ON A CONJECTURE OF LIVINGSTON

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Abstract. In an attempt to resolve a folklore conjecture of Erdös, Livingston conjectured the $\bar{\mathbb{Q}}$-linear independence of logarithms of certain algebraic numbers. We disprove this conjecture, highlighting that a new approach is required to settle Erdös's conjecture.

1. Introduction

In a written communication with Livingston, Erdös [5] conjectured the following:

Conjecture 1. *(Erdös) Let q be a positive integer and f be an arithmetical function, periodic with period q. If $f(n) \in \{-1, 1\}$ when $q \nmid n$ and $f(n) = 0$ otherwise, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

whenever the series is convergent.

In 1965, Livingston [5] attempted to resolve the above conjecture. He predicted that to settle Conjecture 1, one would first have to prove:

Conjecture 2. *(Livingston) Let $q \geq 3$ be a positive integer. The numbers*

$$\left\{ \log\left(2 \sin \frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2} \right\} \text{ and } \pi$$

when q is odd, and

$$\left\{ \log\left(2 \sin \frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2} \right\},\ \pi \text{ and } \log 2$$

when q is even, are linearly independent over the field of algebraic numbers.

The above statement does not depend on the branch of log considered, as the values would only differ by an integer multiple of $2\pi i$.

In this paper, we disprove Livingston's conjecture in the case when q is not prime and show that the conjecture is true when q is prime. More precisely, we prove the following theorems:

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Theorem 1.1. *Conjecture 2 does not hold for $q \geq 6$ and q not prime. In fact, for a composite positive integer $q \geq 6$, the numbers*

$$\left\{ \log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2} \right\}$$

are $\mathbb{Q}$-linearly dependent.

Theorem 1.2. *Let p be an odd prime. The numbers*

$$\left\{ \log\left(2\sin\frac{a\pi}{p}\right) : 1 \leq a \leq \frac{p-1}{2} \right\} \text{ and } \pi$$

are $\bar{\mathbb{Q}}$-linearly independent. Thus, Conjecture 2 is true when the modulus p is prime.

In both the above theorems, log denotes the principal branch. As a corollary of Theorem 1.2, we have

Corollary 1. *Let p be an odd prime and f be an arithmetical function, periodic with period p such that $f(n) \in \{-1, 1\}$ when $p \nmid n$ and $f(n) = 0$ otherwise. Assume that $\sum_{a=1}^{p} f(a) = 0$. Then, only one of the following is true, either*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

or

$$\sum_{a=1}^{p-1} f(a) \cot\left(\frac{a\pi}{p}\right) = \sum_{a=1}^{p-1} f(a) \cos\left(\frac{2\pi ab}{p}\right) = 0,$$

for $1 \leq b \leq (p-1)/2$.

Remark. *Conjecture 2 holds for $q = 4$ because the set $\{1 \leq a < q/2\}$ is a singleton, namely, $a = 1$ and*

$$\log\left(2\sin\frac{\pi}{4}\right) = \log\sqrt{2} \neq 0.$$

2. Preliminaries

This section introduces some notation and fundamental results to be used in the later part of the paper.

2.1. **L-series attached to a periodic arithmetical function.** Let q be a positive integer and f be an arithmetical function that is periodic with period q. We define

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Let us observe that $L(s, f)$ converges absolutely for $\Re(s) > 1$. Since f is periodic,

$$\begin{aligned} L(s, f) &= \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(a + kq)^s} \\ &= \frac{1}{q^s} \sum_{a=1}^{q} f(a) \zeta(s, a/q), \end{aligned}$$

where $\zeta(s,x)$ is the Hurwitz zeta function. For $\Re(s)>1$ and $0<x\le 1$, recall that the Hurwitz zeta function is defined as

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}.$$

In 1882, Hurwitz [4] proved that $\zeta(s,x)$ has an analytic continuation to the entire complex plane except for a simple pole at $s=1$ with residue 1. In particular,

$$\zeta(s,x)=\frac{1}{s-1}-\Psi(x)+O(s-1),$$

where Ψ is the digamma function, which is defined as the logarithmic derivative of the gamma function. This can be used to conclude that $L(s,f)$ can be extended analytically to the entire complex plane except for a simple pole at $s=1$ with residue $\frac{1}{q}\sum_{a=1}^{q}f(a)$. Thus, $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ exists if and only if $\sum_{a=1}^{q}f(a)=0$, which we will assume henceforth.

Let us also note that (2.1) helps us to express $L(1,f)$ as a linear combination of values of the digamma function. Therefore,

$$L(1,f)=-\frac{1}{q}\sum_{a=1}^{q}f(a)\Psi\left(\frac{a}{q}\right). \tag{1}$$

2.2. $L(1,f)$ **as a linear form in logarithm of algebraic numbers.** For a function f that is periodic with period q, define the Fourier transform of f as

$$\hat{f}(k):=\frac{1}{q}\sum_{a=1}^{q}f(a)\zeta_q^{-ak},$$

where $\zeta_q=e^{2\pi i/q}$. This can be inverted using the identity

$$f(n)=\sum_{k=1}^{q}\hat{f}(k)\zeta_q^{kn}. \tag{2}$$

Thus, the condition for convergence of $L(1,f)$, i.e, $\sum_{a=1}^{q}f(a)=0$ can be interpreted as $\hat{f}(q)=0$. Substituting (2) in the expression for $L(s,f)$ we have,

$$\begin{aligned}L(s,f)&=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{k=1}^{q-1}\hat{f}(k)\zeta_q^{kn}.\\&=\sum_{k=1}^{q-1}\hat{f}(k)\sum_{n=1}^{\infty}\frac{\zeta_q^{kn}}{n^s}.\end{aligned} \tag{3}$$

The inner sum converges for $s=1$. To see this, recall the partial summation or the Abel summation formula that says:

Theorem. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers and f be a C^1 function on $\mathbb{R}_{>0}$. For $x>0$, if $A(x):=\sum_{n\le x}a_n$, then*

$$\sum_{1\le n\le x}a_nf(n)=A(x)f(x)-\int_1^x A(t)f'(t)dt.$$

For $1 \leq k \leq q-1$, let $a_n = \zeta_q^{kn}$ and $f(x) = 1/x$. Thus, $A(x) = \sum_{n\leq x} \zeta_q^{kn}$ and the partial summation formula gives us that

$$\sum_{1\leq n\leq x} \frac{\zeta_q^{kn}}{n} = \frac{A(x)}{x} + \int_1^x \frac{A(t)}{t^2} dt. \tag{4}$$

Now, note that for $1 \leq k \leq q-1$,

$$\sum_{n=1}^{q} \zeta_q^{kn} = 0.$$

Hence, the partial sums, $A(x)$ are bounded above by q for all $x > 0$. Therefore, the integral in (4) is absolutely convergent as x tends to infinity. Thus, taking limit as x goes to infinity in (4), we get the convergence of the inner sum in (3) and can conclude that

$$L(1, f) = -\sum_{k=1}^{q-1} \hat{f}(k) \log(1-\zeta_q^k), \tag{5}$$

where log is the principal branch.

2.3. **A simplified expression for $L(1,\chi)$.** If χ is an even Dirichlet character modulo a prime p, then according to (5) the expression for $L(1,\chi)$ is

$$\begin{aligned} L(1,\chi) &= -\sum_{k=1}^{p-1} \widehat{\chi}(k) \log(1-\zeta_p^k) \\ &= -\sum_{k=1}^{\lfloor (p-1)/2 \rfloor} \widehat{\chi}(k) \big[\log(1-\zeta_p^k) + \log(1-\zeta_p^{-k}) \big] \\ &= -\sum_{k=1}^{\lfloor (p-1)/2 \rfloor} \widehat{\chi}(k) \log \big|1-\zeta_p^k\big|^2 \\ &= -\sum_{k=1}^{p-1} \widehat{\chi}(k) \log \big|1-\zeta_p^k\big|, \end{aligned}$$

where $\widehat{\chi}$ denotes the Fourier transform of χ as defined earlier. Let $\tau(\chi)$ denote the Gauss sum associated to χ, i.e,

$$\tau(\chi) = \sum_{a=1}^{p} \chi(a) \zeta_p^a. \tag{6}$$

Hence, the Fourier transform of χ can be evaluated as follows. For every $(k,p)=1$,

$$\begin{aligned}\widehat{\chi}(k) &= \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)\zeta_p^{-ak}\\ &= \frac{1}{p}\sum_{t=1}^{p-1}\chi(-tk^{-1})\zeta_p^{t}\\ &= \frac{\overline{\chi(-k)}}{p}\sum_{t=1}^{p-1}\chi(t)\zeta_p^{t}\\ &= \frac{\overline{\chi(-k)}}{p}\tau(\chi).\end{aligned}$$

Since χ is even, the expression for $L(1,\chi)$ becomes

$$L(1,\chi) = -\frac{\tau(\chi)}{p}\sum_{k=1}^{p}\overline{\chi}(k)\log|1-\zeta_p^k|. \tag{7}$$

Another elementary but important fact about the Gauss sum (6) is that when χ is a non-trivial Dirichlet character modulo p,

$$\tau(\chi)\neq 0. \tag{8}$$

For a proof of the above fact, we refer the reader to [6], Theorem 5.3.3, pg. 76.

2.4. **Baker's theorem about linear forms in logarithm of algebraic numbers.** We will also use an important theorem of Baker (see [1], Theorem 2.1, pg. 10) concerning linear forms in logarithms of algebraic numbers, namely,

Theorem 2.1. *If* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are non-zero algebraic numbers such that* $\log\alpha_1$, $\log\alpha_2$, $\cdots$, $\log\alpha_n$ *are linearly independent over the rationals, then* 1, $\log\alpha_1$, $\log\alpha_2$, $\cdots$, $\log\alpha_n$ *are linearly independent over the field of all-algebraic numbers.*

2.5. **Matrices of the Dedekind type.** Let $\mathfrak{M}$ be an $n\times n$ matrix with complex entries. Let $m_{i,j}$ denote the (i,j)-th entry of $\mathfrak{M}$. Then, $\mathfrak{M}$ is said to be of Dedekind type if there exists a finite abelian group, $G=\{x_1,x_2,\cdots,x_n\}$ and a complex valued function f on G such that

$$m_{i,j} = f(x_i^{-1}x_j),$$

for all $1\leq i,j\leq n$. We will use the following well-known theorem regarding matrices of the Dedekind type:

Theorem 2.2. *Let* $\mathfrak{M}$ *be an* $n\times n$ *matrix of the Dedekind type. For a character* χ *on* G *(a homomorphism of* G *into* $\mathbb{C}^*$*), define*

$$S_\chi := \sum_{s\in G} f(s)\chi(s).$$

Then the determinant of $\mathfrak{M}$ *is equal to*

$$\prod_{\chi} S_\chi,$$

where the product runs over all characters of G. Thus, $\mathfrak{M}$ is invertible if and only if

$$S_\chi \neq 0,$$

for all characters χ of G.

For a proof of the above theorem and an exposition on properties of matrices of the Dedekind type, we refer the reader to [7]. The determinant of a matrix of the Dedekind type is often referred to as a Dedekind determinant.

3. **The approach of Livingston**

Let f be an Erdös function, i.e, $f(n) = \pm 1$ when $q \nmid n$ and $f(n) = 0$ whenever $q|n$. The condition for the existence of $L(1, f)$ implies that

$$\sum_{a=1}^{q} f(a) = \sum_{a=1}^{q-1} f(a) = 0. \tag{9}$$

As seen earlier, $L(1, f)$ can be written as a linear combination of the values of the digamma function. Gauss (see [3], pg. 35-36) proved the following formula for $1 \leq a < q$:

$$\Psi\left(\frac{a}{q}\right) = -\gamma - \log q - \frac{\pi}{2}\cot\left(\frac{a\pi}{q}\right) + \sum_{b=1}^{r}\left\{\cos\left(\frac{2\pi ab}{q}\right)\log\left(4\sin^2\frac{\pi b}{q}\right)\right\} + (-1)^a \log 2\,\frac{1+(-1)^q}{2}, \tag{10}$$

where $r := \lfloor (q-1)/2 \rfloor$.

Substituting (10) in (1), we have

$$L(1, f) = \frac{-1}{q}\left[\sum_{a=1}^{q-1} f(a)\left\{\gamma + \log q + \frac{\pi}{2}\cot\left(\frac{a\pi}{q}\right) - \sum_{b=1}^{r}\left\{\cos\left(\frac{2\pi ab}{q}\right)\log\left(4\sin^2\frac{\pi b}{q}\right)\right\} + (-1)^a \log 2\,\frac{1+(-1)^q}{2}\right\}\right].$$

On simplifying the above expression using (9), we get

$$L(1, f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right) + \frac{2}{q}\sum_{b=1}^{r}\left\{\left[\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi ab}{q}\right)\right]\log\left(2\sin\frac{\pi b}{q}\right)\right\} - T_q, \tag{11}$$

where

$$T_q = \begin{cases} \frac{\log 2}{q}\left(\sum\limits_{k=1}^{q-1}(-1)^k f(k)\right) & \text{if } q \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Let us note that the numbers

$$\cot\left(\frac{a\pi}{q}\right) \text{ and } \cos\left(\frac{2\pi ab}{q}\right)$$

are algebraic for $1 \leq a < q$ and $1 \leq b < q$. Since $f(a) \in \bar{\mathbb{Q}}$ and $f(q) = 0$, we are led to deduce that $L(1, f)$ is an algebraic linear combination of

$$\pi, \log\left(2\sin\frac{\pi}{q}\right), \log\left(2\sin\frac{2\pi}{q}\right), \cdots, \log\left(2\sin\frac{(q-1)\pi}{2q}\right)$$

together with $\log(2)$ when q is even. This led Livingston to predict that if Conjecture 1 were to be true, the above numbers should be linearly independent over $\bar{\mathbb{Q}}$. At this point, we make the following key observation - to conclude Conjecture 1 as an implication of Conjecture 2, one is still required to prove that the resulting relation is non-trivial. That is, if f is an Erdös function, not identically zero, then at least one of

$$\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right), \tag{12}$$

or

$$\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi ab}{q}\right), \quad 1 \leq b \leq r \tag{13}$$

or T_q is not zero. This question is not addressed by Conjecture 2 and hence, Livingston's conjecture is not sufficient to settle the conjecture of Erdös.

Remark. *If f is allowed to take values in $\bar{\mathbb{Q}}$ and q is odd, then there exist a plethora of examples of functions f that are not identically zero but for which* (12) *and* (13) *are both zero for all $1 \leq b \leq r$. These are given by the following theorem from* [2]*:*

Theorem 3.1. *Let $q \geq 3$ be a natural number. Then all odd, algebraically-valued functions f, periodic mod q, for which $L(1, f) = 0$ are given by the totality of linear combinations with algebraic coefficients of the following $\left\lfloor \frac{1}{2}(q-3) \right\rfloor$ functions:*

$$f_l(n) = (-1)^{n-1}\left(\frac{\sin n\pi/q}{\sin \pi/q}\right)^l, \quad \textit{for } l = 3, 5, \cdots, (q-2) \tag{14}$$

when q is odd and

$$f_l(n) = (-1)^{n-1}\left(\frac{\cos n\pi/q}{\cos \pi/q}\right)\left(\frac{\sin n\pi/q}{\sin \pi/q}\right)^l \quad \textit{for } l = 3, 5, \cdots, (q-1)$$

when q is even. The functions are linearly independent and take values in $\mathbb{Q}(\zeta_q)$, i.e, the q-th cyclotomic field.

Each f_l in the above theorem is an odd function. Since $\cos(2\pi ab/q)$ *is an even function for $1 \leq a < q$,* (13) *is zero for all $1 \leq b \leq r$. $T_q = 0$ as q is odd. Thus,*

$$L(1, f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right),$$

which is zero by Theorem 3.1.

4. **Proof of the main theorems**

We make a useful observation before proceeding with the proofs. If q is a positive integer and $1 \leq a < q/2$, then

$$2\sin\frac{a\pi}{q} = \frac{e^{ia\pi/q} - e^{-ia\pi/q}}{i} = ie^{-ia\pi/q}(1-\zeta_q^a), \tag{15}$$

where $\zeta_q = e^{2\pi i/q}$. Since

$$\sin\frac{a\pi}{q} > 0,$$

for $1 \leq a < q/2$ and log denotes the principal branch,

$$\begin{aligned}\log\left(2\sin\frac{a\pi}{q}\right) &= \log\left(\left|1-\zeta_q^a\right|\right) + i0 = \log\left(\left|1-\zeta_q^a\right|\right)\\ &= \log\left(\left|1-\zeta_q^{-a}\right|\right) = \log\left(2\sin\frac{(q-a)\pi}{q}\right).\end{aligned} \tag{16}$$

4.1. **Proof of Theorem 1.1.** We prove the linear dependence of the numbers

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2}\right\}$$

by giving an explicit $\mathbb{Q}$-relation among them.

Proof. Before proceeding, we note that by (16), it suffices to exhibit a relation among logarithms of cyclotomic numbers. Now, since q is not prime, there is a divisor d of q such that $d \neq 1, q$. For such a divisor d, we have the following polynomial identity in $\mathbb{C}[X,Y]$:

$$X^{q/d} - Y^{q/d} = \prod_{j=1}^{q/d}\left(X - \zeta_{q/d}^j Y\right),$$

where $\zeta_{q/d} = e^{2\pi i d/q}$. Substituting $X = 1$ and $Y = \zeta_q^a$ for $(a,q) = 1$, we have

$$1 - e^{2\pi i a/d} = \prod_{j=1}^{q/d}\left(1 - e^{2\pi i(dj/q + a/q)}\right) = \prod_{j=1}^{q/d}\left(1 - e^{2\pi i(a+dj)/q}\right)$$

Thus, taking absolute values of both sides of the above equation gives us

$$\left(\left|1-\zeta_q^{aq/d}\right|\right) = \prod_{j=1}^{q/d}\left(\left|1-\zeta_q^{(a+dj)}\right|\right).$$

Taking logarithms of both sides, we obtain the following $\mathbb{Q}$-linear relation

$$\log\left(\left|1-\zeta_q^{aq/d}\right|\right) - \sum_{j=1}^{q/d}\log\left(\left|1-\zeta_q^{(a+dj)}\right|\right) = 0,$$

for all $1 \leq a < q$ and $(a, q) = 1$ and $d|q$, $d \neq 1, q$. Hence, using (16), we have

$$\log\left(2\sin\left(\frac{aq}{d}\,\frac{\pi}{q}\right)\right) - \sum_{j=1}^{q/d}\log\left(2\sin\frac{(a+dj)\pi}{q}\right) = 0. \tag{17}$$

Since we want a linear relation among

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2}\right\},$$

we will replace $\log(2\sin(b\pi/q))$ by $\log(2\sin((q-b)\pi/q))$ whenever $b \geq q/2$. This is valid by (16). Now, we make the following observations. Suppose that there exists a k such that $1 \leq k < q/2$ and

$$k \equiv a + dj \equiv a + dl \bmod q,$$

for some $1 \leq j, l \leq q/d$ and $j \neq l$. This implies that $q|d(j-l)$, which is impossible since $(j-l) < q/d$. Thus,

$$a + dj \not\equiv a + dl \bmod q, \tag{18}$$

for $1 \leq j, l \leq q/d$ and $j \neq l$. Similarly,

$$-(a + dj) \not\equiv -(a + dl) \bmod q, \tag{19}$$

for $1 \leq j, l \leq q/d$ and $j \neq l$. Suppose there exists a k such that $1 \leq k < q/2$ and

$$k \equiv a + dj \equiv -(a + dl) \bmod q,$$

for $1 \leq j, l \leq q/d$ and $j \neq l$. Thus, $q|(2a + d(j + l))$. Since $d|q$, we have $d|(2a + d(j - l))$, i.e, $d|2a$. But $(a, q) = 1$. Hence, $(a, d) = 1$, which implies that $d|2$. We assumed that $d \neq 1, q$. Therefore, $d = 2$. As a result, we have

$$a + dj \not\equiv -(a + dl) \bmod q, \tag{20}$$

for $1 \leq j, l \leq q/d$ and $j \neq l$ unless $d = 2$.

Thus, for $(a, q) = 1$, $d|q$ and $2 < d < q$, (17) along with (18), (19) and (20) give us a non-trivial $\mathbb{Q}$-relation, namely,

$$\mathfrak{R}_{a,d} := \sum_{1\leq k<q/2} \alpha_k \log\left(2\sin\frac{k\pi}{q}\right) = 0,$$

where α_k is determined as follows:

$$\alpha_k = -1 \text{ if} \begin{cases} \text{either } (aq/d \bmod q) < q/2,\ k \not\equiv aq/d \bmod q \ \&\ k \equiv \pm(a + dj) \bmod q \\ \text{or } (aq/d \bmod q) \geq q/2,\ k \not\equiv -(aq/d) \bmod q \ \&\ k \equiv \pm(a + dj) \bmod q, \end{cases}$$

for some $1 \leq j \leq q/d$,

$$\alpha_k = 1 \text{ if} \begin{cases} \text{either } (aq/d \bmod q) < q/2,\ k \equiv aq/d \bmod q \ \&\ k \not\equiv \pm(a + dj) \bmod q \\ \text{or } (aq/d \bmod q) \geq q/2,\ k \equiv -(aq/d) \bmod q \ \&\ k \not\equiv \pm(a + dj) \bmod q, \end{cases}$$

for some $1 \leq j \leq q/d$ and

$$\alpha_k = 0, \text{ otherwise.}$$

To see that the above relation is non-trivial for q not prime and $q \geq 6$, note that at least one of the following scenarios happens- either $(aq/d \bmod q) < q/2$, in which case for

$k \equiv aq/d \bmod q$, $\alpha_k = \pm 1$, or $(aq/d \bmod q) \geq q/2$, in which case for $k \equiv -(aq/d) \bmod q$, $\alpha_k = \pm 1$.

Hence, the numbers under consideration in Conjecture 2 are $\mathbb{Q}$-linearly dependent. As a result, Livingston's conjecture is false when q is not prime and $q \geq 6$. □

4.2. **Proof of Theorem 1.2.** We use the theory of Dedekind determinants developed in [7] to prove that Conjecture 2 is true when the modulus q is prime.

Proof. Let p be an odd prime. Our aim is to prove that the numbers

$$\left\{ \log\left(2\sin\frac{a\pi}{p}\right) : 1 \leq a \leq \frac{p-1}{2} \right\} \text{ and } \pi$$

are $\bar{\mathbb{Q}}$-linearly independent.

Suppose, to the contrary, that the above numbers have a $\bar{\mathbb{Q}}$-linear relation among them. Thus, there exist algebraic numbers $\beta_0, \beta_1, \cdots, \beta_r$, not all zero, such that

$$\beta_0\pi + \sum_{a=1}^{r} \beta_a \log\left(2\sin\frac{a\pi}{p}\right) = 0, \tag{21}$$

where $r = (p-1)/2$. If $\beta_0 \neq 0$, then (21) does not hold by the following Lemma from [8]:

Lemma 4.1. *If $c_0, c_1, \cdots, c_n$ are algebraic numbers and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are positive algebraic numbers with $c_0 \neq 0$, then*

$$c_0\pi + \sum_{j=1}^{n} c_j \log\alpha_j \neq 0.$$

Thus, β_0 must be zero. Now, if the numbers

$$\left\{ \log\left(2\sin\frac{a\pi}{p}\right) : 1 \leq a \leq \frac{p-1}{2} \right\}$$

are $\mathbb{Q}$-linearly independent, then by Baker's Theorem 2.1, the above numbers are also $\bar{\mathbb{Q}}$-linearly independent. This contradicts our assumption, and hence, the above numbers must satisfy a $\mathbb{Q}$-linear relation. Thus, there exist $b_1, b_2, \cdots, b_r$ such that

$$\sum_{a=1}^{r} b_a \log\left(2\sin\frac{a\pi}{p}\right) = 0. \tag{22}$$

On clearing denominators, we can assume that

$$b_a \in \mathbb{Z},\ 1 \leq a \leq \frac{(p-1)}{2}.$$

Since log denotes the principal branch and $\sin a\pi/p \in \mathbb{R}_{>0}$, (22) gives us the multiplicative relation -

$$\prod_{a=1}^{r}\left(2\sin\frac{a\pi}{p}\right)^{b_a} = 1.$$

Using (15), this relation can be interpreted as a relation among roots of unity and cyclotomic numbers, i.e,

$$\prod_{a=1}^{r}\left(ie^{-ia\pi/p}(1-\zeta_p^a)\right)^{b_a}=1.$$

The above relation can be further simplified by raising both sides of the equation to the $4p$-th power. Since $\left(ie^{-ia\pi/p}\right)^{4p}=1$, we are now left with the simpler multiplicative relation,

$$\prod_{a=1}^{r}\left(1-\zeta_p^a\right)^{B_a}=1, \tag{23}$$

where $B_a := 4pb_a$ and each factor in the product belongs to the cyclotomic field, $\mathbb{Q}(\zeta_p)$.

Let G be the group $\mathbb{Z}/p\mathbb{Z}^*/\{\pm 1\}$. Let $c \in G$ and σ_c be the unique automorphism of $\mathbb{Q}(\zeta_p)$ such that

$$\sigma_c(\zeta_p)=\zeta_p^c.$$

The action of $\sigma_{c^{-1}}$ on (23) gives us

$$\prod_{a=1}^{r}\left(1-\zeta_p^{ac^{-1}}\right)^{B_a}=1.$$

On taking log of the above equation, we obtain the relation

$$\sum_{a=1}^{r}B_a\log\left(2\sin\frac{ac^{-1}\pi}{p}\right)=0, \tag{24}$$

for all $1\le a\le r$ and $1\le c\le r$.

Define an $r\times r$ matrix $\mathfrak{M}$ whose $(a,c)^{\text{th}}$ entry is

$$\log\left(2\sin\frac{ac^{-1}\pi}{p}\right).$$

Thus, (24) can be rewritten as a matrix equation, i.e,

$$\mathfrak{M}v=0,$$

where v the $r\times 1$ column vector with the a^{th}-entry being B_a. Since (22) was a non-trivial relation, $v\neq 0$. This is possible only if the determinant of $\mathfrak{M}$, $\det\mathfrak{M}=0$.

Let $\mathfrak{M}^T$ denote the transpose of $\mathfrak{M}$. Notice that $\mathfrak{M}^T$ is a matrix of the Dedekind type with $\mathfrak{f}: G\to\mathbb{C}$ given by

$$\mathfrak{f}(a)=\log\left(2\sin\frac{a\pi}{p}\right),$$

where G is as defined above. As mentioned in Theorem 2.2, $\mathfrak{M}^T$ is invertible if and only if

$$S_\chi := \sum_{a=1}^{r}\mathfrak{f}(a)\chi(a)\neq 0,$$

for all characters χ of the group G.

Observe that all characters of the group G are precisely the even Dirichlet characters modulo p. Thus, for a non-trivial even Dirichlet character χ, we can use (16) to express S_χ as:

$$\begin{aligned}
S_\chi &= \sum_{a=1}^{r} \chi(a) \log\left(2\sin\frac{a\pi}{p}\right) \\
&= \sum_{a=1}^{r} \chi(a) \log\left(\left|1-\zeta_p^a\right|\right) \\
&= \frac{1}{2}\sum_{a=1}^{p-1} \chi(a) \log\left(\left|1-\zeta_p^a\right|\right) \\
&= -\frac{p}{2\tau(\chi)} L(1,\bar{\chi}),
\end{aligned}$$

where the last equality follows from (7) and (8). By a famous theorem of Dirichlet,

$$L(1,\bar{\chi}) \neq 0.$$

Therefore, $S_\chi \neq 0$ when χ is a non-trivial character on G.

Let χ_0 be the trivial character on G, i.e, χ_0 is the trivial Dirichlet character modulo p. Then the factor S_{χ_0} is

$$\begin{aligned}
S_{\chi_0} &= \sum_{a=1}^{r} \mathfrak{f}(a) \\
&= \sum_{a=1}^{r} \log\left(2\sin\frac{a\pi}{p}\right) \\
&= \sum_{a=1}^{r} \log\left(\left|1-\zeta_p^a\right|\right) \\
&= \frac{1}{2}\log\left(\prod_{a=1}^{p-1}\left|1-\zeta_p^a\right|\right) \\
&= \frac{1}{2}\log p \neq 0,
\end{aligned}$$

where the last equality can be derived by noting that

$$\frac{1-X^p}{1-X} = \sum_{j=0}^{p-1} X^j = \prod_{a=1}^{p-1}(1-\zeta_p^a X),$$

substituting $X = 1$ and taking absolute values of both sides. Thus, $S_{\chi_0} \neq 0$.

Hence, $\mathfrak{M}^T$, and in turn, $\mathfrak{M}$ is invertible. Therefore $v = 0$, which is a contradiction. This proves the theorem. □

Proof of Corollary 1. Suppose that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = L(1,f) = 0.$$

From Theorem 1.2, we see that Conjecture 2 is true in the case under consideration. Thus, the relation obtained from (11), namely,

$$0 = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right) + \frac{2}{q}\sum_{b=1}^{r}\left\{\left[\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi ab}{q}\right)\right]\log\left(2\sin\frac{\pi b}{q}\right)\right\} - T_q, \quad (25)$$

where

$$T_q = \begin{cases} \frac{\log 2}{q}\left(\sum_{k=1}^{q-1}(-1)^k f(k)\right) & \text{if } q \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

is a trivial relation. This proves the corollary.

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