RELATIONS BETWEEN CERTAIN POWER SERIES AND FUNCTIONS INVOLVING ZEROS OF ZETA FUNCTIONS

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Abstract. We study infinite series of the form

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)}\, x^n$$

where $A(t), B(t) \in \mathbb{C}[t]$ are polynomials and $0 < x \leq 1$. We relate these series to other series involving zeros of the Riemann zeta-function. We also discuss functional relations between such power series and the zeros of other zeta-functions.

1. Introduction

In 1735, Euler proved that for $k \in \mathbb{N}$,

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

where B_k is the k-th Bernoulli number given by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k\, t^k}{k!}.$$

Thus, the Bernoulli numbers are rational numbers and we conclude that $\zeta(2k) \in \pi^{2k}\, \mathbb{Q}$. The nature of $\zeta(2k+1)$ however is still shrouded in mystery even though spectacular breakthroughs have been made by Apéry [A] in 1978 who showed that $\zeta(3) \notin \mathbb{Q}$ and by Rivoal [R] in 2000 who showed that for infinitely many k, $\zeta(2k+1) \notin \mathbb{Q}$.

It is thus natural to inquire whether we can evaluate explicitly a series of the form

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)} \tag{1}$$

where $A(t), B(t) \in \mathbb{C}[t]$ are polynomials with deg A < deg B and natural conditions are imposed to ensure that the series converges. We may consider more

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generally power series of the form

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)}\, x^n \tag{2}$$

with $|x| \leq 1$. The goal of this paper is to investigate these series and relate them to series of the form

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, y^\rho \tag{3}$$

where the sum is over non-trivial zeros of the Riemann zeta-function. There is nothing special about the Riemann zeta-function. One could replace it with any other L-function or more generally, a suitable element of the Selberg class. Such a connection was first discovered in a recent paper by S. Gun, M. R. Murty and P. Rath [GMR1] but their focus was on the transcendental nature of such sums. Here, our focus will be more on establishing a curious functional relation between sums of the form (2) and (3).

Returning momentarily to series of the form (1) and (2), we can identify certain cases when these can be evaluated explicitly. For example, if $A(t) \in \overline{\mathbb{Q}}[t]$, $B(t) \in \mathbb{Q}[t]$ where $B(t)$ has simple rational roots, S. D. Adhikari, N. Saradha, T. N. Shorey and R. Tijdeman [ASST] showed that (1) can be written as a linear form in logarithms of algebraic numbers with algebraic coefficients, and so by Baker's theory [B], the sum is transcendental provided it is not zero. They also discussed the transcendence of linear combinations of sums of the form (2) when $x \in \overline{\mathbb{Q}}$ (see Corollary 4.1 of [ASST] as well as Corollary 3.1).

If $B(t)$ does not have simple rational roots, the situation becomes more complicated, as can be inferred by the fact that $\zeta(3)$ or generally $\zeta(2k+1)$ fall into this category. The first serious investigation of such series was initiated by M. R. Murty and C. Weatherby [MW1] as well as S. Gun, M. R. Murty and P. Rath [GMR2]. In [MW1], the authors study (among other things)

$$\sum_{n \in \mathbb{Z}} \frac{A(n)}{B(n)} \tag{4}$$

and derive general results and explicit evaluations. In particular, Euler's evaluation of $\zeta(2k)$ is a special case of their work. A stunning example is given by the following:

$$\sum_{n \in \mathbb{Z}} \frac{1}{An^2 + Bn + C} = \frac{2\pi}{\sqrt{D}} \left(\frac{e^{2\pi\sqrt{D}/A - 1}}{e^{2\pi\sqrt{D}/A} - 2\cos(\pi B/A) e^{\pi\sqrt{D}/A} + 1} \right) \tag{5}$$

is transcendental if $A, B, C \in \mathbb{Z}$ and $-D = B^2 - 4AC < 0$.

More generally, one can evaluate explicitly

$$\sum_{n \in \mathbb{Z}} \frac{1}{(An^2 + Bn + C)^k}$$

and deduce transcendence results [MW2]. A critical role is played by a theorem of Nesterenko [N] that states that π and $e^{\pi\sqrt{D}}$ are algebraically independent. Thus, (5) is a transcendental number.

The essential idea animating much of the work in [MW1] and [MW2] is the following. Writing $A(X)/B(X)$ as a partial fraction, (and assuming for now that $B(X)$ has only simple zeros) we are led to sums of the form

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+\alpha_i} \tag{6}$$

where the α_i are roots of $B(X)$. This is $\pi\cot\pi\alpha_i$ deduced from the classical cotangent expression

$$\pi\cot\pi z = \sum_{n\in\mathbb{Z}} \frac{1}{n+z}, \quad z \notin \mathbb{Z}. \tag{7}$$

Of course, we must make some assumptions about the α_i and also understand the convergence in (6) and (7) as a limit:

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{N\to\infty} \sum_{|n|\le N} f(n).$$

By successive differentiation of (7) one can handle the case when $B(X)$ has multiple roots as well. These considerations lead one to explicit evaluations of series of the form (4). To go further into the study, one needs to invoke some algebraic number theory as well as a celebrated conjecture of Gelfond and Schneider, namely that if α is an algebraic number with $\alpha \neq 0, 1$ and β is an algebraic irrational number of degree d, then the $d-1$ numbers

$$\alpha^{\beta}, \alpha^{\beta^2}, \cdots, \alpha^{\beta^{d-1}} \tag{8}$$

are algebraically independent. A result of Diaz [D] states that the transcendence degree of the field generated by the numbers in (8) over $\overline{\mathbb{Q}}$ is at least $[(d+1)/2]$. When $d=2$, this is the famous Gelfond-Schneider theorem resolving a problem of Hilbert's list of 23 problems presented at the 1900 congress of mathematics in Paris. The case $d=3$ was also known earlier and is due to Gelfond. Invoking these results, we can deduce various transcendence theorems. We refer the reader to [MW1] for precise details.

In [PP], the authors consider (1), where the sum is over $n \ge 1$:

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)}$$

and (via partial fractions) are led to the study of the series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n+\alpha_i} \tag{9}$$

and more generally

$$\sum_{n=1}^{\infty} \frac{1}{(n+\alpha_i)^k}. \tag{10}$$

The fundamental idea in their work is the recognition that (9) is essentially the digamma function $\Psi(\alpha_i)$ and (10) is related to the k-th derivative $\Psi^{(k)}(\alpha_i)$. More precisely, we have

$$\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$$

and

$$-\Psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

where γ is Euler's constant. The digamma function $\Psi(x)$ appears in the constant term of the Laurent series expansion of the Hurwitz zeta-function at $s = 1$. Recall that for $0 < x \leq 1$, the Hurwitz zeta-function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

has the expansion:

$$\zeta(s,x) = \frac{1}{s-1} - \Psi(x) + O(s-1).$$

Thus, one can prove without difficulty that

$$\sum_{n=1}^{\infty} \frac{A(n)}{B(n)},$$

where $B(t)$ has simple zeros $\alpha_1, \alpha_2, \cdots, \alpha_r$ (say), is essentially a linear combination of $\Psi(\alpha_i)$ (see Theorem 10 of [MS]). If the α_i are rational numbers, a classical theorem of Gauss discovered in 1813 shows that for $(a,q) = 1$,

$$\Psi\left(\frac{a}{q}\right) = -\gamma - \log 2q - \frac{\pi}{2} \cot \frac{\pi a}{q} + 2 \sum_{0<j\leq q/2} \cos \frac{2\pi a j}{q} \left(\log \sin \frac{\pi j}{q} \right), \tag{11}$$

see for example [MS, pg. 300]. If however the zeros are neither simple, nor rational then there are considerable difficulties in evaluating in "closed form" the value of the sum and in ascertaining its algebraic or transcendental nature. For instance, if the roots of $B(t)$ are rational, but not simple, the value of the sum can be given as a linear combination of special values of the Hurwitz zeta-function at rational arguments. In [GMR2], the authors used the Chowla-Milnor conjecture regarding the $\mathbb{Q}$-linear independence of

$$\zeta\left(k, \frac{a}{q}\right) \; 1 \leq a < q, \; (a,q) = 1.$$

The nature of the Hurwitz zeta-function at irrational arguments is unknown and (to our knowledge) there has been scant attention given to such questions.

In this paper, we offer a new perspective on sums of the form (1) and (2) and relate such a study to cognate sums involving the zeros of the Riemann zeta-function. As will be explained below, one could replace the Riemann zeta-function by any L-function or more generally by an element of the Selberg class. To keep the prerequisites of this paper to a minimum, we do not do this here but indicate in our concluding remarks what needs to be modified and what can be expected.

Such sums involving zeros of the Riemann zeta-function are intricately connected to the Laurent series expansion of its logarithmic derivative. A case in point is Li's criterion for the Riemann hypothesis obtained by X.-J. Li [Li] in 1997. More specifically, let

$$\lambda_n := \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right),$$

where the sum is over non-trivial zeros of the Riemann zeta-function. Then the Riemann hypothesis is equivalent to the positivity of λ_n for all $n \in \mathbb{N}$. Furthermore, if

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + \sum_{j=0}^{\infty} \eta_j (s-1)^j, \tag{12}$$

then it was shown in [BL] that

$$\lambda_n = -\sum_{j=1}^{n} \left[\binom{n}{j}\eta_{j-1}\right] + 1 - (\log 4\pi + \gamma)\frac{n}{2} + \sum_{j=2}^{n} (-1)^j \binom{n}{j}(1-2^{-j})\zeta(j). \tag{13}$$

The study of η_j's is highly important for a variety of reasons. They enter into our understanding of Li's criterion for the Riemann hypothesis to hold as expressed by the formula (13) above. In this context, Coffey [Co] writes

$$\lambda_n = 1 - \frac{n}{2}(\gamma + \log 4\pi) + S_1(n) + S_2(n),$$

where

$$S_1(n) = \sum_{j=2}^{n} (-1)^j \binom{n}{j}\left(1 - \frac{1}{2^j}\right)\zeta(j)$$

and

$$S_2(n) = -\sum_{j=1}^{n} \binom{n}{j}\eta_{j-1},$$

where η_j's are as defined in (12). He shows that for $n \geq 2$,

$$\frac{1}{2}(n(\log n + \gamma - 1) + 1) \leq S_1(n) \leq \frac{1}{2}(n(\log n + \gamma + 1) - 1),$$

so that

$$\lambda_n = \frac{1}{2}\, n\log n + S_2(n) + O(n).$$

Now, Bombieri and Lagarias [BL] have shown that to deduce the Riemann hypothesis, it suffices to show that for any $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that

$$\lambda_n \geq -c(\epsilon) e^{\epsilon n},$$

for every $n \geq 1$. In other words, the Riemann hypothesis can be deduced from a good estimate for $S_2(n)$ involving the η_j's.

A generalized version of the η_j-coefficients appear in our analysis of sums of the form (3). In particular, let s_0 be a pole of logarithmic derivative of the Riemann zeta-function, i.e, $s_0 = 1$, $-2n$ for some $n \in \mathbb{N}$ or ρ for a non-trivial zero ρ of $\zeta(s)$. We define the *generalized η_j-coefficients* by

$$-\frac{\zeta'}{\zeta}(s) = \frac{R(s_0)}{s - s_0} + \sum_{j=0}^{\infty} \eta_j(s_0)\,(s - s_0)^j, \tag{14}$$

where $R(s_0)$ is the residue at $s = s_0$ of $-\,\zeta'/\zeta$. It is easy to see that $R(s_0) = 1$ if $s_0 = 1$ and $R(s_0) = -1$ if $s_0 = -2n$ for some $n \in \mathbb{N}$. If $s_0 = \rho$ for some non-trivial zero ρ of the Riemann zeta-function, $-R(\rho)$ is simply the order of the zero ρ and there is a folklore conjecture that the non-trivial zeros of $\zeta(s)$ are simple and thus, $R(\rho) = -1$. When $s_0 = 1$, these are the classical η-coefficients defined by Laurent series expansion of $-\,\zeta'/\zeta$ around $s = 1$ as given in (12). These generalized η-coefficients enter into formulas stated in Theorem 1.3 below in a fundamental way if $B(t)$ has a simple zero at s_0 which also happens to be either 1 or a zero of $\zeta(s)$. In order to understand these coefficients concretely, we derive an integral representation and a limit formula for these constants in a more general setting.

Proposition 1.1. Let

$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series, absolutely convergent on $\Re(s) > 1$. Suppose that for any $A > 0$,

$$S(x) := \sum_{n \leq x} a_n = \delta\, x + E(x), \tag{15}$$

for some $\delta \in \mathbb{R}$ and

$$E(x) = O\left(\frac{x}{(\log x)^A}\right). \tag{16}$$

Then, by partial summation, $f(s)$ can be analytically continued to $\Re(s) \geq 1$, with a possible simple pole at $s = 1$ and one can write its Laurent series expansion around $s = 1$ as

$$f(s) = \frac{\delta}{s-1} + \sum_{j=0}^{\infty} \eta_j(1, f)(s-1)^j.$$

Then,

$$\eta_0(1, f) = \delta + \int_1^\infty \frac{E(t)}{t^2}\, dt,$$

and for $j \geq 1$,

$$\eta_j(1, f) = \frac{(-1)^j}{j!} \int_1^\infty \frac{E(t)}{t^2} \Big((\log t)^j - j(\log t)^{j-1} \Big)\, dt. \tag{17}$$

Further, for $j \geq 1$,

$$\eta_j(1, f) = \frac{(-1)^j}{j!} \left\{ \lim_{x \to \infty} \left[\sum_{n \leq x} \frac{a_n (\log n)^j}{n} \right] - \delta\, \frac{(\log x)^{j+1}}{j+1} \right\}.$$

Thus, the generalized η_j-coefficients defined in the context of $-\zeta'/\zeta$ by (14) are nothing but $\eta_j(s_0) := \eta_j(s_0, -\zeta'/\zeta)$ in the above notation.

On the other hand, the constants $\eta_j(1, \zeta)$ (known as Stieltjes constants) were first introduced by Stieltjes (see [Na, pg.161]), who proved that

$$\eta_j(1, \zeta) = \frac{(-1)^j}{j!} \left\{ \lim_{x \to \infty} \left[\sum_{n \leq x} \frac{(\log n)^j}{n} \right] - \frac{(\log x)^{j+1}}{j+1} \right\},$$

in a letter to Hermite in 1885. This formula seems to have been rediscovered by Briggs and Chowla in 1955 (see [Na, pg. 163]). Clearly, this result can be stated in a more general setting, as is seen in Proposition 1.1.

Since this paper focuses on the logarithmic derivative of the Riemann zeta-function, we will use $\eta_j(s_0)$ for $\eta_j(s_0, -\zeta'/\zeta)$ and η_j for $\eta_j(1, -\zeta'/\zeta)$ to simplify notation. For the sake of clarity, we state the special case for $f(s) = -\zeta'/\zeta(s)$ separately below.

Proposition 1.2. Let η_j be as defined in (12). Let $\Lambda(n)$ denote the von-Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha,\ \alpha \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Then, for $j \geq 1$,

$$\eta_j = \frac{(-1)^j}{j!} \left\{ \lim_{x \to \infty} \left[\sum_{n \leq x} \frac{\Lambda(n)(\log n)^j}{n} \right] - \frac{(\log x)^{j+1}}{j+1} \right\}.$$

The main theorem of this paper is the following.

Theorem 1.3. *Let $x > 1$ and $A(t)$, $B(t) \in \mathbb{C}[t]$ with $B(t)$ having simple zeros, α_1, α_2, $\cdots$ α_r. First suppose that none of the α_i equal 1, a non-trivial zero of $\zeta(s)$ or $-2n$ ($n \in \mathbb{N}$).*

Then, if x is not a prime power,

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho} + \sum_{n=1}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n} - \frac{xA(1)}{B(1)}$$

$$= - \sum_i \lambda_i x^{\alpha_i} \left(\sum_{n \leq x} \frac{\Lambda(n)}{n^{\alpha_i}} \right) - \sum_i \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i), \tag{19}$$

where

$$\frac{A(t)}{B(t)} = \sum_i \frac{\lambda_i}{t - \alpha_i}.$$

Now, suppose $\alpha_1 = \rho_0$, a non-trivial zero of $\zeta(s)$ and none of the α_j, $2 \leq j \leq r$ are equal to 1, ρ or $-2n$. Then,

$$\sum_{\rho \neq \rho_0} \frac{A(\rho)}{B(\rho)}\, x^{\rho} + \sum_{n=1}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n} - \frac{xA(1)}{B(1)}$$

$$= - \sum_i \lambda_i x^{\alpha_i} \left(\sum_{n \leq x} \frac{\Lambda(n)}{n^{\alpha_i}} \right) - \sum_{i \neq 1} \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i) - \tag{20}$$

$$x^{\rho_0} \left(\sum_{i \neq 1} \frac{\lambda_i\, x^{\alpha_i}}{\rho_0 - \alpha_i} \right) + \lambda_1\, x^{\rho_0}\ \eta_0(\rho_0) + \lambda_1 x^{\rho_0} \log x.$$

Similarly, if $\alpha_1 = -2m$ for some $m \in \mathbb{N}$ and none of the α_j, $2 \leq j \leq r$ are equal to 1, ρ or $-2n$, then,

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho} + \sum_{\substack{n=1, \\ n \neq m}}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n} - \frac{xA(1)}{B(1)}$$

$$= - \sum_i \lambda_i x^{\alpha_i} \left(\sum_{n \leq x} \frac{\Lambda(n)}{n^{\alpha_i}} \right) - \sum_{i \neq 1} \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i) + \tag{21}$$

$$x^{-2m} \left(\sum_{i \neq 1} \frac{\lambda_i x^{\alpha_i}}{2m + \alpha_i} \right) + \lambda_1\, x^{-2m}\ \eta_0(-2m) + \lambda_1 x^{-2m} \log x,$$

and when $\alpha_1 = 1$ and none of the α_j, $2 \leq j \leq r$ are equal to 1, ρ or $-2n$, then,

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho} + \sum_{n=1}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n} - x \sum_{i \neq 1} \frac{\lambda_i}{1 - \alpha_i}$$

$$= - \sum_i \lambda_i x^{\alpha_i} \left(\sum_{n \leq x} \frac{\Lambda(n)}{n^{\alpha_i}} \right) - \sum_{i \neq 1} \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i) + \lambda_1\, x\, \eta_0(1) + \lambda_1\, x \log x. \tag{22}$$

Generally, if $B(t)$ has a subset of zeros which are equal to 1 or a zero of $\zeta(s)$, one modifies (19) *in the appropriate way.*

If x is a prime power, then the sum

$$\sum_{n\le x}\frac{\Lambda(n)}{n^{\alpha_i}}$$

on the right hand side of (19), (20), (21) *and* (22) *is replaced by*

$$\sum_{n<x}\frac{\Lambda(n)}{n^{\alpha_i}}+\frac{1}{2}\frac{\Lambda(x)}{x^{\alpha_i}}.$$

2. **Preliminaries**

In our discussion, a fundamental role is played by:

Lemma 2.1. If $x>1$, $x\neq p^m$ (p prime) then

$$\sum_{n\le x}\frac{\Lambda(n)}{n^s}=-\frac{\zeta'}{\zeta}(s)+\frac{x^{1-s}}{1-s}-\sum_{\rho}\frac{x^{\rho-s}}{\rho-s}+\sum_{n=1}^{\infty}\frac{x^{-2n-s}}{2n+s}, \tag{23}$$

provided $s\neq 1$, $s\neq\rho$, $s\neq -2n$ for any $n\in\mathbb{N}$.

If $s=1$, a non-trivial zero ρ_0 of $\zeta(s)$ or $-2m$ for some $m\in\mathbb{N}$, then the right hand side of (23) should be replaced by

$$\eta_0(1)+\log x-\sum_{\rho}\frac{x^{\rho-1}}{\rho-1}+\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n+1},$$

or

$$\eta_0(\rho_0)+\log x+\frac{x^{1-\rho_0}}{1-\rho_0}-\sum_{\rho\neq\rho_0}\frac{x^{\rho-\rho_0}}{\rho-\rho_0}+\sum_{n=1}^{\infty}\frac{x^{-2n-\rho_0}}{2n+\rho_0},$$

or

$$\eta_0(-2m)+\log x+\frac{x^{1+2m}}{1+2m}-\sum_{\rho}\frac{x^{\rho+2m}}{\rho+2m}+\sum_{\substack{n=1,\\ n\neq m}}^{\infty}\frac{x^{-2n+2m}}{2n-2m},$$

respectively.

If $x=p^m$, the left hand side must be corrected by the term $\Lambda(x)/2$.

Proof. This follows by the standard method of contour integration. The statement of the first part is found in several places (e.g. [IK, pg. 566] where we caution the reader that there is a typo in (25.21) in which the second "=" symbol on the right hand side should be a minus sign). Since no proof is available in the English language, we now give it.

We use Perron's formula in the following form [T, pg. 60]. Let

$$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},\ \ \sigma=\Re(s)>1,$$

where $a_n = O(\Phi(n))$, $\Phi(n)$ is increasing and assume

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O\bigg(\frac{1}{(1-\sigma)^\alpha}\bigg)$$

for some $\alpha \geq 0$, as $\sigma \to 1^+$. If $c > 0$ and $\sigma + c > 1$, x is not an integer and N is the nearest integer to x, we have

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w}\, dw \;+\; O\bigg(\frac{x^c}{T(\sigma+c-1)^\alpha}\bigg) + O\bigg(\frac{\Phi(2x) x^{1-\sigma} \log x}{T}\bigg) + O\bigg(\frac{\Phi(N) x^{1-\sigma}}{T|x-N|}\bigg).$$

If x is an integer, then

$$\sum_{n=1}^{x-1} \frac{a_n}{n^s} + \frac{a_x}{2x^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w}\, dw \;+\; O\bigg(\frac{x^c}{T(\sigma+c-1)^\alpha}\bigg) + O\bigg(\frac{\Phi(2x) x^{1-\sigma} \log x}{T}\bigg) + O\bigg(\frac{\Phi(x) x^{-\sigma}}{T}\bigg),$$

for any $T > 0$. We apply this to

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Since $\Lambda(n)$ is supported on prime powers, we deduce

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \bigg(-\frac{\zeta'}{\zeta}(s+w)\bigg) \frac{x^w}{w}\, dw \;+\; O\bigg(\frac{x^c}{T(\sigma+c-1)^\alpha}\bigg) + O\bigg(\frac{x^{1-\sigma} \log^2 x}{T}\bigg) + O\bigg(\frac{\log(N)\, x^{1-\sigma}}{T|x-N|}\bigg)$$

if x is not a prime power, whereas if x is a prime power, the last term is replaced by

$$O\bigg(\frac{x^{-\sigma} \log x}{T}\bigg).$$

Here c is chosen so that $c + \sigma > 1$.

The integral is evaluated using Cauchy's residue theorem as follows. If R denotes the oriented rectangle with vertices $c - iT$, $c + iT$, $-U + iT$ and $-U - iT$, with U large and unequal to an integer, then by the residue theorem,

$$\begin{aligned} &\frac{1}{2\pi i} \int_R \bigg(-\frac{\zeta'}{\zeta}(s+w)\bigg) \frac{x^w}{w}\, dw \\ &= -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} + \sum_{n=1}^{U} \frac{x^{-2n-s}}{2n+s} \end{aligned}$$

because

$$-\frac{\zeta'}{\zeta}(s+w)\,\frac{x^w}{w}$$

has poles at $w = 0$, $1-s$, $\rho - s$ and $-2n-s$ with $|\Im(\rho)| \le T$ and $n \le U$ in the rectangle R. We want to let T, $U \to \infty$ but before we do that, we need to estimate the line integrals along the other three edges of the rectangle. Of course, we must choose T so that it is not the ordinate of a zero of $\zeta(s)$. But these estimates are quite standard (see [Mu, Exercise 7.2.4]).

The key point is to know for this suitable choice of T, we have

$$\left| -\frac{\zeta'}{\zeta}(s+w)\right| = O(\log^2 T)$$

which leads to the final estimate of

$$O\left(\frac{x^c \log^2 T}{T \log x}\right)$$

for the horizontal line integrals. For the vertical line integral, we have an estimate of

$$O\left(\frac{\log U}{U}\cdot\frac{T}{x^T}\right)$$

as seen in [Mu, pg. 392]. We let $U \to \infty$ first and then let $T \to \infty$ to deduce the final result. This completes the proof of Lemma 2.1 if $s \neq 1$, ρ or $-2n$ for $n \in \mathbb{N}$.

If $s = 1$, ρ_0 or $-2m$ for some $m \in \mathbb{N}$, we take limits of both sides as $s \to 1$, ρ_0 or $-2m$. We illustrate this using the case of $s = 1$ since the analysis for $s = \rho_0$ or $-2m$ is similar. Taking the limit of right-hand side of (23) as $s \to 1$, we obtain

$$-\sum_{\rho}\frac{x^{\rho-1}}{\rho-1} + \sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n+1} + \lim_{s\to 1}\left(\frac{x^{1-s}}{1-s} - \frac{\zeta'}{\zeta}(s)\right).$$

Now, using (14) at $s_0 = 1$, $R(s_0) = 1$ and

$$\frac{x^{1-s}}{1-s} = \frac{-1}{s-1} + \log x + O(1-s),$$

we see that the limit evaluates to $\eta_0(1) + \log x$ and thus, Lemma 2.1 is proved. $\square$

Incidentally, $\eta_0(1) = -\gamma$ and this is not difficult to see because

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

3. **Proofs of the Propositions**

In this section, we give proofs of the Propositions in Section 1.

3.1. **Proof of Proposition 1.1.** Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Then the usual partial summation method gives,

$$\begin{aligned} f(s) &= s \int_1^{\infty} \frac{S(x)}{x^{s+1}}\, dx \\ &= \frac{\delta\, s}{s-1} + s \int_1^{\infty} \frac{S(x) - \delta\, x}{x^{s+1}}\, dx. \end{aligned}$$

By our hypothesis, the integral on the right hand side converges absolutely for $\Re(s) \geq 1$. Thus, we can derive the Laurent expansion at $s = 1$ using this integral. Writing $E(x) = S(x) - \delta x$, we find

$$\begin{aligned} s \int_1^{\infty} \frac{E(x)}{x^{s+1}}\, dx &= \big((s-1)+1\big) \int_1^{\infty} \frac{E(x)}{x^2} \sum_{j=0}^{\infty} \frac{(-1)^j \,(\log x)^j}{j!}\, (s-1)^j\, dx \\ &= \int_1^{\infty} \frac{E(x)}{x^2} \sum_{j=0}^{\infty} \frac{(-1)^j \,(\log x)^j}{j!} (s-1)^{j+1}\, dx + \\ &\qquad \int_1^{\infty} \frac{E(x)}{x^2} \sum_{j=0}^{\infty} \frac{(-1)^j \,(\log x)^j}{j!}\, (s-1)^j\, dx \\ &= \int_1^{\infty} \frac{E(x)}{x^2}\, dx + \\ &\qquad \sum_{j=1}^{\infty} \frac{(-1)^j}{j!}\, (s-1)^j \int_1^{\infty} \frac{E(x)}{x^2} \Big((\log x)^j - j(\log x)^{j-1} \Big)\, dx, \end{aligned}$$

the interchange of summation and integral being justified by the absolute convergence of the integral at $s = 1$. Thus, we see that an integral representation for the Laurent series coefficients at $s = 1$ of a general Dirichlet series can be obtained.

On the other hand, analysis with the help of partial summation gives

$$\begin{aligned} \sum_{n \leq x} \frac{a_n (\log n)^j}{n} &= \frac{S(x)\, (\log x)^j}{x} + \int_1^x \frac{S(t)}{t^2} \Big((\log t)^j - j(\log t)^{j-1} \Big)\, dt \\ &= \frac{S(x)\, (\log x)^j}{x} + \delta \int_1^x \frac{(\log t)^j}{t}\, dt - j\, \delta \int_1^x \frac{(\log t)^{j-1}}{t}\, dt + \\ &\qquad \int_1^x \frac{E(t)}{t^2} \Big((\log t)^j - j(\log t)^{j-1} \Big)\, dt, \end{aligned}$$

by (15). Hence, we deduce that

$$\sum_{n\le x}\frac{a_n(\log n)^j}{n} = \delta\,\frac{(\log x)^{j+1}}{j+1} + \int_1^\infty \frac{E(t)}{t^2}\Big((\log t)^j - j(\log t)^{j-1}\Big)\,dt + \left(\frac{S(x)\,(\log x)^j}{x} - \delta\,(\log x)^j\right) + \mathcal{E}(x),$$

where

$$\mathcal{E}(x) = \int_x^\infty \frac{E(t)}{t^2}\Big((\log t)^j - j(\log t)^{j-1}\Big)\,dt.$$

Note that $\mathcal{E}(x)$ is the tail of a convergent integral by (16) and since $j \ge 1$ and therefore, tends to zero as $x \to \infty$. Moreover, the third term on the right hand side also goes to zero as $x \to \infty$ by (15). On comparison with (17), the proposition is proved.

3.2. **Proof of Proposition 1.2.** Let $\Lambda(n)$ denote the von-Mangoldt function given by (18). Recall that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^\infty \frac{\Lambda(n)}{n^s},$$

for $\Re(s) > 1$ and that

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + \sum_{j=0}^\infty \eta_j(s-1)^j.$$

Let

$$\psi(x) := \sum_{n\le x}\Lambda(n).$$

Then the prime number theorem (see [Mu, Theorem 4.2.9]) gives

$$\psi(x) = x + E(x),$$

with

$$E(x) = O\Big(x\exp\left(-c\,(\log x)^{1/10}\right)\Big),$$

for some positive constant c. Thus, we can apply Proposition 1.1 to obtain the result.

4. **Proof of the Main Theorem**

First suppose that x is not a prime power. Without loss of generality, we may assume that $B(t)$ is monic. For the moment, we suppose that $B(t)$ has only simple zeros and none of which are equal to a non-trivial zero or a pole of the Riemann zeta-function, or $-2n$ for some natural number n. We write using partial fractions

$$\frac{A(t)}{B(t)} = \sum_i \frac{\lambda_i}{t-\alpha_i} \tag{24}$$

where $\lambda_i = A(\alpha_i)/B'(\alpha_i)$. Then

$$\sum_\rho \frac{A(\rho)}{B(\rho)} x^\rho = \sum_i \lambda_i \Big(\sum_\rho \frac{x^\rho}{\rho - \alpha_i} \Big).$$

We analyze the inner sum using the lemma: by (23),

$$\sum_\rho \frac{x^{\rho-\alpha}}{\rho - \alpha} = -\frac{\zeta'}{\zeta}(\alpha) + \frac{x^{1-\alpha}}{1-\alpha} + \sum_{n=1}^\infty \frac{x^{-2n-\alpha}}{2n+\alpha} - \sum_{n \le x} \frac{\Lambda(n)}{n^\alpha}.$$

We put $\alpha = \alpha_i$, multiply by $\lambda_i x^{\alpha_i}$ and sum over the i to get

$$\sum_\rho \frac{A(\rho)}{B(\rho)} x^\rho = -\sum_i \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i) \; + \; \frac{xA(1)}{B(1)} \; - \sum_i \lambda_i x^{\alpha_i} \Big(\sum_{n \le x} \frac{\Lambda(n)}{n^{\alpha_i}} \Big) + \sum_{n=1}^\infty x^{-2n} \Big(\sum_i \frac{\lambda_i}{2n + \alpha_i} \Big).$$

Note that using (24),

$$\sum_i \frac{\lambda_i}{2n + \alpha_i} = -\frac{A(-2n)}{B(-2n)}.$$

This proves that

$$\begin{aligned} &\sum_\rho \frac{A(\rho)}{B(\rho)}\, x^\rho \; + \sum_{n=1}^\infty \frac{A(-2n)}{B(-2n)} \Big(\frac{1}{x} \Big)^{2n} \\ &= \frac{xA(1)}{B(1)} - \sum_i \lambda_i x^{\alpha_i} \Big(\sum_{n \le x} \frac{\Lambda(n)}{n^{\alpha_i}} \Big) - \sum_i \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i). \end{aligned}$$

An appropriate modification gives the case when x is a prime power where we need to add

$$\frac{1}{2}\, \Lambda(x) \, \big(\sum_i \lambda_i \big).$$

If we assume that deg $A \le$ deg $B - 2$, then one can deduce from the partial fraction decomposition of $A(X)/B(X)$ that $\sum_i \lambda_i = 0$.

Finally, when $B(t)$ has a zero at 1, ρ_0 or $-2m$, the terms in the summation have to be adjusted appropriately. Since the analysis of the three cases is identical, we demonstrate this in the case $\alpha_1 = 1$, α_j is not 1, ρ or $-2n$ for $j = 2, \cdots, r$. For x not a prime power, Lemma 2.1 gives

$$\sum_\rho \frac{x^{\rho-1}}{\rho - 1} = \eta_0(1) + \log x + \sum_{n=1}^\infty \frac{x^{-2n-1}}{2n+1} - \sum_{n \le x} \frac{\Lambda(n)}{n}.$$

Thus, modifying the previous argument for α_1 as above, we get

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho} \;+\; \sum_{n=1}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n} \;-\; x \sum_{i\neq 1} \frac{\lambda_i}{1-\alpha_i}$$
$$= -\;\sum_{i} \lambda_i x^{\alpha_i} \left(\sum_{n\leq x} \frac{\Lambda(n)}{n^{\alpha_i}} \right) \;-\; \sum_{i\neq 1} \lambda_i x^{\alpha_i} \frac{\zeta'}{\zeta}(\alpha_i) +\; \lambda_1\, x\, \eta_0(1) + \lambda_1\, x \log x.$$

This completes proof of the main theorem.

5. **Connections with other L-series**

As noted in [GMR1], our study of series of the form (3) can be expanded to the realm of the Selberg class. Before amplifying the general setting, let us focus on two specific cases.

If χ is a non-principal Dirichlet character mod q which is even (that is $\chi(-1) = 1$), then our earlier discussion extends mutatis mutandis to this case also with only one minor modification. Since $L(s,\chi)$ does not have a pole at $s = 1$, the analog of Lemma 2.1 becomes

$$\sum_{n\leq x} \frac{\Lambda(n)\chi(n)}{n^s} = -\,\frac{L'}{L}(s,\chi) - \;\sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s} \tag{25}$$

where the second sum on the right hand side is over the non-trivial zeros of $L(s,\chi)$. Our main theorem modified to deal with this case becomes

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho} \;+\; \sum_{n=1}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n}$$
$$= \;-\;\sum_{i} \lambda_i x^{\alpha_i} \left(\sum_{n\leq x} \frac{\Lambda(n)\chi(n)}{n^{\alpha_i}} \right) \;-\; \sum_{i} \lambda_i x^{\alpha_i} \frac{L'}{L}(\alpha_i,\chi),$$

if x is not a prime power. As noted earlier, if x is a prime power, the sum

$$\sum_{n\leq x} \frac{\Lambda(n)\chi(n)}{n^s}$$

must be replaced by

$$\sum_{n< x} \frac{\Lambda(n)\chi(n)}{n^{\alpha_i}} + \frac{1}{2}\frac{\Lambda(x)}{x^{\alpha_i}}\chi(x).$$

Since the method of contour integration employed in the proof of Lemma 2.1 goes through with little change, we leave the details to the reader.

If χ is an odd character, the analogous derivation needs some modification but only in one step. The trivial zeros of $L(s,\chi)$ are now at the negative odd integers

$-1, -3, -5, \cdots$ so that the last term on the right hand side of (25) changes to

$$\sum_{n=0}^{\infty} \frac{x^{-2n-1-s}}{2n+1+s}$$

and the analog of Theorem 1.3 becomes

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho} + \sum_{n=1}^{\infty} \frac{A(-2n-1)}{B(-2n-1)} \left(\frac{1}{x}\right)^{2n+1}$$
$$= -\sum_{i} \lambda_i x^{\alpha_i} \left(\sum_{n \le x} \frac{\Lambda(n)\chi(n)}{n^{\alpha_i}} \right) - \sum_{i} \lambda_i x^{\alpha_i} \frac{L'}{L}(\alpha_i, \chi),$$

for $x > 1$, x not a prime power. If x is a prime power, we need to make the same modification as we made earlier.

These theorems extend smoothly to elements of the Selberg class. We will not adumbrate the properties of this class here, but refer the reader to the exposition in [GMR1] where the authors study sums of the form

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho}$$

when ρ runs over the non-trivial zeros of a fixed element F of the Selberg class. The essential point to note is that the nature of the second term in the appropriate analogue of (25) is determined by the trivial zeros of $F(s)$. These sums will often not be of the form

$$\sum_{n=1}^{\infty} \frac{A(-2n)}{B(-2n)} \left(\frac{1}{x}\right)^{2n}, \ \sum_{n=1}^{\infty} \frac{A(-2n-1)}{B(-2n-1)} \left(\frac{1}{x}\right)^{2n+1} \ \text{or} \ \sum_{n=1}^{\infty} \frac{A(-n)}{B(-n)} \left(\frac{1}{x}\right)^{n} \tag{26}$$

because these terms are determined by the nature of Γ-factors in the functional equation of $F(s)$. Only in special cases does the second term on the left-hand side of the analogue of (25) take such a simple form. For instance, if the Γ-factor in the functional equation is

$$\Gamma\left(\frac{s}{2}\right), \ \Gamma\left(\frac{s+1}{2}\right) \ \text{or} \ \Gamma(s),$$

then the second term on the left-hand side is one of the expressions in (26) respectively.

We have already seen the first two cases of these Γ-factors arising in the case of $L(s, \chi)$ with χ even or odd. The case of $\Gamma(s)$ emerges if $F(s)$ is the L-function attached to a Hecke eigenform. This case leads to an expression of the form

$$\sum_{n=1}^{\infty} \frac{A(-n)}{B(-n)} \left(\frac{1}{x}\right)^{n}.$$

In these formulas, we have assumed that $x > 1$. One could analyze the case $x \to 1^+$ and derive corresponding results.

As noticed in [GMR1], one can investigate the case $0 < x < 1$ of the series

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)}\, x^{\rho}$$

in a similar manner. By the method used in the proof of Lemma 2.1, one can show that if $s \neq 0, 1, 3, 5, \cdots$ and $1/x$ is not a prime power, then

$$\begin{aligned}&\sum_{n \leq 1/x} \frac{\Lambda(n)}{n^{1-s}} \\ &= -\frac{\zeta'}{\zeta}(1-s) + \frac{x^{-s}}{s} + \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{2n+1-s}}{2n+1-s}.\end{aligned}$$

An appropriate adjustment of the left hand side is needed if $1/x$ is a prime power. Clearly, similar results can be derived for elements of the Selberg class.

What these results suggest is a method (perhaps) to analyze relations that may exist among special values of logarithmic derivative of the Riemann zeta function. If we consider (say) $L(s, \chi)$ with χ even, then one could explore any rational linear combination of special values of the logarithmic derivatives of $L(s, \chi)$ as χ varies. These investigations we relegate to a future occasion.

6. **Concluding Remarks**

Our study here opens a new line of investigation regarding on the one hand sums of the form (1) and generally (2) relating them to sums of the form (3) expressing a functional relation. On the other hand, this relation involves linear forms in logarithms as well as the η_j-coefficients of a general kind.

Scattered throughout the literature are various (seemingly unrelated) investigations and it is hoped that these disparate researches can be brought into a cohesive unity that will illuminate our understanding about these sums and perhaps shed some light on the Riemann hypothesis.

To give one example of related results in the literature, we state here a fascinating formula found by Ihara, Murty and Shimura [IMS].

Let K be an algebraic number field and write

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for its Dedekind zeta function. Let χ be a primitive Dirichlet character. Modifying the (confusing) notation of [IMS], we define

$$L_K(s,\chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}.$$

It is well-known that $L_K(s,\chi)$ extends to an entire function if $\chi \neq \chi_0$, the principal character. Define γ_K to be the constant term divided by the residue of the Laurent expansion of $\zeta_K(s)$ at $s=1$. Set

$$\gamma^*_{K,\chi} = \begin{cases} \gamma_K + 1, & \text{if } \chi = \chi_0, \\ \\ \frac{L'_K(1,\chi)}{L_K(1,\chi)} & \text{if } \chi \neq \chi_0, \end{cases}$$

and define for $x > 1$,

$$\Phi_{K,\chi}(x) := \frac{1}{x-1} \sum_{N(\mathfrak{p})^k \leq x} \left(\frac{x}{N\mathfrak{p}^k} - 1 \right) \chi(N\mathfrak{p}) \log(N\mathfrak{p}).$$

Then, for $x > 1$,

$$\gamma^*_{K,\chi} = \delta_\chi \log x - \Phi_{K,\chi}(x) + \frac{1}{x-1} \sum_{\rho} \frac{x^\rho - 1}{\rho(1-\rho)}$$
$$+ \frac{a}{2} F_1(x) + \frac{a'}{2} F_3(x) + r_2 F_2(x),$$

where $\delta_\chi = 1$ or 0 depending on whether $\chi = \chi_0$ or not, ρ runs over non-trivial zeros of $L_K(s,\chi)$, a is the number of real places of K where χ is unramified, a' is the number of real places of K where χ is ramified, $r_1 = a + a'$ (resp. r_2) is the total number of real (resp. complex) places of K and

$$F_1(x) = \log \frac{x+1}{x-1} + \frac{2}{x-1} \log \frac{x+1}{2},$$

$$F_3(x) = \log \frac{x^2}{x^2-1} + \frac{2}{x-1} \log \frac{2x}{x+1},$$

and

$$F_2(x) = \log \frac{x}{x-1} + \frac{\log x}{x-1},$$

see [IMS, Theorem 1]. This formula can be deduced by our general methodology discussed in earlier sections of this paper. The novelty here is the meaning of the expression on the right hand side.

Related to this, the authors in [IMS] also derive the following. Let d_K be the discriminant of K, F_χ be the conductor of χ and put $d_\chi = |d_K| N(F_\chi)$. Let

$$\alpha_{K,\chi} = \frac{1}{2} \log d_\chi$$

and

$$\beta_{K,\chi} - \left(\frac{a+r_2}{2}\right)(\gamma + \log 4\pi) - \left(\frac{a'+r_2}{2}\right)(\gamma + \log \pi).$$

Then,

$$\gamma^*_{K,\chi} = \sum_\rho \frac{1}{1-\rho} - \alpha_{K,\chi} - \beta_{K,\chi}.$$

In particular, one deduces that for $x > 1$, x algebraic,

$$\gamma^*_{K,\chi} - \frac{1}{x-1}\sum_\rho \frac{x^\rho - 1}{\rho(1-\rho)}$$

is a linear form in logarithms of algebraic numbers. Baker's theory implies that this is a transcendental number if it is non-zero, related to the theme of [GMR1].

This raises a series of cognate questions, the foremost being the non-vanishing of $L'_K(1,\chi)$. Indeed, in [MM], it was shown that if $K = \mathbb{Q}$ and χ is the quadratic character associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ $(d > 0)$, such that $L'(1,\chi) = 0$, then e^γ is transcendental. The vanishing or non-vanishing of $L'(1,\chi)$ has received very little attention in the literature and these remarks indicate that the problem is worthy of serious study.

We also signal the importance of related themes discovered by A. P. Guinand [G] and his doctoral student I. C. Chakravarty [C]. Special cases of the functional relation we derived in this paper can be found in [G], where curiously the author assumes the Riemann hypothesis. They also study the "secondary zeta-functions" defined as

$$\sum_{\gamma>0} \gamma^{-s},$$

where γ runs through the imaginary parts of the non-trivial zeros of $\zeta(s)$. They derive analytic continuation and functional equation of such series.

These researches reveal that there are further patterns to explore and embrace into a larger theory. We relegate this to the future.

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