1), we have another expression for the value $L(1,f)$, namely,

$$L(1,f) = -\sum_{a=1}^{N-1}\widehat{f}(a)\,\log\left(1-e^{2\pi i a/N}\right). \tag{3}$$

For more details, we refer the reader to [16, Section 2.2, pg. 14].

Motivated by Dirichlet's theorem that $L(1,\chi) \neq 0$ for any non-principal Dirichlet character χ, the study of non-vanishing of the value $L(1,f)$ was initiated by S. Chowla in the 1960s and was later developed by several authors beginning with the work of Baker, Birch and Wirsing [2]. In this paper, we are interested in studying the $\mathbb{Q}$-vector spaces

$$\begin{aligned} F(N) &:= \left\{f:\mathbb{Z}\to\mathbb{Q}\,\middle|\, f(n+N)=f(n) \text{ for all } n\in\mathbb{Z}\right\}, \\ F^{(0)}(N) &:= \left\{f\in F(N)\,\middle|\, \sum_{a=1}^{N} f(a)=0\right\}, \text{ and} \\ \mathscr{O}(N) &:= \left\{f\in F^{(0)}(N)\,\middle|\, L(1,f)=0\right\}. \end{aligned} \tag{4}$$

Our analysis remains unchanged if the functions $f \in F(N)$ are allowed to take values in a number field K, which is disjoint from the N-th cyclotomic field $\mathbb{Q}(e^{2\pi i/N})$. For simplicity however, we restrict ourselves to the case when the functions are rational valued.

From the perspective of (2), our focus then is the study of $\mathbb{Q}$-linear relations among values of the digamma function $\psi(a/N)$, with $1 \leq a \leq N$. Inspired by this objective, we define the following $\mathbb{Q}$-vector spaces:

$$\begin{aligned} \mathscr{D}(N) &:= \mathbb{Q}\text{-span of } \left\{\psi\left(\frac{a}{N}\right) : 1\leq a\leq N\right\}, \\ \mathscr{D}^{(0)}(N) &:= \mathbb{Q}\text{-span of } \left\{L(1,f) : f\in F^{(0)}(N)\right\}. \end{aligned}$$

By equation (2), $\mathscr{D}^{(0)}(N) \subseteq \mathscr{D}(N)$ and the question arises if this containment is strict. Since $-\psi(1) = \gamma$ is Euler's constant, it is an element of $\mathscr{D}(N)$. However, it is unlikely that it is an element of $\mathscr{D}^{(0)}(N)$ and thus we conjecture that the containment is strict. In fact, we will provide evidence for the conjecture that

$$\mathscr{D}(N) = \mathscr{D}^{(0)}(N) \oplus \gamma\,\mathbb{Q}.$$

In Section 4, we will show that if there is an $N \geq 2$ such that $\mathscr{D}^{(0)}(N) = \mathscr{D}(N)$, then for every $M \geq 2$ which is coprime to N, the containment is strict. That is, $\mathscr{D}^{(0)}(M) \subsetneq \mathscr{D}(M)$.

An expanded discussion of these vector spaces and methods of viewing this conjecture are found in section 4. In the penultimate section of this paper, we describe a Galois action on $\mathscr{O}(N)$ and study this representation.

But we begin with the problem of finding explicit bases for the vector spaces (4). To this end, we will use earlier works of Okada [13], [14]. In [13, Theorem 10], the following theorem was proved.

Theorem 1.1 (Okada)**.** *Fix a positive integer $N \geq 2$ and let $\omega(N)$ be the number of distinct prime factors of N and ϕ the Euler totient function. Fix a function $f \in F(N)$. Then the value $L(1, f)$ exists and equals zero, i.e., $f \in \mathscr{O}(N)$, if and only if $(f(1), f(2), \cdots, f(N))$ is a solution to the following system of $\phi(N) + \omega(N)$ linear equations with rational coefficients:*

$$x_n + \sum_{\substack{a=1, \\ 1<(a,N)<N}}^{N} x_a\, A(a,n) + \frac{x_N}{\phi(N)} = 0, \qquad \textit{for each } 1 \leq n \leq N,\ (n,N) = 1 \tag{5}$$

$$\sum_{a=1}^{N} x_a \frac{\varepsilon(a,p)}{\varepsilon(p,p)} = 0, \qquad \textit{for each prime } p \mid N,$$

where $A(a,n)$ and $\varepsilon(a,p)$ are rational numbers defined as

$$A(a,n) = \sum_{m \in M(a,n)} \frac{1}{m}, \tag{6}$$

with M being the monoid generated by prime divisors of N, $M(a,n) = \{m \in M\ :\ mn \equiv a \bmod N\}$ and

$$\varepsilon(a,p) = \begin{cases} v_p(N) + \dfrac{1}{p-1} & \textit{if } v_p(a) \geq v_p(N), \\ v_p(a) & \textit{if } v_p(a) < v_p(N). \end{cases}$$

Here $v_p(a)$ is the power of the prime p that divides a.

An important point to note here is that the convergence condition (1) is included in the above $\phi(N) + \omega(N)$ linear equations. It is also worth remarking that $A(N,n) = 1/\phi(N)$ when n is coprime to N as is easily calculated. Then (5) can be expressed in the slightly abbreviated form:

$$x_n + \sum_{\substack{a=1, \\ 1<(a,N)\leq N}}^{N} x_a\, A(a,n) = 0, \qquad \text{for each } 1 \leq n \leq N,\ (n,N) = 1. \tag{7}$$

The rationality of the numbers $A(a,n)$ for $(n,N) = 1$ is not immediate from the definition above. Okada obtained an alternate expression for these numbers in [14] to deduce their rationality. We discuss this in detail in Section 3.

In principle, Theorem 1.1 gives us a characterization of when relations between special values of the digamma function can hold. In practice however, this is not the case. For example, there is the celebrated conjecture of Erdős [8]:

Conjecture 1.2. *Let f be an arithmetic function, periodic with period $N \geq 2$ such that $f(n) = \pm 1$ for $n \not\equiv 0 \pmod N$ and $f(N) = 0$. Then*

$$L(1,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

If (1) does not hold, then the value of the L-function is clearly non-zero, and so we may focus on the case when (1) holds. Looking at (1) modulo 2, we must clearly have N odd and so we need only consider the cases $N \equiv 1 \pmod 4$ and $N \equiv 3 \pmod 4$. In the latter case, the first author and Saradha proved the conjecture (see Theorem 7 of [12]). The case $N \equiv 1 \pmod 4$ remains open and Theorem 1.1 does not seem to help in resolving the issue, though it provides a wealth of relations among the values of a function f with $L(1,f) = 0$. Erdős's conjecture can be viewed as part of a larger question about simple relations among the values of the digamma function. To determine when such relations hold is a problem of equal and independent interest.

We will refer to the $\mathbb{Q}$-vector space $\mathscr{O}(N)$ defined in (4) as the *Okada space*. We first find an explicit basis for the Okada space. Using this basis, we extend previous theorems of Baker-Birch-Wirsing [2, Theorem 1] and Murty-Saradha [11, Theorem 4]. Taking the analysis further, we describe the $\mathbb{Q}$-vector space spanned by the values $L(1,f)$ as f ranges over functions in $F^{(0)}(N)$. Moreover, we initiate a systematic study of a variant of Erdős's conjecture. We conclude by providing an algebraic perspective to this study by recognizing $\mathscr{O}(N)$ as a rational linear representation of the group $(\mathbb{Z}/N\mathbb{Z})^*$.

2. An explicit basis for $\mathscr{O}(N)$

We say that a residue class $r \pmod N$ is *free* if r is not coprime to N and is not represented by a prime divisor of N. Using (7), the system of equations in Theorem 1.1 becomes

$$\begin{aligned} x_n &= -\sum_{p|N} A(p,n)\, x_p - \sum_{r\, free} A(r,n)\, x_r, \quad (n,N) = 1 \\ x_p &= -\sum_{p|r,\, r\, free} \frac{\varepsilon(r,p)}{\varepsilon(p,p)}\, x_r, \qquad p|N. \end{aligned}$$

Inserting the second set of formulas for x_p into the first set of formulas and interchanging summation, we get

$$\begin{aligned} x_n &= \sum_{r\, free} \Big\{ -A(r,n) + \sum_{p|(r,N)} A(p,n) \frac{\varepsilon(r,p)}{\varepsilon(p,p)} \Big\}\, x_r, \qquad (n,N) = 1 \\ x_p &= -\sum_{p|r,\, r\, free} \frac{\varepsilon(r,p)}{\varepsilon(p,p)}\, x_r, \qquad p|N. \end{aligned} \tag{8}$$

Thus, the following reordering of residue classes (mod N) suggests itself. First group together the residue classes that are coprime to N, then those represented by prime divisors of N and then the free residue classes, each in increasing order. On doing so, the system of equations in Theorem 1.1 can be interpreted as an $N \times N$ matrix equation, in *reduced row echelon form.*

Immediately, we see that the null space of the matrix or the Okada space $\mathscr{O}(N)$ has dimension $N - \phi(N) - \omega(N)$ where $\omega(N)$ is the number of distinct prime divisors of N. This observation enables us to write down an explicit basis for the Okada space. When N is prime, the Okada space is zero dimensional and so, henceforth, we will tacitly assume N is composite.

For each free residue class $r \pmod N$, let e_r be the column vector that satisfies $e_r(r) = 1$ and $e_r(b) = 0$ for every other free residue class b and which in addition satisfies (8). It is clear that these $N - \phi(N) - \omega(N)$ functions, e_r as r ranges over the free residue classes, are linearly independent and hence form a basis for the Okada space.

We now determine the functions $\{e_r : r \text{ - free}\}$ explicitly. Since e_r is only supported on $r \pmod N$ among the free residue classes, the coefficient of x_r in each set of equations is precisely $e_r(n)$ and $e_r(p)$ respectively. We then see that all the functions in the Okada space $\mathscr{O}(N)$ are given by making arbitrary selections for x_r with r free, and then determining x_n, for $(n, N) = 1$ and x_p for $p|N$, by the above system (8). From Theorem 1.1, we can compute the values $\varepsilon(r, p)$ and $\varepsilon(p, p)$ to be:

$$\varepsilon(p,p) = \begin{cases} 1 + \dfrac{1}{p-1} & \text{if } p \,||\, N \\ \\ 1 & \text{otherwise,} \end{cases} \qquad \varepsilon(r,p) = \begin{cases} v_p(N) + \dfrac{1}{p-1} & \text{if } v_p(r) \geq v_p(N) \\ \\ v_p(r) & \text{otherwise.} \end{cases}$$

Therefore we have two natural cases, namely, when $p \,||\, N$ and when $p^2 \mid N$. In the case when $p \,||\, N$, $\varepsilon(p, p) = p/(p-1)$, $\varepsilon(r, p) = 0$ for free classes 'r' such that $p \nmid r$ and $\varepsilon(r, p) = p/(p-1)$ for free classes 'r' with $p \mid r$. Hence, when $p \,||\, N$, we have that

$$e_r(p) = \begin{cases} -1 & \text{if } p \mid r, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Now consider the case when $p^2 \mid N$, so that $\varepsilon(p, p) = 1$. Thus, $e_r(p) = -\varepsilon(r, p)$, that is,

$$e_r(p) = \begin{cases} -v_p(N) - \dfrac{1}{p-1} & \text{if } v_p(r) \geq v_p(N), \\ \\ -v_p(r) & \text{if } v_p(r) < v_p(N). \end{cases} \tag{10}$$

In particular, $e_N(p) = -1$ if $p \,||N$ and

$$e_N(p) = -v_p(N) - \frac{1}{p-1}$$

otherwise.

Furthermore, if n is coprime to N, we can determine $e_r(n)$ as follows. From equation (7), we can write,

$$e_r(n) + \sum_{p|N} e_r(p)A(p, n) + e_r(r)A(r, n) = 0,$$

from which we deduce

$$e_r(n) = -A(r, n) - \sum_{p|N} e_r(p)A(p, n).$$

We summarise our discussion in the following table:

Table 1: Values of the basis functions e_r at coprime and prime arguments

n with $(n,N)=1$	$p \,\|\, N$	$p^2 \| N$
$e_r(n) = -A(r,n) - \sum_{p\|N} e_r(p)\,A(p,n)$	$e_r(p) = \begin{cases} -1 & \text{if } p \mid r, \\ 0 & \text{otherwise.} \end{cases}$	$e_r(p) = \begin{cases} -v_p(N) - \dfrac{1}{p-1} & \text{if } v_p(r) \geq v_p(N), \\ -v_p(r) & \text{if } v_p(r) < v_p(N). \end{cases}$

Thus, we have proved:

Theorem 2.1. *Let $N \geq 2$ be a composite integer. Then the functions e_r, as 'r' ranges over free residue classes (mod N), form a basis of the Okada space $\mathscr{O}(N)$. For a free residue class r, $e_r(r) = 1$ and $e_r(b) = 0$ if $b \neq r$ is free. For each prime $p \mid N$, the value $e_r(p)$ is given by* (9) *and* (10) *depending on whether $p \,\|\, N$ or $p^2 \mid N$ respectively. Finally, for every n coprime to N, we have*

$$e_r(n) = -A(r,n) - \sum_{p|N} e_r(p)\,A(p,n)$$

Here $A(r,n)$ is given by (6)*.*

As the set $\{e_r : r \pmod N \text{ is free}\}$ forms a basis of the Okada space $\mathscr{O}(N)$, any function f mod N for which $L(1,f) = 0$ can be expressed as

$$f(n) = \sum_{r\, free} f(r)\, e_r(n). \tag{11}$$

We apply this deduction in sections that follow.

3. Computing the numbers $A(r,n)$

The numbers $A(p,n)$ and $A(r,n)$ when n is coprime to N make an appearance in the explicit basis constructed in Theorem 2.1. More specifically, for any n coprime to N, we can rewrite (11) using Theorem 2.1, as:

$$f(n) = -\sum_{r\, free} f(r)A(r,n) - \sum_{p|N} A(p,n) \left(\sum_{r\, free} f(r) e_r(p) \right).$$

The inner sum in the second sum above equation can be recognized as $f(p)$ using (11). Therefore, we have

$$f(n) = -\sum_{r\, free} f(r)A(r,n) - \sum_{p|N} f(p)A(p,n).$$

Thus, the numbers $A(r,n)$ for all free residue classes r warrant explicit computation and careful study, which we initiate below.

The calculation of the sum $A(r,n)$ is straightforward in one particular case.

Theorem 3.1. *Let n be coprime to N. Suppose that $r \pmod N$ is a free residue class satisfying $v_p(r) < v_p(N)$ for every prime $p \mid N$. Writing $(r,N) = t$, we have*

$$A(r,n) = \begin{cases} 1/t & \text{if } n \equiv r/t \,(\text{mod } N/t) \\ \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Proof. Recall that

$$A(r,n) = \sum_{m \in M(r,n)} \frac{1}{m},$$

where $M(r,n)$ is the set of all $m \in M$ such that $mn \equiv r \pmod N$. Since n is coprime to N, this means that $t|m$ and writing $m = tm'$, we have $m'n \equiv r/t \pmod{N/t}$. The condition $v_p(r) < v_p(N)$ for every prime $p \mid N$ translates to the fact that the prime divisors of N/t are the same as the prime divisors of N.

Now m' is in the monoid generated by the prime divisors of N. Thus, our condition on $v_p(r)$ implies that the only solution of the congruence $m'n \equiv r/t \bmod (N/t)$ is $m' = 1$. This means that $n \equiv r/t \pmod{N/t}$. This proves (12). □

The following useful corollaries can be extracted from Theorem 3.1.

Corollary 3.2. *Let $N = p^k$ with $k \geq 2$ for a prime p. Let $r \pmod N$ be a non-zero free residue class. If $t = (r,N)$, then*

$$A(r,n) = \begin{cases} 1/t & \text{if } n \equiv r/t \,(\text{mod } N/t) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.3. *If $p^2|N$ for some prime p, then*

$$A(p,n) = \begin{cases} 1/p & \text{if } n \equiv 1 \,(\text{mod } N/p) \\ \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

A natural question is to evaluate $A(p,n)$ for primes p such that $p \,||\, N$. Again, following Okada's algorithm, we write $m = pm'$ and deduce $m'n \equiv 1 \pmod{N/p}$. But this time, m' being in the monoid generated by the prime divisors of N, we see that it must be a power of p. In particular, n has to be in the cyclic subgroup generated by $p \bmod N/p$, otherwise, $A(p,n) = 0$. Let $u = u_p(N/p)$ be the order of $p \pmod{N/p}$. Let w be the unique number with $0 \leq w < u$ such that $p^w n \equiv 1 \pmod{N/p}$. Then

$$A(p,n) = \frac{1}{p} \sum_{j=0}^{\infty} \frac{1}{p^{w+ju}} = \frac{1}{p^{w+1}} \frac{p^u}{p^u - 1} = \frac{p^{u-w-1}}{p^u - 1}.$$

We therefore deduce

$$A(p,n) = \begin{cases} \dfrac{p^{u-w-1}}{p^u - 1} & \text{if } n \in \langle p \rangle \text{ in } (\mathbb{Z}/(N/p)\mathbb{Z})^* \\ \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

This proves:

Theorem 3.4. *When n is coprime to N, the numbers $A(p,n)$ are given by (13) and (14) corresponding to the cases $p^2 \mid N$ and $p \,||\, N$ respectively.*

In absence of an explicit evaluation of $A(r,n)$, we explore alternative representations of these numbers, which give us further insight into their nature. In [13] and [14], Okada required only the rationality of the numbers $A(r,n)$. To do so, he proved that if $t=(r,N)$ and

$$S_t := \{p \text{ - prime} : p \mid N,\ p \nmid (N/t)\} = \{p \text{ - prime} : p \mid N,\ v_p(r) \geq v_p(N)\}, \tag{15}$$

then

$$A(r,n) = \frac{1}{t}\prod_{p\in S_t}\left(1-\frac{1}{p^{\phi(N)}}\right)^{-1}\sum_{m\in M_t(N)}\frac{\delta(r,n,m)}{m},$$

where $M_t(N) = \left\{\prod_{p\in S_t} p^{\alpha(p)} : 0 \leq \alpha(p) < \phi(N)\right\}$ and

$$\delta(r,n,m) = \begin{cases} 1 & \text{if } r \equiv mnt \bmod N, \\ 0 & \text{otherwise.}\end{cases}$$

In the theorem below, we obtain another expression for the numbers $A(r,n)$ which appears more viable to deduce their arithmetic properties.

Theorem 3.5. *Let n be coprime to N and $r \pmod N$ be a free residue class with $t=(r,N)$. Let S_t be the set defined in (15). Then*

$$t\,A(r,n) = \frac{1}{\phi(N/t)}\sum_{\chi}\overline{\chi}(r/t)\chi(n)\prod_{p\in S_t}\left(1-\frac{\chi(p)}{p}\right)^{-1}, \tag{16}$$

where the summation runs over Dirichlet characters $\chi \pmod{N/t}$.

Proof. The congruence $mn \equiv r \pmod N$ simplifies to $m'n \equiv r/t \pmod{N/t}$ where $m=tm'$. Since $(r/t, N/t)=1$, we see that m' must belong to M_t, the monoid generated by primes in S_t. We can use Dirichlet characters $\pmod{N/t}$ to sift our elements, to obtain

$$t\,A(r,n) = \frac{1}{\phi(N/t)}\sum_{m'\in M_t}\frac{1}{m'}\sum_{\chi}\overline{\chi}(r/t)\chi(m'n) = \frac{1}{\phi(N/t)}\sum_{\chi}\overline{\chi}(r/t)\chi(n)\prod_{p\in S_t}\left(1-\frac{\chi(p)}{p}\right)^{-1},$$

which proves the theorem. □

Theorem 3.5 gives an alternate proof of Theorem 3.1 because under the conditions of the latter theorem, the set S_t is empty and the result is now immediate from the orthogonality of Dirichlet characters. Another consequence of Theorem 3.5 is the corollary below.

Corollary 3.6. *With $t=(r,N)$, we have*

$$|A(r,n)| \leq \frac{1}{\phi(t)}.$$

Proof. The sum on the right hand side of (16) is bounded by

$$\prod_{p\in S_t}\left(1-\frac{1}{p}\right)^{-1} \leq \frac{t}{\phi(t)}$$

from which the corollary follows. □

An additional noteworthy feature of Theorem 3.5 is that if n and m are coprime to N and we write $t = (r, N)$, then $n \equiv m \pmod{N/t}$ implies $A(r, n) = A(r, m)$. It is also worth noting that $A(r, n) = A(-r, -n)$ which is evident from the definition as well.

It is easy to deduce rationality of the numbers $A(r, n)$ from Theorem 3.5 simply by noting that the right hand side of (16) is invariant under Galois automorphisms. Our formula also makes patently clear that the power of 2 in the denominator of $A(r, n)$ emerges not only from the presence of $\phi(N/t)$ but also from primes p appearing in the product.

4. **Applications of Theorem 2.1**

As a consequence of the explicit basis obtained in Theorem 2.1, we can deduce non-vanishing of $L(1, f)$ and linear independence of values of the digamma function. Furthermore, the arithmetic properties of the numbers $A(r, n)$ seem to hold the key towards establishing variants of Erdős's conjecture. We discuss these results below.

4.1. **Extension of the Baker-Birch-Wirsing theorem.** In the 1960s, S. Chowla asked the following question: suppose that f is a rational-valued function, periodic with an odd prime period p satisfying (1) and $f(p) = 0$. Then is it true that $L(1, f) \neq 0$? This problem was answered in the affirmative by Baker, Birch and Wirsing [2, Theorem 1] using Baker's theory of linear forms in logarithms of algebraic numbers. namely, they used the following theorem (see [1, Chapter 2]).

Theorem 4.1. *Let α_1, α_2, $\cdots$, α_n be non-zero algebraic numbers such that $\log(\alpha_1)$, $\log(\alpha_2)$, $\cdots$, $\log(\alpha_n)$ are $\mathbb{Q}$-linearly independent. Then 1, $\log(\alpha_1)$, $\log(\alpha_2)$, $\cdots$, $\log(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.*

More specifically, Baker, Birch and Wirsing proved that:

Theorem (Baker-Birch-Wirsing)**.** *Let f be a rational-valued function, periodic with period N and not identically zero. Suppose that $f(a) = 0$ for every a satisfying $1 < (a, N) < N$ and that (1) holds. Then $L(1, f) \neq 0$.*

As a consequence of Okada's Theorem 1.1 and the explicit basis we obtained in Theorem 2.1, we deduce the following extension of the Baker-Birch-Wirsing theorem.

Theorem 4.2. *If $f \in F^{(0)}(N)$ is such that $f(r) = 0$ for all free residue class $r \pmod{N}$, then $L(1, f) \neq 0$ unless $f \equiv 0$. Similarly, if $f \in F^{(0)}(N)$ is such that $f(r) = 0$ for all non-zero free residue classes $r \pmod{N}$, then $L(1, f) = 0$ if and only if $f = f(N)\, e_N$.*

Proof. Suppose that $L(1, f) = 0$. From (11), we have that $f(n) = \sum_{r\, free} f(r)\, e_r(n)$. The conclusions are now evident. □

In the Baker-Birch-Wirsing theorem, the support of f is contained in the coprime residue classes (mod N) along with the zero residue class. In our theorem above, the support is contained in the coprime residue classes (mod N) along with the residue classes $p \pmod{N}$ as p ranges over the prime divisors of N.

One can obtain the Baker-Birch-Wirsing theorem from Theorem 4.2. Indeed, suppose that f is as in the Baker-Birch-Wirsing theorem. By Theorem 4.2, $f(n) = f(N)e_N(n)$ for all $n \in \mathbb{Z}$. By Okada's theorem, if N is a prime, $\mathscr{O}(N)$ is empty. Thus, one can assume that N is not a prime.

Let p be a prime divisor of N. Then, $f(p) = 0$, which implies that $f(N)\, e_N(p) = 0$. However, it can be seen from Table 1 that $e_N(p) \neq 0$. Hence, $f(N) = 0$ implying that f is identically zero.

4.2. **Extension of the Murty-Saradha theorem.** In [11], the first author and N. Saradha explored the connection between vanishing of the value $L(1, f)$ and linear relations among values of the digamma function, $\psi(a/N)$ with $1 \leq a \leq N$, by virtue of (2). Using the Baker-Birch-Wirsing theorem, they showed:

Theorem (Murty-Saradha)**.** *The numbers $\psi(a/N)+\gamma$ with $1 \leq a < N$, $(a, N) = 1$ are $\mathbb{Q}$-linearly independent.*

Using Theorem 4.2, we obtain the following generalization.

Theorem 4.3. *Let $N > 1$. Let the set $\mathcal{S}$ consist of the $\phi(N) + \omega(N) - 1$ numbers*

$$\left\{\psi\left(\frac{a}{N}\right) + \gamma \,:\, 1 \leq a \leq N,\, (a, N) = 1\right\}$$

together with all but one element of

$$\left\{\psi\left(\frac{p}{N}\right) + \gamma \,:\, p \text{ - prime, } p \mid N\right\}.$$

Then the elements of $\mathcal{S}$ are linearly independent over $\mathbb{Q}$.

Proof. Suppose that

$$\sum_{(a,N)=1} c_a \left(\psi\left(\frac{a}{N}\right) + \gamma\right) + \sum_{p \mid N} c_p \left(\psi\left(\frac{p}{N}\right) + \gamma\right) = 0,$$

is a rational linear relation, with rational numbers c_a for $(a, N) = 1$ and c_p with $p \mid N$, with at least one of the c_p's zero.

Define the function f, periodic modulo N as $f(a) = c_a$ when $(a, N) = 1$, $f(p) = c_p$ when p is a prime such that $p \mid N$ and $f(N) = -\sum_{(a,N)=1} c_a - \sum_{p \mid N} c_p$. Then f satisfies (1) and

$$0 = \sum_{(a,N)=1} \left(c_a\, \psi\left(\frac{a}{N}\right)\right) + \sum_{p \mid N} \left(c_p\, \psi\left(\frac{p}{N}\right)\right) + \left(-\sum_{(a,N)=1} c_a - \sum_{p \mid N} c_p\right) \psi(1) = \sum_{a=1}^{N} f(a)\, \psi\left(\frac{a}{N}\right),$$

as $\psi(1) = -\gamma$. By (2), the above relation implies that $L(1, f) = 0$. Since $f = 0$ on all the non-zero free residue classes, Theorem 4.2 implies that $f(n) = f(N) e_N(n)$ for all $n \in \mathbb{Z}$.

If the number excluded from the set is $\psi(p/N) + \gamma$ for some prime divisor p of N, then $c_p = 0$. That is, $0 = f(p) = f(N) e_N(p)$. The value $e_N(p) \neq 0$, which can be immediately seen from Table 1. Thus, we deduce that $f(N) = 0$, which in turn implies that f is identically 0. So the relation among values of the set $\mathcal{S}$ was a trivial relation to begin with.
□

The above theorem leads to the natural question of whether *all* the numbers

$$\left\{\psi\left(\frac{a}{N}\right) + \gamma \,: 1 \leq a \leq N\, (a, N) = 1\right\} \cup \left\{\psi\left(\frac{p}{N}\right) + \gamma \,:\, p \text{ - prime, } p \mid N\right\}$$

are $\mathbb{Q}$-linearly independent. During the course of the proof of Theorem 4.3, we proved that any $\mathbb{Q}$-linear relation among the above numbers gives a function f, periodic mod N such that

$L(1,f)=0$ and $f=f(N)e_N$ and vice a versa. Thus, the function e_N gives rise to an explicit $\mathbb{Q}$-linear relation among the above numbers, that is,

$$\sum_{1\le a\le N,\,(a,N)=1} e_N(a)\,\left(\psi\left(\frac{a}{N}\right)+\gamma\right)+\sum_{p\mid N} e_N(p)\,\left(\psi\left(\frac{p}{N}\right)+\gamma\right)=0.$$

For an integer $N\ge 2$, let

$$\begin{aligned}\mathscr{D}(N)&:=\mathbb{Q}\text{-span of }\left\{\psi\left(\frac{a}{N}\right)\,:\,1\le a\le N\right\},\\ \mathscr{D}^{(0)}(N)&:=\mathbb{Q}\text{-span of }\left\{L(1,f)\,:\,f\in F^{(0)}(N)\right\},\\ \mathscr{D}_{\mathcal{S}}(N)&:=\mathbb{Q}\text{-span of numbers in }\mathcal{S}.\end{aligned}\tag{17}$$

Note that every element of the form $\psi(a/N)+\gamma$ for $1\le a<N$ is in $\mathscr{D}^{(0)}(N)$. Indeed, if $f \pmod N$ is defined as $f(a)=1$, $f(N)=-1$ and $f(b)=0$ if $b\not\equiv a, N \bmod N$, then $f\in F^{(0)}(N)$, $-NL(1,f)=\psi(a/N)+\gamma$ is in $\mathscr{D}^{(0)}(N)$. Thus, $\mathscr{D}_{\mathcal{S}}(N)\subseteq\mathscr{D}^{(0)}(N)\subseteq\mathscr{D}(N)$. An alternate formulation of Theorem 4.3 is that

$$\dim\mathscr{D}_{\mathcal{S}}(N)=\phi(N)+\omega(N)-1.\tag{18}$$

The Murty-Saradha theorem implies that $\dim\mathscr{D}(N)\ge\phi(N)$. For $N>1$, Theorem 4.3 improves the Murty-Saradha bound to give $\dim\mathscr{D}(N)\ge\phi(N)+\omega(N)-1$. On the other hand, the fact that $e_r\in\mathscr{O}(N)$ for any free residue class $r \pmod N$ implies that $L(1,e_r)=0$, which translates to

$$\psi\left(\frac{r}{N}\right)=-\sum_{(a,N)=1} e_r(a)\,\psi\left(\frac{a}{N}\right)-\sum_{p\mid N} e_r(p)\,\psi\left(\frac{p}{N}\right)$$

by (2). Therefore, the numbers $\psi(a/N)$ when $(a,N)=1$ together with $\psi(p/N)$ for $p\mid N$ span $\mathscr{D}(N)$. Hence, we have that

$$\phi(N)+\omega(N)-1\le\dim\mathscr{D}(N)\le\phi(N)+\omega(N).$$

4.3. **Linear independence of digamma values.** The preceding discussion naturally leads one to ask whether $\mathscr{D}_{\mathcal{S}}(N)=\mathscr{D}^{(0)}(N)$ and whether $\mathscr{D}^{(0)}(N)=\mathscr{D}(N)$. Towards answering these questions, define the linear map $\mathscr{L}:F(N)\to\mathscr{D}(N)$ given by

$$\mathscr{L}(f)=-\frac{1}{N}\sum_{a=1}^{N} f(a)\,\psi\left(\frac{a}{N}\right).$$

Observe that if $f\in F^{(0)}(N)$, then by (2), $\mathscr{L}(f)=L(1,f)$. Therefore, we also have the linear map $\mathscr{L}^{(0)}:F^{(0)}(N)\twoheadrightarrow\mathscr{D}^{(0)}(N)$, with $\mathscr{L}^{(0)}=\mathscr{L}\big|_{F^{(0)}}(N)$. Moreover, the Okada space $\mathscr{O}(N)$ is precisely $\ker(\mathscr{L}^{(0)})$. Therefore, by the rank and nullity theorem of linear algebra, we obtain that

$$\dim\mathscr{O}(N)+\dim\mathscr{D}^{(0)}(N)=\dim F^{(0)}(N)=N-1.$$

By Theorem 2.1, $\dim\mathscr{O}(N)=N-\phi(N)-\omega(N)$, giving us

$$\dim\mathscr{D}^{(0)}(N)=\phi(N)+\omega(N)-1.$$

Now, $\mathscr{D}_{\mathcal{S}}(N)\subseteq\mathscr{D}^{(0)}(N)$, with both the vector spaces having equal dimension from (18). Thus,

$$\mathscr{D}_{\mathcal{S}}(N)=\mathscr{D}^{(0)}(N).$$

This proves the following.

Theorem 4.4. *Let $N \geq 2$. Then the set $\mathcal{S}$ defined in Theorem 4.3 forms a basis for the $\mathbb{Q}$-vector space $\mathscr{D}^{(0)}(N)$ defined in* (17).

It will be of significant interest to find a constructive proof of the above theorem.

The question that remains to be addressed is whether the containment $\mathscr{D}^{(0)}(N) \subseteq \mathscr{D}(N)$, or equivalently, $\mathscr{O}(N) \subseteq \ker(\mathscr{L})$ is strict. By considering the dimensions of the spaces involved, it is clear that exactly one of the two, $\mathscr{D}^{(0)}(N) \subseteq \mathscr{D}(N)$ or $\mathscr{O}(N) \subseteq \ker(\mathscr{L})$ is strict.

To begin with, observe that all non-zero elements of $\mathscr{D}^{(0)}(N)$ are transcendental. Indeed, by (3), any element of $\mathscr{D}^{(0)}(N)$ can be expressed as a linear form in logarithms of $(1 - e^{2\pi i a/N})$ for $1 \leq a < N$ with coefficients in $\mathbb{Q}(e^{2\pi i/N})$ and by Baker's theorem, any non-vanishing linear form in logarithms of algebraic numbers is transcendental.

Note that every element of $\mathscr{D}(N)$ can be expressed as a sum of an element from $\mathscr{D}^{(0)}(N)$ and a rational multiple of γ. Indeed, if $\alpha = (-1/N)\sum_{a=1}^{N} c_a\, \psi(a/N) \in \mathscr{D}(N)$, then

$$\alpha = -\frac{1}{N}\sum_{a=1}^{N-1} c_a \left(\psi\left(\frac{a}{N}\right) + \gamma\right) + \frac{1}{N}\left(\sum_{a=1}^{N} c_a\right)\gamma = L(1,f) + \frac{1}{N}\left(\sum_{a=1}^{N} c_a\right)\gamma$$

by (2), for the function $f \in F^{(0)}(N)$ defined as $f(a) = c_a$ when $1 \leq a < N$ and $f(N) = -\sum_{a=1}^{N-1} c_a$. Therefore,

$$\mathscr{D}(N) = \mathscr{D}^{(0)}(N) + \gamma\,\mathbb{Q},$$

and the question of whether the above sum is a direct sum depends on whether γ is an element of $\mathscr{D}^{(0)}(N)$.

More specifically, a non-zero complex number α is said to be a *Baker period* if α can be expressed as a linear form in logarithms of algebraic numbers with algebraic coefficients. Then by (3), all non-zero elements of $\mathscr{D}^{(0)}(N)$ are recognized as Baker periods.

Suppose we have $\mathscr{D}^{(0)}(N) = \mathscr{D}(N)$, which means that $\gamma \in \mathscr{D}^{(0)}(N)$ is a Baker period and hence, is transcendental.

However, a conjecture of Kontsevich and Zagier [6] predicts that γ is not a Baker period. Thus, under the assumption of this conjecture, one would obtain that $\mathscr{D}^{(0)}(N) \subsetneq \mathscr{D}(N)$ and $\ker(\mathscr{L}) = \mathscr{O}(N)$. In other words, the only $\mathbb{Q}$-linear relations among the digamma values

$$\psi\left(\frac{a}{N}\right) \qquad \text{with} \qquad 1 \leq a \leq N,$$

arise from the vanishing of the value $L(1,f)$. Moreover, in this case, a generating set for such relations is given by the functions $\{e_r : r \pmod N \text{ - free}\}$.

Unconditionally, we obtain an infinite collection of natural numbers M such that $\mathscr{D}^{(0)}(M) \subsetneq \mathscr{D}(M)$. Towards this result, we prove the proposition below.

Proposition 4.5. *For two integers M, $N \geq 2$ that are coprime, $\mathscr{D}^{(0)}(M) \cap \mathscr{D}^{(0)}(N) = \{0\}$.*

Proof. Suppose that $0 \neq \delta \in \mathscr{D}^{(0)}(M) \cap \mathscr{D}^{(0)}(N)$. Then there exists $f \in F^{(0)}(M)$ and $g \in F^{(0)}(N)$ such that $L(1, f) = L(1, g) = \delta$. Using (3), we have

$$L(1, f) = -\sum_{a=1}^{M-1} \widehat{f}(a) \log\left(1 - e^{2\pi i a/M}\right) = L(1, g) = -\sum_{a=1}^{N-1} \widehat{g}(a) \log\left(1 - e^{2\pi i a/N}\right).$$

Let $A \subseteq \{1, 2, \cdots, M-1\}$ and $B \subseteq \{1, 2, \cdots, N-1\}$ be such that

$$\mathbb{S}_M := \left\{\log(1 - e^{2\pi i a/M}) \,:\, a \in A\right\} \text{ and } \mathbb{S}_N := \left\{\log(1 - e^{2\pi i b/N}) \,:\, b \in B\right\}$$

are maximal $\mathbb{Q}$-linearly independent subsets of

$$\left\{\log(1 - e^{2\pi i a/M}) \,:\, 1 \leq a \leq M-1\right\} \text{ and } \left\{\log(1 - e^{2\pi i b/N}) \,:\, 1 \leq b \leq N-1\right\}$$

respectively. Thus, we can write

$$L(1, f) = \sum_{a \in A} c_a \log\left(1 - e^{2\pi i a/M}\right), \;\; L(1, g) = \sum_{b \in B} d_b \log\left(1 - e^{2\pi i b/N}\right),$$

and $L(1, f) = L(1, g)$ gives a non-trivial $\overline{\mathbb{Q}}$-linear relation among the numbers in T_M and T_N. Therefore, by Baker's theorem (Theorem 4.1), there exists a non-trivial $\mathbb{Q}$-linear relation among the above numbers. That is, there exist integers n_a and n_b, not all zero, such that

$$\prod_{a \in A} \left(1 - e^{2\pi i a/M}\right)^{n_a} = \prod_{b \in B} \left(1 - e^{2\pi i b/N}\right)^{n_b} = \alpha. \tag{19}$$

Now the left hand side of (19) is an element in $\mathbb{Q}(e^{2\pi i/M})$ whereas the right hand side belongs to $\mathbb{Q}(e^{2\pi i/N})$. Since $\gcd(N, M) = 1$, $\mathbb{Q}(e^{2\pi i/M}) \cap \mathbb{Q}(e^{2\pi i/N}) = \mathbb{Q}$. Therefore, α is rational, in particular, it is a rational integer.

Furthermore, the product on the left hand side can only be divisible by prime factors of M and the product on the right hand side can only be divisible by prime factors of N. By the coprimality of M and N, we deduce that $\alpha = \pm 1$. Thus,

$$\prod_{a \in A} \left(1 - e^{2\pi i a/M}\right)^{2n_a} = \prod_{b \in B} \left(1 - e^{2\pi i b/N}\right)^{2n_b} = 1$$

gives a non-trivial $\mathbb{Q}$-linear relation among the numbers in $\mathbb{S}_M$ as well as $\mathbb{S}_N$, which is a contradiction. □

As a direct consequence of the above proposition, we obtain the following theorem.

Theorem 4.6. *Suppose there exists an integer $N \geq 2$ such that $\mathscr{D}^{(0)}(N) = \mathscr{D}(N)$. Then for every positive integer $M \geq 2$ which is coprime to N, $\mathscr{D}^{(0)}(M) \subsetneq \mathscr{D}(M)$.*

Proof. If $\mathscr{D}^{(0)}(M) = \mathscr{D}(M)$ and $\mathscr{D}^{(0)}(N) = \mathscr{D}(N)$ for two coprime integers M and N, then $\{0\} \subsetneq \mathscr{D}^{(0)}(M) \cap \mathscr{D}^{(0)}(N)$ as $\gamma \in \mathscr{D}(N) \cap \mathscr{D}(M)$. This contradicts Proposition 4.5. □

This theorem proves that there exists an integer $N_0 \geq 2$ such that for all integers $M \geq 2$ coprime to N_0, $\mathscr{D}^{(0)}(M) \subsetneq \mathscr{D}(M)$.

Now $\mathscr{D}^{(0)}(N)$ consists of Baker-periods. Thus, if γ is not a Baker period, then

$$\mathscr{D}(N) = \mathscr{D}^{(0)}(N) \oplus \gamma\,\mathbb{Q}.$$

By Theorem 4.3, this gives that the elements of $\mathcal{S} \cup \{\gamma\}$ form a basis for the space $\mathscr{D}(N)$.

An alternate basis for $\mathscr{D}(N)$ is obtained in the theorem below. We note a few useful properties of the digamma function before proceeding. The digamma function satisfies a (generalized) distributional relation (see for example, formula (24) in [9]) of the form

$$\psi(x) = \log m + \frac{1}{m} \sum_{j=0}^{m-1} \psi\left(\frac{x+j}{m}\right),$$

for any natural number m. If we put $x = 1$, and $m = N$, we see that

$$\psi(1) = \log N + \frac{1}{N} \sum_{j=0}^{N-1} \psi\left(\frac{1+j}{N}\right),$$

and whence

$$\Big\{\psi(1) - \log N\Big\} N = \sum_{j=1}^{N} \psi\left(\frac{j}{N}\right).$$

On partitioning the sum on the right hand side according to the gcd (j, N) and applying Möbius inversion, we deduce:

$$\sum_{(a,N)=1} \psi(a/N) = \sum_{d|N} \mu(d) \frac{N}{d} \left(\psi(1) - \log \frac{N}{d}\right) = \psi(1)\phi(N) - \sum_{d|N} \mu(d) \frac{N}{d} \log \frac{N}{d}. \tag{20}$$

We are now ready to prove:

Theorem 4.7. *If γ is not a Baker period, then the following $\phi(N) + \omega(N)$ numbers*

$$\widetilde{\mathcal{S}} = \left\{\psi\left(\frac{a}{N}\right), \log p \,:\, (a, N) = 1 \quad \text{and} \quad p \mid N\right\}$$

form a basis for the $\mathbb{Q}$-vector space $\mathscr{D}(N)$.

Proof. We first observe that $\log p \in \mathscr{D}^{(0)}(N)$ for $p \mid N$. Indeed, if $f \in F(p) \subseteq F(N)$ such that $L(s, f) = (1 - p^{1-s})\zeta(s)$, then $L(1, f) = \log p$.

As observed earlier, if γ is not a Baker period, then $\mathscr{D}^{(0)}(N) \subsetneq \mathscr{D}(N)$ and thus, $\dim \mathscr{D}(N) = \phi(N) + \omega(N)$. It therefore suffices to show that $\mathscr{D}(N)$ is contained in the $\mathbb{Q}$-span of elements in $\widetilde{\mathcal{S}}$.

Let $\alpha \in \mathscr{D}(N)$ be $\alpha = -N^{-1} \sum_{a=1}^{N} c_a \, \psi(a/N)$. We can re-write the element α as

$$\alpha = -\frac{1}{N} \left(\sum_{a=1}^{N-1} c_a \, \psi\left(\frac{a}{N}\right) + \left(-\sum_{a=1}^{N-1} c_a\right) \psi(1) \right) - \frac{1}{N} \left(\sum_{a=1}^{N} c_a\right) \psi(1).$$

The first term in the above summation is $L(1, f)$ when f is defined as $f(a) = c_a$ for $1 \leq a < N$ and $f(N) = -\sum_{a=1}^{N-1} c_a$. Hence, $\alpha = L(1, f) + ((\sum_{a=1}^{N} c_a)/N)\, \gamma$.

By Lemma 1 of [14], there is a function $g \in F(N)$, supported only on the coprime residue classes (mod N) and functions $g_p \in F(N/p)$ for each prime $p \mid N$ such that

$$L(s, f) = L(s, g) + \sum_{p|N} \left(1 - \frac{p}{p^s}\right) L(s, g_p).$$

Let $\rho_{g_p} = pN^{-1}\sum_{a=1}^{N/p} g(a)$ be the residue of $L(s, g_p)$ as $s = 1$. Then we get

$$L(1,f) = L(1,g) + \sum_{p|N} \rho_{g_p} \log p.$$

In other words, $L(1,f)$ is contained in the $\mathbb{Q}$-span of $\widetilde{\mathcal{S}}$. By (20), $\psi(1)$ also belongs to the span of $\widetilde{\mathcal{S}}$. Hence, α is in the $\mathbb{Q}$-span of $\widetilde{\mathcal{S}}$, which proves the theorem. □

4.4. **Variant of the Erdős conjecture.** Erdős's conjecture (Conjecture 1.2) is perhaps the most tantalizing question in the context of non-vanishing of $L(1,f)$ which remains unresolved. In [15], it was proved by the second author that the conjecture holds with 'probability' 1, that is, for 100% of Erdős functions f, $L(1,f) \neq 0$. Despite several independent approaches, the case $N \equiv 1 \bmod 4$ of the conjecture is still open.

In order to gain insight into the problem, it is prudent to study generalized variants of the Erdős conjecture. With the help of exhaustive computer search, Sz. Tengely [20, Theorem 1] found an explicit example of a function f, periodic modulo 36 with $f(n) = \pm 1$ for *all* $n \in \mathbb{Z}$ such that $L(1,f) = 0$. A proof of this fact using the distribution relation (4.3) was given by Kh. Pilehrood and T. Pilehrood in [17]. Moreover, Tengely showed that $N = 36$ is the smallest period for which such a function f exists. This raises the interesting question: can we classify N for which there exist functions $f \pmod N$ with $f(n) = \pm 1$ for all $n \in \mathbb{Z}$ such that $L(1,f) = 0$. Such a function f will give rise to relations of the form

$$\sum_{n=1}^{N} f(a)\psi(a/N) = 0, \qquad f(a) = \pm 1,\ 1 \leq a \leq N, \qquad \sum_{a=1}^{N} f(a) = 0. \tag{21}$$

Since $f(n) = \pm 1$ and $\sum_{a=1}^{N} f(a) = 0$, N has to necessarily be even for (21) to hold. In this direction, we have the following theorem.

Theorem 4.8. *Suppose that $N \geq 2$ is even. If $N \equiv 2 \bmod 4$, then for any function $f \pmod N$ such that $f(n) = \pm 1$ for all $n \in \mathbb{Z}$, $L(1,f) \neq 0$.*

Proof. Let $N = 2M$ with M odd. Suppose to the contrary that there exists $f \in \mathscr{O}(N)$ with $f(n) = \pm 1$. Using the basis derived in Theorem 2.1, we have $f(n) = \sum_{r-free} f(r)\, e_r(n)$. In particular, $f(2) = \sum_{r-free} f(r)\, e_r(2)$.

We are in the case $2 \,||\, N$. By Table 1, $e_r(2) = -1$ when $2 \mid r$ and 0 otherwise. Hence,

$$f(2) = -\sum_{\substack{r-free,\\ 2|r}} f(r).$$

The number of even free residue classes is $(N/2) - 1 = M - 1$, which is even. Therefore, the right hand side of the above equation will be even, whereas the left hand side is odd, leading to a contradiction.
□

This theorem can alternatively be proved by an argument similar to the proof of Erdős's conjecture when $N \equiv 3 \bmod 4$ by the first author and N. Saradha in [12, Theorem 7]. We relegate the investigation of the case $N \equiv 0 \bmod 4$ of the variant of Erdős's conjecture to future research.

Following arguments along the same lines as that of [15, Theorem 1.3], we show that for 100% of functions f mod N, relations of the type (21) do not exist. More specifically,

Theorem 4.9. *For an even integer $N \geq 2$, let*

$$E_N := \left\{ f \in F^{(0)}(N) \,:\, f(n) \in \{-1, 1\} \text{ for all } n \in \mathbb{Z} \right\}$$
$$V_N := \{ f \in E_N \,:\, L(1, f) = 0 \}.$$

Then $|V_N| \leq 2^{N-\phi(N)-\omega(N)}$ and

$$\lim_{M \to \infty} \frac{\sum\limits_{\substack{2 \leq N \leq M, \\ N \text{ - even}}} |V_N|}{\sum\limits_{\substack{2 \leq N \leq M, \\ N \text{ - even}}} |E_N|} = 0.$$

Proof. Define an equivalence relation $\sim$ on E_N by setting $f \sim g$ iff

$$f(r) = g(r), \text{ for all free residue classes } r \bmod N.$$

By Theorem 2.1, if $f \in \mathscr{O}(N)$, then for all $g \in E_N$, $g \neq f$ and $g \sim f$, $g \notin \mathscr{O}(N)$. Indeed, if $g \sim f$ and $g \in \mathscr{O}(N)$, then for any $n \in \mathbb{Z}$,

$$g(n) = \sum_{r free} g(r)\, e_r(n) = \sum_{r free} f(r)\, e_r(n) = f(n).$$

Thus, the number of functions in V_N is at most equal to the number of equivalence classes, $|E_N/\sim|$.

Furthermore, $|E_N| = \binom{N}{N/2}$ as the condition (1) forces every $f \in E_N$ to take the values 1 and -1 equally often. Let $n_r = N - \phi(N) - \omega(N)$ be the number of free residue classes mod N. We consider two cases.

(a) $n_r < N/2$: The number of free residue classes where $f \in E_N$ can take the value 1 ranges from 0 to n_r. Hence,

$$|E_N/\sim| = \sum_{j=0}^{n_r} \binom{n_r}{j} = 2^{n_r}.$$

(b) $n_r \geq N/2$: The number of free residue classes where $f \in E_N$ can take the value 1 ranges from $n_r - (N/2)$ to $N/2$. Thus,

$$|E_N/\sim| = \sum_{j=n_r-(N/2)}^{N/2} \binom{n_r}{j} \leq 2^{n_r}.$$

In either case, we obtain that $|V_N| \leq 2^{N-\phi(N)-\omega(N)}$.

Now we use the bounds

$$\phi(N) \gg \frac{N}{\log\log N} \text{ and } \binom{N}{N/2} \gg \frac{2^N}{\sqrt{N}}$$

from [10, Theorem 2.9] and [15, pg. 11] together with the following standard fact (see [18, Problem 70, pg. 16]): let a_n and b_n be sequences satisfying the conditions that $b_n > 0$ for all $n \in \mathbb{N}$, $b_1 + b_2 + \cdots$ diverges and $\lim_{n\to\infty} a_n/b_n = s$. Then

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = s.$$

Taking $a_N = 2^{N-\left(\frac{N}{\log\log N}\right)}$ and $b_N = 2^N/\sqrt{N}$ gives the theorem. $\square$

Thus, relations of the type (21) are 'rare'.

The existence of a function $f \pmod{36} \in \mathscr{O}(36)$ with $f(n) = \pm 1$ for all $n \in \mathbb{Z}$ insinuates that it is worth investigating if Conjecture 1.2 also has a counterexample.

5. The Okada space $\mathscr{O}(N)$ as a rational representation

For every h coprime to N, let $\sigma_h : F(N) \to F(N)$ be the linear operator given by $\sigma_h(f)(n) := f(hn)$. One can easily check that this defines a linear action of the group $(\mathbb{Z}/N\mathbb{Z})^*$ on $F(N)$, or in other words, we have a *rational linear representation* of the group $(\mathbb{Z}/N\mathbb{Z})^*$,

$$\rho : (\mathbb{Z}/N\mathbb{Z})^* \to GL\left(F(N)\right)$$

with $\rho(h)(f) := \sigma_h(f)$. Define $t_a \in F(N)$ by

$$t_a(n) = \begin{cases} 1 & \text{if } a \equiv n \bmod N, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\{t_a : a \bmod N\}$ forms a standard basis for the vector space $F(N)$ and $\sigma_h(t_a) = t_{ha}$, where the subscripts are interpreted as residue classes modulo N. Thus, if χ_ρ is the character of the representation ρ, then

$$\chi_\rho(h) = \mathrm{Tr}\left(\sigma_h\right) = \sum_{\substack{m=1, \\ m(1-h)\equiv 0 \bmod N}}^{N} 1.$$

Furthermore, as a consequence of standard theory of representations of finite groups (see [19, Chapter 12]), the space $F(N)$ can be decomposed into irreducible representations.

The Okada space can be identified as an important subrepresentation of ρ. Indeed, a crucial step in the proof of the Baker-Birch-Wirsing theorem is the following lemma (see [2, Lemma 2]).

Lemma. *Suppose $f \in F^{(0)}(N)$ is such that $L(1, f) = 0$. Then $L(1, \sigma_h(f)) = 0$ for every $1 \leq h \leq N$ with $(h, N) = 1$.*

In other words, if $f \in \mathscr{O}(N)$, then $\sigma_h(f) \in \mathscr{O}(N)$ for each coprime residue class $h \pmod N$. Therefore, we obtain the representation

$$\rho^{(0)} : (\mathbb{Z}/N\mathbb{Z})^* \to GL\left(\mathscr{O}(N)\right) \tag{22}$$

defined as $\rho^{(0)}(h)(f) := \sigma_h(f)$.

The action of $\rho^{(0)}$ on the basis elements from Theorem 2.1 can be written down precisely.

Proposition 5.1. *Fix a residue class $h \pmod N$, with $h \not\equiv 1 \bmod N$. Let $\overline{h} \pmod N$ be such that $h\overline{h} \equiv 1 \bmod N$. If $r \not\equiv hl \bmod N$ for any prime $l \mid N$, then*

$$\sigma_h(e_r) = e_{\overline{h}r} + \sum_{p \mid N} e_r(p)\, e_{\overline{h}p}. \tag{23}$$

If $r \equiv hl \bmod N$ for some prime $l \mid N$, then

$$\sigma_h(e_{hl}) = -e_{\overline{h}l}. \tag{24}$$

Proof. Since the functions e_r for a free residue class 'r' form a basis for $\mathscr{O}(N)$, $\sigma_h(e_r) \in \mathscr{O}(N)$ when $(h, N) = 1$. Let $r \pmod N$ be a free residue class. By equation (11),

$$\sigma_h(e_r) = \sum_{a\, free} e_r(ha)\, e_a.$$

Now consider the case when $r \not\equiv hl \bmod N$, for any prime $l \mid N$. If $ha \pmod N$ is free, then $e_r(ha) = 0$ unless $ha \equiv r \bmod N$, when it is equal to 1. By the condition on r, $\overline{h}r \pmod N$ is free, and so we get that the coefficient of $e_{\overline{h}r}$ is 1. On the other hand, if $ha \equiv p \bmod N$ for some $p \mid N$, then $e_r(p)$ is the coefficient of $e_{\overline{h}p}$. This gives equation (23).

Now suppose $r \equiv hl \bmod N$, for some $l \mid N$. Then $\sigma_h(e_r) = \sum_{a\, free} e_{hl}(ha)\, e_a$. Observe that $ha \pmod N$ either represents a free residue class or a prime divisor of N. If $ha \pmod N$ is free, then $e_{hl}(ha)$ is zero unless $l \equiv a \bmod N$, which contradicts the fact that a is free. If $ha \equiv p \pmod N$ for a prime $p \mid N$, then the coefficient of e_a becomes $e_{hl}(p)$. Thus, we have

$$\sigma_h(e_{hl}) = \sum_{p \mid N} e_{hl}(p)\, e_{\overline{h}p},$$

If $p \,||\, N$, then $e_{hl}(p) = -1$ if $p = l$ and 0 otherwise. Similarly, if $p^2 \mid N$, $v_p(N) \geq 2$, $v_p(hl) = 1$ if $p = l$ and 0 otherwise. Therefore, in both the cases, we obtain (24). □

One can use this basis to write down the character of the representation $\rho^{(0)}$.

Proposition 5.2. *Let $\rho^{(0)}$ be the rational representation defined in* (22) *and $\chi_{\rho^{(0)}}$ be its character. Then*

$$\chi_{\rho^{(0)}}(h) = \begin{cases} \phi(N) + \omega(N) - 1 & \text{if } h = 1, \\ -\omega(N) + 1 & \text{if } h \equiv -1 \bmod N,\ 2|N,\ N > 4, \\ -\omega(N) & \text{if } h \equiv -1 \bmod N,\ (2, N) = 1 \text{ or } N = 2,\ 4, \\ \sum\limits_{p|N} e_r(p) - \sum\limits_{\substack{p|N, \\ h^2 \equiv 1 \bmod \frac{N}{p}}} 1 + \sum\limits_{\substack{r\, free, \\ r(1-\overline{h}) \equiv 0 \bmod N}} 1 & \text{otherwise.} \end{cases}$$

Proof. Fix a residue class $h \pmod N$ with $(h, N) = 1$ and $h \not\equiv 1 \bmod N$. Since $\sigma_h : \mathscr{O}(N) \to \mathscr{O}(N)$ is a linear transformation, one can write a matrix, say A_h, for σ_h with respect to the basis $\{e_r : r \pmod N \text{ - free}\}$. Then $\chi_\rho(h) = \mathrm{Tr}(A_h)$, can be calculated as a sum of the diagonal terms in A_h, that is, the sum of the coefficient of e_r in $\sigma_h(e_r)$.

Suppose $r \equiv hl \bmod N$, with l varying over prime divisors of N. Then by (24), $\sigma_h(e_r)$ will contribute towards $\mathrm{tr}\,(A_h)$ if and only if

$$r \equiv \overline{h}^2 r \bmod N \iff hl(1 - \overline{h}^2) \equiv 0 \bmod N \iff h \equiv \overline{h} \bmod \frac{N}{l}.$$

Now assume that $r \not\equiv hl \bmod N$ for any prime $l \mid N$, in which case, $\sigma_h(e_r)$ is given by (23). Thus, we have a non-zero contribution from coefficient of e_r to $\mathrm{tr}\,(A_h)$ if either $r \equiv \overline{h}p \bmod N$ for some $p \mid N$ or if $r \equiv \overline{h}r \bmod N$. If $h \equiv -1 \bmod N$, then there are no such free classes $r \pmod N$ unless N is even and $r = N/2$. The proposition now follows. □

Since $\rho^{(0)}$ is a subrepresentation of ρ, Maschke's theorem implies that there exists a $(\mathbb{Z}/N\mathbb{Z})^*$-invariant complement of $\mathscr{O}(N)$ in $F(N)$. That is, there exists a subspace $\mathscr{I}(N)$, invariant under the action of $(\mathbb{Z}/N\mathbb{Z})^*$, such that

$$F(N) = \mathscr{O}(N) \oplus \mathscr{I}(N).$$

6. **Concluding Remarks**

The values $\psi(a/N)$ are naturally related to the Euler-Lehmer constants, $\gamma(a, N)$ introduced by Lehmer [7] as analogs of Euler's constant γ for arithmetic progressions. In particular, we have the relationship (see [7, Theorem 7])

$$\gamma(a, N) = -\frac{1}{N}\left(\psi\left(\frac{a}{N}\right) + \log N\right),$$

Thus, our results regarding the vector spaces spanned by the digamma values also have implications for vector spaces spanned by the Euler-Lehmer constants, which have been independently studied.

There have been earlier researches into the structure of the space $\mathscr{D}^{(0)}(N)$, most notably [2], [11], [12], [14], [13], [16] and [4]. In the latter paper, the problem was studied through the lens of the Bass theorem [3] (as corrected by Ennola [5]) giving relations among the so-called cyclotomic numbers $(1 - e^{2\pi i a/N})$.

In this paper, we have initiated a new approach to study the space $\mathscr{D}^{(0)}(N)$ by constructing an explicit basis for $\mathscr{O}(N)$. We have underlined that $\mathscr{D}^{(0)}(N) \subseteq \mathscr{D}(N)$ and the question of whether this is a proper containment is equivalent to the question of whether Euler's constant γ is a Baker period. In other words, is γ a special value $L(1, f)$ for some periodic arithmetical function f? Most likely, it is not and so we conjecture that $\mathscr{D}^{(0)}(N)$ is a proper subspace of $\mathscr{D}(N)$.

Acknowledgments

The authors thank Abhishek Bharadwaj, Anup Dixit, Sanoli Gun and Purusottam Rath for valuable feedback on an earlier version of this paper.

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