VARIANTS OF ERDŐS-KAC VIA TAUBERIAN THEOREMS

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Abstract. In this note, we establish a version of the Erdős-Kac theorem on monoids, where the uniform measure is replaced by a harmonic measure with completely multiplicative weights. This allows the deduction of several known and interesting Erdős-Kac type results as special cases of our theorem. For instance, we deduce the existence of Beurling generalized number systems where the analog of the prime number theorem fails, the uniform analog of Erdős-Kac is not known, but the harmonic analog of Erdős-Kac is true. Our approach expands upon that of M. Cranston and T. Mountford, and relies on the application of a Tauberian theorem for general Dirichlet series.

1. Introduction

The value of analytical methods in the study of numbers can be witnessed in its most fundamental form in Euler's demonstration of the infinitude of primes. To deduce this, he considered the limit as $x \to 1^+$ in the identity

$$\prod_{p\text{ - prime}} \left(1 - \frac{1}{p^x}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad \text{for} \qquad x > 1,$$

and used the divergence of the harmonic series. Following Euler, number theory gained further impetus with the introduction of complex analytical tools by Riemann in his seminal 1859 paper. This approach ultimately paved way for the proof of the prime number theorem by Hadamard and de la Vallée Poussin at the close of the nineteenth century. The next twenty years were marked by the infusion of many new ideas, especially stemming from the works of Ramanujan. Each of these strands of thought led to profound mathematical insights which are being investigated even today. The circle method, which arose in the work of Hardy and Ramanujan on the partition function, and modular forms, whose study was motivated by Ramanujan's conjectures on the τ-function, are two such instances.

Among these was a result of Hardy and Ramanujan on the 'normal' number of primes factors of a natural number. For $n \in \mathbb{N}$, let $\omega(n)$ denote the number of distinct prime factors of n. It is not hard to see that $\omega(n)$ can oscillate from 1 to approximately $\frac{\log n}{\log\log n}$ for primorials. However, Hardy and Ramanujan [15] showed that except for at most $o(N)$ many $n \le N$, $\omega(n)$ is 'very close' to $\log\log n$. They formulated this as $\omega(n)$ having *normal order* $\log\log n$. In 1934, Turán [36] gave a simpler proof of the Hardy-Ramanujan result, essentially by computing the 'mean' and the 'variance' of $\omega(n)$.

This probabilistic perspective was recognized by Mark Kac, who anticipated that the central limit theorem, a fundamental phenomenon in probability theory, may also have incarnations in

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the theory of numbers. In 1940, together with P. Erdős [12], one of the masters of sieve theory, they established that

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

has the standard normal or Gaussian distribution in the limit. In particular, they proved that

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le n \le N : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \le x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

This result is regarded as the cornerstone of classical probabilistic number theory and has been widely studied in the literature. For more details on the intriguing story of how the Erdős-Kac theorem came about, see [10, page 24].

The proof of Erdős and Kac relied on a clever application of the Brun sieve to count the number of natural numbers without small prime factors, and was rather involved. Till date, many different proofs, a plethora of generalizations and variations of the Erdős-Kac theorem have appeared (for instance, [23], [31], [14], [6], [16]). One of the earliest alternative proofs was given by Billingsley [2], who reduced the deduction to an application of the central limit theorem with the Lyapunov-Lindeberg condition and the method of moments in probability. This proof has been at the heart of axiomatizations such as that by Y. R. Liu [24] for monoids and by R. Murty, K. Murty and S. Pujahari [30] for $\omega(a_m)$, where a_m's are certain sequences of arithmetic significance. The proof of a localized version of the theorem, namely, when $\omega(n)$ is replaced by $\omega_y(n) = \sum_{p|n, p\le y} 1$ by A. B. Dixit and R. Murty [7] also builds upon the ideas of Billingsley. A more transparent sieve-theoretic proof, again relying on the method of moments, was given by Granville and Soundararajan [13]. This theorem was also studied from the dynamical viewpoint by Loyd [25]. A different direction of study is a weighted analogue of the Erdős-Kac theorem, studied by Elliott [11] for the usual divisor function as the weight, Khan-Milinovich-Subedi [20] when the weight is the k-fold divisor function and by Tenenbaum [35] and Elboim-Gorodetsky [9] for multiplicative weights. In order to maintain brevity, we refrain from describing the various generalizations here and instead urge the reader to see references in the above articles.

In this paper, we focus on a recent approach towards the Erdős-Kac theorem by M. Cranston and T. Mountford [5]. They first establish a central limit theorem for $\omega(X_s)$ where X_s is a random variable with the Riemann zeta-distribution, and then apply the Wiener-Ikehara Tauberian theorem to 'transfer' the result to the classical setting. However, the use of the Wiener-Ikehara Tauberian theorem requires an assumption on analytic continuation of the corresponding Dirichlet series, which is not justified in their paper, resulting in a lacuna in their proof[1]. In this paper, we instead apply Karamata's version of the Hardy-Littlewood Tauberian theorem to deduce Erdős-Kac with respect to the harmonic distribution. More specifically, our main theorem will imply that if $H_N := \sum_{n=1}^{N} n^{-1}$ and h_N denotes the harmonic measure on $[N] = \{1, 2, \cdots, N\}$ given by $h_N(\{n\}) = H_N^{-1} \times (1/n)$, then

$$\lim_{N\to\infty} h_N \left(\left\{ n \in [N] : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \le x \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

[1] On contacting the authors, they informed us that they are aware of the gap and are working on fixing it. They are also aware that their method gives an analog of the Erdős-Kac theorem for the harmonic distribution, as expanded upon in this article.

In order to present a unified account of the various settings in which this method is applicable, our theorem is formulated for monoids endowed with a completely multiplicative size function and a completely multiplicative weight function. Unfortunately, the complete multiplicativity cannot be easily replaced by multiplicativity due to technical conditions. We elaborate upon this point in the last section. Another setting which is amenable for the application of the current method is when the support of the measures considered is 'squarefree'. We study this scenario in a separate section. Thus, the two main theorems in this paper are as below.

Fix a countably infinite set of elements, say $\mathcal{P}$, to be treated as the set of prime numbers. Let $\mathcal{M}$ be the monoid generated by elements in $\mathcal{P}$, written multiplicatively. That is,

$$\mathcal{M} = \left\{ m = \prod_{p \in \mathcal{P}} p^{\nu_p(m)} \,:\, \nu_p(m) \in \mathbb{Z}_{\geq 0},\ \nu_p(m) = 0 \text{ for all but finitely many } p \right\}.$$

One can define

$$\omega_{\mathcal{M}}(m) = \#\{p \in \mathcal{P} \,:\, \nu_p(m) > 0\}.$$

Fix a norm map, that is, a monoid homomorphism $\|\cdot\| : \mathcal{M} \to \mathbb{R}_{\geq 1}$ with $\|1_{\mathcal{M}}\| = 1$ and denote by $\mathcal{F}_N := \{m \in \mathcal{M} \,:\, \|m\| \leq N\}$.

For the first theorem, let $\alpha : \mathcal{M} \to \mathbb{R}_{\geq 0}$ be a completely multiplicative function, namely, $\alpha(p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}) = \alpha(p_1)^{k_1} \alpha(p_2)^{k_2} \cdots \alpha(p_t)^{k_t}$ for $p_1, p_2, \cdots, p_t \in \mathcal{P}$ and $k_1, k_2, \cdots, k_t \in \mathbb{N}$. Define

$$H_{\alpha,N} := \sum_{m \in \mathcal{F}_N} \frac{\alpha(m)}{\|m\|},$$

and the harmonic measure of weight α on $\mathcal{F}_N$ as

$$h_{\alpha,N}\left(\{m\}\right) = \left(\frac{\alpha(m)}{\|m\|}\right) \times \frac{1}{H_{\alpha,N}} \qquad \text{for all} \qquad m \in \mathcal{F}_N.$$

We also associate a zeta-function to the above setup as

$$\zeta_{\alpha,\mathcal{M}}(s) := \sum_{m \in \mathcal{M}} \frac{\alpha(m)}{\|m\|^s} = \lim_{N \to \infty} \sum_{m \in \mathcal{F}_N} \frac{\alpha(m)}{\|m\|^s}.$$

Note that this function can also be expressed as an Euler product

$$\zeta_{\alpha,\mathcal{M}}(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{\alpha(p)}{\|p\|^s}\right)^{-1}$$

in the region of convergence. Throughout this paper, we assume that $\zeta_{\alpha,\mathcal{M}}(s)$ converges absolutely in the half plane $\mathrm{Re}(s) > 1$.

With this notation, we prove the following generalized Erdős-Kac theorem.

Theorem 1.1. *Let $\alpha : \mathcal{M} \to \mathbb{R}_{\geq 0}$ be a completely multiplicative function such that*

(i) $\lim_{s \to 1^+} (s-1)\, \zeta_{\alpha,\mathcal{M}}(s) = \kappa$ *for some* $\kappa \in \mathbb{R}$, $\kappa \neq 0$.

(ii) $|\alpha(p)| \leq \|p\|^{\theta}$ *for some* $0 \leq \theta < 1/2$ *for all* $p \in \mathcal{P}$.

Then

$$\lim_{N \to \infty} h_{\alpha,N}\left(\left\{ m \in \mathcal{F}_N \,:\, \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \leq x \right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

for every real number x.

One can re-interpret the left hand side in the above limit as

$$\frac{1}{H_{\alpha,N}} \sum_{\substack{m\in\mathcal{F}_N \\ \frac{\omega_{\mathcal{M}}(m)-\log\log\|m\|}{\sqrt{\log\log\|m\|}}\le x}} \frac{\alpha(m)}{\|m\|},$$

where the weight $\alpha(m)/\|m\|$ is equal to 1 in the classical Erdős-Kac theorem.

To state our second main theorem, let $\mathcal{P}$, $\mathcal{M}$, $\|\cdot\|$ and $\omega_{\mathcal{M}}$ be as before. Let $\eta : \mathcal{M} \to \mathbb{C}$ be a multiplicative function supported on the 'squarefree' elements of $\mathcal{M}$, that is, if $m = p_1 p_2 \cdots p_t$, then $\eta(m) = \eta(p_1)\eta(p_2)\cdots\eta(p_t)$, and if $m = \prod_{p\in\mathcal{P}} p^{\nu_p(m)}$ with $\nu_p(m) \ge 2$ for some $p \in \mathcal{P}$, then $\eta(m) = 0$. Let $H_{\eta,N}$, $h_{\eta,N}$, $\zeta_{\eta,\mathcal{M}}(s)$ be defined as in the previous setup. We assume that $\zeta_{\eta,\mathcal{M}}(s)$ converges in $\mathrm{Re}(s) > 1$. Then we prove the following.

Theorem 1.2. *Let $\eta : \mathcal{M} \to \mathbb{R}_{\ge 0}$ be a multiplicative function supported on the squarefree elements of $\mathcal{M}$ such that*

(i) $\lim_{s\to 1^+}(s-1)\,\zeta_{\eta,\mathcal{M}}(s) = \kappa$ for some $\kappa \in \mathbb{R}$, $\kappa \ne 0$.
(ii) $|\eta(p)| \le \|p\|^{\theta}$ for some $0 \le \theta < 1/2$ for all $p \in \mathcal{P}$.

Then

$$\lim_{N\to\infty} h_{\eta,N}\left(\left\{m \in \mathcal{F}_N \,:\, \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \le x\right\}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

for every real number x.

We will first outline the corollaries of the above theorems in the next section. In Section 3, we discuss the two important Tauberian theorems for general Dirichlet series which are relevant to our discussion, namely the Hardy-Littlewood-Karamata Tauberian theorem and the Wiener-Ikehara Tauberian theorem. Sections 4 and 5 are devoted to the proofs of Theorem 1.1 and Theorem 1.2 respectively. Although both the proofs have similar arguments, there are some technical differences which we highlight. The last section is devoted to some essential remarks.

2. **Corollaries of the main theorems**

In this section, we delineate the various special cases of our main results, the classical analogues of which are scattered across the literature.

As consequences of Theorem 1.1, we derive the following interesting corollaries.

(a) *The classical Erdős-Kac theorem:*
Set $\mathcal{P}$ = the set of all prime numbers, $M = \mathbb{N}$, $\|\cdot\| = |\cdot|$ the usual absolute value and $\alpha(p) = 1$ for all $p \in \mathcal{P}$. Then $\zeta_{\alpha,\mathcal{M}}(s) = \zeta(s)$, the Riemann zeta function, which has an analytic continuation to the complex plane except for a simple pole at $s = 1$ with residue 1. Thus, Theorem 1.1 in this setup gives the classical Erdős-Kac theorem, albeit with the harmonic measure on $\{1, 2, \cdots, N\}$. One can also choose $\alpha(m)$ as any completely multiplicative weight satisfying the hypothesis.

(b) *Erdős-Kac theorem with a restricted set of primes:*
Choose $\mathcal{P}$ to be a subset of prime numbers with positive Dirichlet density, namely

$$\lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}} p^{-s}}{\sum_{p\text{ - prime}} p^{-s}} = \kappa > 0.$$

For example, one can consider the set of primes in an arithmetic progression $a \bmod q$, in which case $\kappa = 1/\varphi(q)$ by Dirichlet's theorem on primes in arithmetic progressions. For such a subset, choose $\alpha(p) = 1$ for simplicity. Then,

$$\zeta_{\mathcal{M}}(s) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \text{for} \qquad \mathrm{Re}(s) > 1.$$

Taking the log of both sides gives

$$\log \zeta_{\mathcal{M}}(s) = \sum_{p\in\mathcal{P}} -\log\left(1 - \frac{1}{p^s}\right) = \sum_{p\in\mathcal{P}} \sum_{n=1}^{\infty} \frac{1}{n\, p^{ns}} = \sum_{p\in\mathcal{P}} \frac{1}{p^s} + \sum_{p\in\mathcal{P}} \sum_{n=2}^{\infty} \frac{1}{n\, p^{ns}}.$$

It is not difficult to show that the second term in the last equality above converges as $s \to 1^+$ (see the proof of Lemma 4.1 in this paper). A similar computation as above for the Riemann zeta-function implies

$$\log \zeta_{\mathcal{M}}(s) = \log \zeta(s) \left(\frac{\sum_{p\in\mathcal{P}} p^{-s} + O(1)}{\sum_{p\text{ - prime}} p^{-s} + O(1)} \right) \qquad \text{as} \qquad s \to 1^+.$$

Exponentiating and using the behaviour of $\zeta(s)$ near $s = 1$ gives

$$\lim_{s\to 1^+} (s-1)\, \zeta_{\mathcal{M}}(s) = \kappa.$$

Thus, Theorem 1.1 is applicable, giving an Erdős-Kac theorem in this scenario as well.

(c) *Erdős-Kac theorem for Beurling primes:*
In 1937, A. Beurling [1] introduced 'generalized primes' in order to highlight the essential properties of the natural number system which lead to the prime number theorem. He defined 'generalized primes' $\mathcal{P}$ as any countable subset of the real numbers $1 < p_1 \le p_2 \le p_3 \le \cdots$ with $p_n \to \infty$ as $n \to \infty$. A Beurling system or 'generalized numbers' $\mathcal{B}$ is the (multi)-set of all finite products of generalized primes. Note that different product representations of the same real number are considered as distinct elements in this system.

Let $N_{\mathcal{B}}(x) = \sum_{\substack{n\in\mathcal{B}\\ n\le x}} 1$ and $N_{\mathcal{P}}(x) = \sum_{\substack{p\in\mathcal{P}\\ p\le x}} 1$ be the counting functions of the generalized numbers and generalized primes respectively. Beurling showed that if

$$N_{\mathcal{B}}(x) = Ax + O\left(\frac{x}{(\log x)^{\lambda}}\right) \qquad \text{as} \qquad x \to \infty,$$

for $\lambda > 3/2$, then

$$N_{\mathcal{P}}(x) \sim \frac{x}{\log x} \qquad \text{as} \qquad x \to \infty.$$

Moreover, he demonstrated the existence of a system with $\lambda = 3/2$ in the asymptotic of $N_{\mathcal{B}}(x)$ for which the prime number theorem does not hold.

Beurling's result was followed by a flurry of activity, where in several classical number theory results were derived for Beurling primes. In this context, it is natural to ask whether there is an analog of the Erdős-Kac theorem for a Beurling system. This has been studied in detail by M. Rupert [32] in his masters thesis. He proved

Theorem (M. Rupert)**.** *Let $\mathcal{P}$ be a set of Beurling primes, $\mathcal{B}$ be the corresponding Beurling system such that*

$$N_{\mathcal{B}}(x) = Ax + O\left(x^{\theta}\right) \qquad \text{for some} \qquad A > 0 \quad \text{and} \quad 0 < \theta < 1.$$

Let $\omega_{\mathcal{B}}(n)$ be the number of prime factors of n. Then

$$\lim_{x\to\infty} \frac{1}{N_{\mathcal{B}}(x)} \# \left\{ n \in \mathcal{B},\, n \le x \,:\, \frac{\omega_{\mathcal{B}}(n) - \log\log n}{\sqrt{\log\log n}} \le y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt.$$

This is an Erdős-Kac theorem with the uniform distribution on $\{n \in \mathcal{B}, n \le x\}$.

As an immediate consequence of Theorem 1.1, we derive an Erdős-Kac theorem for a Beurling system equipped with the harmonic distribution.

Corollary 2.1. *Let $\mathcal{P}$ be a set of generalized primes and $\mathcal{B}$ be the corresponding Beurling system. Suppose that*

$$N_{\mathcal{B}}(x) = Ax + o(x) \qquad \text{as} \qquad x \to \infty.$$

Let $H_{\mathcal{B},x} = \sum_{n\in\mathcal{B},n\le x} 1/n$ and $h_{\mathcal{B},x}$ denote the harmonic measure on $\{n \in \mathcal{B}, n \le x\}$. Then,

$$\lim_{x\to\infty} \frac{1}{H_{\mathcal{B},x}}\, h_{\mathcal{B},x}\left(\left\{ n \in \mathcal{B},\, n \le x \,:\, \frac{\omega_{\mathcal{B}}(n) - \log\log n}{\sqrt{\log\log n}} \le y \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt.$$

To deduce the above corollary, simply take $\mathcal{P}$ to be the set of Beurling primes in question, $\mathcal{M} = \mathcal{B}$ and $\|\cdot\|$ to be the usual absolute value on the real numbers. Consider the Beurling zeta-function

$$\zeta_{\mathcal{B}}(s) = \sum_{n\in\mathcal{B}} \frac{1}{n^s}.$$

Using partial summation (see Lemma 3.1 below), one can show that $\zeta_{\mathcal{B}}(s)$ converges in the half plane $\mathrm{Re}(s) > 1$. Furthermore, by applying an Abelian theorem for Dirichlet series (see [34, Theorem 7.2]), one can deduce

$$\lim_{s\to 1^+} (s-1)\, \zeta_{\mathcal{B}}(s) = A$$

from the asymptotics for $N_{\mathcal{B}}(x)$. Therefore, Theorem 1.1 is now applicable by choosing $\alpha(p_j) = 1$ for each $p_j \in \mathcal{P}$. Thus, we obtain an Erdős-Kac theorem with harmonic distribution for a Beurling system which may not necessarily satisfy the prime number theorem!

(d) *Erdős-Kac theorem for ideals in number fields:*
Let $K/\mathbb{Q}$ be a finite extension and $\mathcal{O}_K$ be its ring of integers. Let the set of prime ideals in $\mathcal{O}_K$ be $\mathcal{P}$. It is known that every non-zero ideal of $\mathcal{O}_K$ can be uniquely factorized as a product of prime ideals. Let $\omega_K(\mathfrak{A})$ be the number of distinct prime ideals dividing $\mathfrak{A}$. Furthermore, the norm map, $\mathfrak{A} \mapsto N(\mathfrak{A}) = |\mathcal{O}_K/\mathfrak{A}|$ is a monoid homomorphism to the natural numbers.

The Dedekind zeta-function associated to K is defined as

$$\zeta_K(s) = \sum_{0\neq\mathfrak{A}\subseteq\mathcal{O}_K} \frac{1}{N(\mathfrak{A})^s}.$$

It can be shown that $\zeta_K(s)$ converges in $\mathrm{Re}(s) > 1$ and can be analytically continued to the entire complex plane except for a simple pole at $s = 1$ with residue ρ_K (say). Hence,

$$\lim_{s\to 1^+} (s-1)\,\zeta_K(s) = \rho_K.$$

Thus, an application of Theorem 1.1 is possible with $\alpha(\mathfrak{p}) = 1$ for all non-zero prime ideals $\mathfrak{p}$. This gives an Erdős-Kac theorem over number fields, that is, for $\omega_K(\mathfrak{a})$. See [28] for the prerequisites from algebraic number theory.

(e) *Erdős-Kac theorem for polynomials over a finite field:*

Let q be a prime power and $\mathbb{F}_q$ be the finite field with q-elements. Consider $\mathbb{F}_q[t]$, the polynomial ring over $\mathbb{F}_q$ with one variable. Take $\mathcal{P}$ to be the set of monic irreducible polynomials in $\mathbb{F}_q[t]$, $\mathcal{M}$ to be the set of all monic polynomials and $\mathcal{M}_d$ to be the set of all monic polynomials of degree d in $\mathbb{F}_q[t]$. Note that $\#\mathcal{M}_d = q^d$.

For any $f \in \mathcal{M}$, define $\|f\| := q^{\deg f}$, which is clearly a monoid homomorphism. Denote by

$$\zeta_{\mathbb{F}_q[t]}(s) := \sum_{f\in\mathcal{M}} \frac{1}{q^{s\deg(f)}}.$$

This sum above can be re-expressed as

$$\zeta_{\mathbb{F}_q[t]}(s) = \sum_{f\in\mathcal{M}} \sum_{d=0}^{\infty} \frac{1}{q^{sd}} = \sum_{d=0}^{\infty} \frac{\#\mathcal{M}_d}{q^{ds}} = \sum_{d=0}^{\infty} \frac{1}{q^{(s-1)d}} = \frac{1}{1-\frac{1}{q^{s-1}}}.$$

From this computation, it is evident that $\zeta_{\mathbb{F}_q[t]}(s)$ converges for $\mathrm{Re}(s) > 1$, has an analytic continuation to the entire complex plane except for a simple pole at $s = 1$ with residue $1/\log q$. Hence,

$$\lim_{s\to 1^+} (s-1)\,\zeta_{\mathbb{F}_q[t]}(s) = \frac{1}{\log q}.$$

We can now apply Theorem 1.1 with $\alpha(f) = 1$ for each $f \in \mathcal{P}$ to derive an Erdős-Kac theorem for polynomials in $\mathbb{F}_q[t]$.

On the other hand, Theorem 1.2 implies a variant of all the above Erdős-Kac theorems, when m varies over only squarefree elements in the monoid. In particular, we can conclude that

$$\lim_{N\to\infty} \frac{1}{(6/\pi^2)\log N} \sum_{\substack{n\leq N \\ n \text{ - squarefree} \\ \frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\leq y}} \frac{1}{n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\,dt.$$

In other words, $(\omega(n) - \log\log n)/\sqrt{\log\log n}$ with harmonic distribution has the standard normal distribution in the limit.

3. **Tauberian theorems**

In analysis, Tauberian theorems originated from the effort to obtain a converse for a result by Abel on power series. Let $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of real numbers such that the series

$$\sum_{n=0}^{\infty} a_n\, x^n$$

converges on the interval $(-1,1)$ and defines the function $f(x)$. Abel showed that if $\sum_{n=0}^{\infty} a_n$ converges and equals L, then $\lim_{x\to 1^-} f(x)$ also exists and equals L. The converse of this statement is not true in general. One can take $a_n = (-1)^n$ so that $f(x) = (1+x)^{-1}$, $\lim_{x\to 1^-} f(x) = 1/2$ but the series $\sum_{n=0}^{\infty} a_n$ diverges.

In 1897, A. Tauber [33] gave a sufficient condition on the a_n's for the converse of Abel's theorem to hold, namely, that

$$\lim_{n\to\infty} n\, a_n = 0.$$

Since then, 'Tauberian theorems' have received great attention, as is recorded in the excellent survey by J. Korevaar [21].

We will be particularly interested in Tauberian theorems applied to Dirichlet series. In this context, there are two main results that are pertinent to our discussion: the Hardy-Littlewood Tauberian theorem and the Wiener-Ikehara Tauberian theorem. Since our application involves general Dirichlet series ($\sum_{n\geq 1} a_n\, \lambda_n^{-s}$ rather than $\sum_{n\geq 1} a_n\, n^{-s}$), we will prove these two statements below assuming the corresponding preparatory results.

3.1. **The Hardy-Littlewood theorem via Karamata's theorem:** Let a_n's be a sequence of complex numbers and λ_n's be a sequence of positive real numbers such that $1 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. Set

$$A(x) := \sum_{\substack{n \\ \lambda_n \leq x}} a_n.$$

Denote by

$$L(s,\underline{a}) := \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$$

the associated general Dirichlet series. Throughout this section, we assume that $L(s,\underline{a})$ converges absolutely in the half plane $\mathrm{Re}(s) > 1$.

We will use the following general formulation of the partial (or Abel) summation. See [4, Theorem 6, Chapter VII] for a proof.

Lemma 3.1. *Let b_n's be a sequence of complex numbers, β_n's be a sequence of non-negative real numbers such that $0 \leq \beta_1 < \beta_2 < \beta_3 < \cdots$ and $B(x) := \sum_{\beta_n \leq x} b_n$. Let $\varphi : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuously differentiable function. Then for $x > \beta_1$,*

$$\sum_{\beta_n \leq x} b_n\, \varphi(\beta_n) = B(x)\, \varphi(x) - \int_{\beta_1}^{x} B(t)\, \varphi'(t)\, dt.$$

Thus, we can express the Dirichlet series as a Laplace-Stieltjes integral.

Lemma 3.2. *Let $\mathcal{A}(t) := A(e^t)$. Then for $Re(s) > 1$,*

$$L(s,\underline{a}) = \int_0^{\infty} e^{-st}\, d\mathcal{A}(t).$$

Proof. For any real number $x > 1$, we apply Lemma 3.1 with $b_n = a_n$, $\beta_n = \log(\lambda_n)$ and $\varphi(t) = e^{-st}$. This gives

$$\sum_{\lambda_n \le e^x} \frac{a_n}{\lambda_n^s} = \mathcal{A}(x)\, e^{-sx} + s \int_0^x \mathcal{A}(t)\, e^{-st}\, dt.$$

Taking the limit as $x \to \infty$, the first term on the right hand side tends to zero as the series converges, and we get

$$L(s, \underline{a}) = s \int_0^\infty \mathcal{A}(t)\, e^{-st}\, dt = \int_0^\infty e^{-st}\, d\mathcal{A}(t),$$

where the second equality is obtained using integration by parts. This proves that a general Dirichlet series is a Laplace-Stieltjes transform of the summatory function $\mathcal{A}(t)$. □

In the early twentieth century, the focus shifted on establishing Tauberian theorems for Laplace-Stieltjes integral as above, since power series and Dirichlet series are both specializations of these integrals. To prove an analogue of the Hardy-Littlewood theorem for general Dirichlet series, we assume a Tauberian theorem of Karamata from 1931 for such integrals. We refer the reader to [34, Section 7.3] for a detailed proof of Karamata's theorem.

Theorem 3.1 (Karamata)**.** *Let $B(t)$ be a non-decreasing function such that the integral*

$$F(\sigma) := \int_0^\infty e^{-\sigma t}\, dB(t)$$

converges for all $\sigma > 0$. Assume that there exist two real numbers $c \ge 0$ and $\nu > 0$ such that

$$F(\sigma) = \frac{c + o(1)}{\sigma^\nu} \qquad as \qquad \sigma \to 0^+.$$

Then we have

$$B(x) = \frac{c + o(1)}{\Gamma(\nu + 1)}\, x^\nu \qquad as \qquad x \to +\infty.$$

From the above theorem, we immediately deduce the following version of the Hardy-Littlewood theorem, which is sufficient for our applications. This also appears in a paper of Heilbronn and Landau [18]. For the most general version, see [35, Corollary 7.9].

Theorem 3.2 (Hardy-Littlewood-Karamata, weak version)**.** *Let a_n's be a sequence of real numbers and λ_n's be positive real numbers such that $1 = \lambda_1 \le \lambda_2 \le \lambda_3 \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. Assume that*

(a) $a_n \ge 0$ for all $n \in \mathbb{N}$
(b) the Dirichlet series

$$F(s) = \sum_{n=1}^\infty \frac{a_n}{\lambda_n^s}$$

converges for Re$(s) > 1$, and there exists a real number c such that

$$F(\sigma) = \frac{c + o(1)}{(\sigma - 1)} \qquad as \qquad \sigma \to 1^+.$$

Then we have

$$\sum_{\lambda_n \le x} \frac{a_n}{\lambda_n} = (c + o(1)) \log x \qquad as \qquad x \to +\infty.$$

Proof. Let $b_n = a_n/\lambda_n$, $\mathcal{B}(t) = \sum_{\lambda_n \le e^t} b_n$ and $G(s) := \sum_{n=1}^{\infty} b_n/n^s$. By the hypotheses, $\mathcal{B}(t)$ is non-decreasing and $G(s) = F(s+1)$ converges in $\mathrm{Re}(s) > 0$. Thus, Lemma 3.2 implies that

$$G(\sigma) = \int_0^{\infty} e^{-\sigma t}\, d\mathcal{B}(t) \qquad \text{for} \qquad \sigma > 0.$$

Moreover, $G(\sigma) = (c + o(1))/\sigma$ as $\sigma \to 0^+$. Therefore by Theorem 3.1, $\mathcal{B}(x) = (c + o(1))x$ as $x \to +\infty$. Replacing x by $\log x$ implies the result. □

A corollary of the above theorem is the corresponding statement when a_n's are a sequence of complex numbers, that are bounded below.

Corollary 3.1. *Let a_n be a sequence of complex numbers and λ_n's be as above. Assume that*

(a) $\mathrm{Re}(a_n)$, $\mathrm{Im}(a_n)$ are both $\ge -K$ for all $n \in \mathbb{N}$ and some positive real number $K \in \mathbb{R}$

(b) the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}$$

converges for $\mathrm{Re}(s) > 1$ and there exist real numbers c and d such that

$$F(\sigma) = \frac{c + id + o(1)}{(\sigma - 1)} \qquad as \qquad \sigma \to 1^+.$$

Then we have

$$\sum_{\lambda_n \le x} \frac{a_n}{\lambda_n} = (c + id + o(1)) \log x \qquad as \qquad x \to +\infty.$$

Proof. We apply Theorem 3.2 to the Dirichlet series associated to the real part and imaginary part of the sequence $(a_n)_n$, shifted by a multiple of the Riemann zeta-function $\zeta(s)$. In particular, let $c_n = \mathrm{Re}(a_n)$, $d_n = \mathrm{Im}(a_n)$ and $(\gamma_n)_n$ be the sequence $\{\gamma_n\} = \{\lambda_n\} \cup \{n\}$, arranged in an increasing order. Set

$$\widetilde{c_n} = \begin{cases} c_m & \text{if } \gamma_n = \lambda_m \text{ for some } m, \gamma_n \notin \mathbb{N} \\ K & \text{if } \gamma_n = m \text{ for some } m \in \mathbb{N}, \gamma_n \notin \{\lambda_n\} \\ c_k + K & \text{if } \gamma_n = \lambda_k = m \text{ for some } k,\, m \in \mathbb{N}. \end{cases}$$

Similarly, define $\widetilde{d_n}$. Since, c_n, $d_n \ge -K$, $\widetilde{c_n} \ge 0$ and $\widetilde{d_n} \ge 0$. One can now apply Theorem 3.2 to the series

$$C(s) = \sum_{n=1}^{\infty} \frac{\widetilde{c_n}}{\gamma_n^s} \qquad \text{and} \qquad D(s) = \sum_{n=1}^{\infty} \frac{\widetilde{d_n}}{\gamma_n^s},$$

and obtain the claim using the standard estimate

$$\sum_{n \le x} \frac{1}{n} = \log x + O(1).$$

□

3.1.1. **The Wiener-Ikehara Tauberian theorem via complex analysis:** In 1931, S. Ikehara [19], a student of N. Wiener proved a Tauberian theorem which led to significant simplification in the proof of the prime number theorem (without the error term). The Wiener-Ikehara theorem gives an asymptotic for

$$\sum_{n \le x} a_n \text{ as opposed to } \sum_{n \le x} \frac{a_n}{n}$$

in the Hardy-Littlewood theorem. However, the Wiener-Ikehara theorem requires the stronger assumption of analytic behaviour of the function on the boundary. Several proofs of the theorem are available in the literature. For instance, see [38], [21], [37].

We obtain the following version of the Wiener-Ikehara theorem by assuming a complex analytic Tauberian theorem proved by A. Vatwani in [37, Theorem 1] using an idea by D. Newman. The statement below follows from a straightforward replacement of the sequence $(n)_{n\in\mathbb{N}}$ by $(\lambda_n)_{n\in\mathbb{N}}$ in the proofs of [37, Theorem 2] and [37, Corollary 3].

Theorem 3.3 (Wiener-Ikehara)**.** *Let $(a_n)_n$ be a sequence of complex numbers and $(\lambda_n)_n$ be a sequence of non-negative real numbers such that $1 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ and $\lambda_n \to \infty$ as $n \to \infty$. Let $A(x) = \sum_{\lambda_n \le x} a_n$. Assume that the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}.$$

converges absolutely in $Re(s) > 1$. Suppose there exists a function $G(s) = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n^s}$, with non-negative coefficients such that

(a) $|a_n| \le b_n$ for all $n \in \mathbb{N}$
(b) $G(s)$ is absolutely convergent for $Re(s) > 1$
(c) the function $G(s)$ (resp. $F(s)$) extends meromorphically to the region $Re(s) \ge 1$, having no poles except for a simple pole at $s = 1$ with residue R (resp r)
(d) $B(x) := \sum_{\lambda_n \le x} b_n = O(x)$.

Then, as $x \to \infty$,

$$A(x) = rx + o(x).$$

Remark. *It is important to note here that for $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ as in Theorem 3.2, even if we have additionally that the limit*

$$\lim_{\sigma \to 1^+} F(\sigma + it) - \frac{c}{\sigma - 1 + it}$$

exists, uniformly for all $t \in [-R, R]$ for some $R > 0$, one can only derive upper and lower bounds for $\sum_{\lambda_n \le x} a_n$ in the sense of Chebyshev. This has been investigated by Heilbronn and Landau [17]. Thus, the conclusion in Theorem 3.3 necessarily requires a stronger hypothesis than in Theorem 3.2.

4. **Completely Multiplicative Weights**

The aim of this section is to prove Theorem 1.1. Towards this, we define a zeta-distribution on the monoid $\mathcal{M}$, and first prove a central limit theorem for a random variable with this distribution.

Recall that $\mathcal{M}$ is a monoid generated by the 'prime' elements $p \in \mathcal{P}$. There exists a completely multiplicative function $\alpha : \mathcal{M} \to \mathbb{R}_{\ge 0}$ and a 'norm' map $\|\cdot\| : \mathcal{M} \to \mathbb{R}_{\ge 0}$. Associated to this data, we have the zeta-function

$$\zeta_{\alpha,\mathcal{M}}(s) := \sum_{m \in \mathcal{M}} \frac{\alpha(m)}{\|m\|^s}.$$

By hypothesis (i) of Theorem 1.1, we have that

$$\lim_{s \to 1^+} (s-1)\, \zeta_{\alpha,\mathcal{M}}(s) = \kappa.$$

Since $\alpha(m)$ are non-negative, we can apply the Hardy-Littlewood-Karamata theorem (Corollary 3.1) to deduce that

$$H_{\alpha,N} = \sum_{m\in\mathcal{F}_N} \frac{\alpha(m)}{\|m\|} = (\kappa + o(1)) \log N. \tag{1}$$

4.1. **Central Limit Theorem for the zeta-distribution.** Throughout this section, we work under the hypothesis of Theorem 1.1.

Associated to the zeta-function $\zeta_{\alpha,\mathcal{M}}(s)$, one can define a probability distribution on the monoid $\mathcal{M}$, via the random variable X_s for each $s > 1$, given by

$$\Pr(X_s = m) = \left(\frac{\alpha(m)}{\|m\|^s}\right) \times \frac{1}{\zeta_{\alpha,\mathcal{M}}(s)}.$$

We first observe that $\omega_{\mathcal{M}}(X_s)$ can be expressed as a product involving independent random variables. This is because one can write

$$X_s = \prod_{p\in\mathcal{P}} p^{c_p(s)},$$

where $c_p(s) \in \mathbb{Z}_{\geq 0}$ for each $p \in \mathcal{P}$ and $s > 1$. Hence, we have

$$\omega_{\mathcal{M}}(X_s) = \sum_{p\in\mathcal{P}} \mathbb{1}_{c_p(s)\geq 1},$$

where $\mathbb{1}_{\{\ldots\}}$ denotes the indicator function. Observe that

$$\Pr(c_p(s) \geq k) = \frac{1}{\zeta_{\alpha,\mathcal{M}}(s)} \sum_{m\in\mathcal{M}} \frac{\alpha(p^k m)}{\|p^k m\|^s} = \left(\frac{\alpha(p)}{\|p\|^s}\right)^k. \tag{2}$$

Moreover, for any $p_1, p_2, \cdots, p_t \in \mathcal{P}$ and $k_1, k_2, \cdots, k_t \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\Pr\big(c_{p_1}(s) \geq k_1, c_{p_2}(s) \geq k_2, \cdots, c_{p_t}(s) \geq k_t\big) &= \frac{1}{\zeta_{\alpha,\mathcal{M}}(s)} \sum_{m\in\mathcal{M}} \frac{a(p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} m)}{\|p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} m\|^s} \\ &= \prod_{j=1}^{t} \Pr\left(c_{p_j}(s) \geq k_j\right).\end{aligned}$$

Therefore, the random variables $c_p(s)$'s are independent as p varies over $\mathcal{P}$ and $\omega_{\mathcal{M}}(X_s)$ is expressed as a sum of independent random variables. To establish a central limit theorem in this context, we resort to Lévy's continuity theorem and characteristic functions. It is conceivable that alternate approaches towards the same goal exist, which we leave for the interested reader to pursue.

For any $s > 1$, define

$$\mathbb{P}_{\mathcal{M}}(s) := \sum_{p\in\mathcal{P}} \frac{\alpha(p)}{\|p\|^s}.$$

Before we proceed, observe the following.

Lemma 4.1. *As* $s \to 1^+$,

$$\log \zeta_{\alpha,\mathcal{M}}(s) = \mathbb{P}_{\mathcal{M}}(s) + O(1).$$

Proof. The Euler product of $\zeta_{\alpha,\mathcal{M}}(s)$ gives

$$\log \zeta_{\mathcal{M},a}(s) = \sum_{p\in\mathcal{P}} -\log\left(1 - \frac{\alpha(p)}{\|p\|^s}\right) = \sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty} \frac{\alpha(p)^n}{n\,\|p\|^{ns}} = \mathbb{P}_{\mathcal{M}}(s) + \sum_{p\in\mathbb{P}}\sum_{n=2}^{\infty} \frac{\alpha(p)^n}{n\,\|p\|^{ns}}.$$

The second term above converges as $s \to 1^+$. Indeed, we see that for s sufficiently close to 1,

$$\begin{aligned}\sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty} \frac{|\alpha(p)|^n}{n\|p\|^{ns}} &\leq \sum_{p\in\mathcal{P}} \frac{|\alpha(p)|^2}{\|p\|^{2s}} \sum_{n=2}^{\infty} \frac{|\alpha(p)|^{n-2}}{\|p\|^{(n-2)s}} \\ &= \sum_{p\in\mathcal{P}} \frac{|\alpha(p)|^2}{\|p\|^{2s}} \times \frac{1}{\left(1 - \frac{|\alpha(p)|}{\|p\|}\right)} \\ &\ll \sum_{p\in\mathcal{P}} \frac{\|p\|^{2\theta}}{\|p\|^2},\end{aligned}$$

because $|\alpha(p)| \leq \|p\|^\theta$ for $0 < \theta < 1/2$, so that $\|p\|^{\theta-1} \to 0$ as $\|p\| \to \infty$. Therefore, for all but finitely many $p \in \mathcal{P}$, $1 - (|\alpha(p)|\,.\,\|p\|^{-1}) \geq 1/2$. As $\theta < 1/2$, the last series converges, proving the claim. □

Thus, we see that $\lim_{s\to 1^+} \mathbb{P}_{\mathcal{M}}(s) = \infty$.

Proposition 4.1. *The mean and variance of the random variable $\omega_{\mathcal{M}}(X_s)$ are given by*

$$\mathbb{E}\left[\omega_{\mathcal{M}}(X_s)\right] = \mathbb{P}_{\mathcal{M}}(s) \qquad \mathit{var}(\omega_{\mathcal{M}}(X_s)) = \mathbb{P}_{\mathcal{M}}(s) + O(1)$$

as $s \to 1^+$.

Proof. To compute the mean, note that

$$\mathbb{E}\left[\omega_{\mathcal{M}}(X_s)\right] = \sum_{p\in\mathcal{P}} \mathbb{E}\left[\mathbb{1}_{c_p(s)\geq 1}\right] = \sum_{p\in\mathcal{P}} \frac{\alpha(p)}{\|p\|^s} = \mathbb{P}_{\mathcal{M}}(s),$$

by equation (2). The variance can be computed equally easily as

$$\begin{aligned}\mathrm{var}(\omega_{\mathcal{M}}(X_s)) &= \mathbb{E}\left[\omega_{\mathcal{M}}(X_s)^2\right] - \mathbb{E}\left[\omega_{\mathcal{M}}(X_s)\right]^2 \\ &= \sum_{\substack{p,q\in\mathcal{P}\\ p\neq q}} \mathbb{E}\left[\mathbb{1}_{c_p(s)\geq 1}, \mathbb{1}_{c_q(s)\geq 1}\right] + \sum_{p\in\mathcal{P}} \mathbb{E}\left[\mathbb{1}_{c_p(s)\geq 1}\right] - \mathbb{P}_{\mathcal{M}}(s)^2 \\ &= \sum_{p,q\in\mathcal{P}} \left(\mathbb{E}\left[\mathbb{1}_{c_p(s)\geq 1}\right]\mathbb{E}\left[\mathbb{1}_{c_q(s)\geq 1}\right]\right) + \mathbb{P}_{\mathcal{M}}(s) - \mathbb{P}_{\mathcal{M}}(s)^2 \\ &= \mathbb{P}_{\mathcal{M}}(s) - \sum_{p\in\mathcal{P}} \frac{\alpha(p)^2}{\|p\|^{2s}},\end{aligned}$$

where the second series converges for $s > 1/2$ as $\alpha(p) < \|p\|^{1/2}$. □

Proposition 4.2. *The characteristic function of $\omega_{\mathcal{M}}(X_s)$ is given by*

$$\Phi_s(t) = \prod_{p\in\mathcal{P}} \left(1 + \left[\left(e^{it} - 1\right) \frac{\alpha(p)}{\|p\|^s}\right]\right).$$

Proof. The characteristic function of $\omega_{\mathcal{M}}(X_s)$ is $\mathbb{E}\left[e^{it\omega_{\mathcal{M}}(X_s)}\right]$. Using the independence of the c_p's, we get

$$\begin{aligned}\Phi_s(t) = \mathbb{E}\left[e^{it\sum_{p\in\mathcal{P}}\mathbb{1}_{c_p(s)\geq 1}}\right] &= \prod_{p\in\mathcal{P}}\mathbb{E}\left[e^{it\,\mathbb{1}_{c(p)(s)\geq 1}}\right]\\ &= \prod_{p\in\mathcal{P}}\left(e^{it}\left(\frac{\alpha(p)}{\|p\|^s}\right)+\left(1-\frac{\alpha(p)}{\|p\|^s}\right)\right)\\ &= \prod_{p\in\mathcal{P}}\left(1+\left[\left(e^{it}-1\right)\frac{\alpha(p)}{\|p\|^s}\right]\right).\end{aligned}$$

Note that for each $t\in\mathbb{R}$, the above infinite product converges uniformly on compact subsets of the half plane $\mathrm{Re}(s)>1$. □

We now establish an Erdős-Kac theorem for $\omega_{\mathcal{M}}(X_s)$,

Theorem 4.3. *As $s\to 1^+$,*

$$\frac{\omega_{\mathcal{M}}(X_s)-\mathbb{P}_{\mathcal{M}}(s)}{\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}\xrightarrow{d}\mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution with mean 0 and variance 1.

Proof. This is proved using Lévy's continuity theorem and establishing the point-wise convergence of the respective characteristic functions. By Proposition 4.2, we get

$$\mathbb{E}\left[\exp\left(\frac{it\left(\omega_{\mathcal{M}}(X_s)-\mathbb{P}_{\mathcal{M}}(s)\right)}{\sqrt{\mathbb{P}_{\mathcal{M}}}(s)}\right)\right]=e^{-it\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}\,\mathbb{E}\left[\exp\left(\frac{it\,\omega_{\mathcal{M}}(X_s)}{\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}\right)\right]=e^{-it\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}\,\Phi_s\left(\frac{t}{\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}\right).$$

As $|\alpha(p)|<2\|p\|$, we can write

$$\begin{aligned}\Phi_s(t) &= \prod_{p\in P}\exp\left(\log\left(1+\left[\left(e^{it}-1\right)\frac{\alpha(p)}{\|p\|^s}\right]\right)\right)\\ &= \exp\left(-\sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty}\left(-\frac{\alpha(p)}{\|p\|^s}\right)^n\frac{\left(e^{it}-1\right)^n}{n}\right)\\ &= \exp\left(\mathbb{P}_{\mathcal{M}}(s)(e^{it}-1)\right)\times\exp\left(-\sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty}\left(-\frac{\alpha(p)}{\|p\|^s}\right)^n\frac{\left(e^{it}-1\right)^n}{n}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{it\left(\omega_{\mathcal{M}}(X_s)-\mathbb{P}_{\mathcal{M}}(s)\right)}{\sqrt{\mathbb{P}_{\mathcal{M}}}(s)}\right)\right] &= \exp\left(\mathbb{P}_{\mathcal{M}}(s)(e^{it/\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}-1)-it\sqrt{\mathbb{P}_{\mathcal{M}}(s)}\right)\\ &\quad\times\exp\left(-\sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty}\left(-\frac{\alpha(p)}{\|p\|^s}\right)^n\frac{\left(e^{it/\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}-1\right)^n}{n}\right).\end{aligned}\tag{3}$$

Using Taylor series expansion, we have

$$\mathbb{P}_{\mathcal{M}}(s)\left(e^{it/\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}-1\right)-it\sqrt{\mathbb{P}_{\mathcal{M}}(s)}=-\frac{t^2}{2}+O\left(\frac{1}{\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}\right),$$

as $s \to 1^+$. Furthermore, since the series

$$-\sum_{p \in \mathcal{P}} \sum_{n=2}^{\infty} \left(-\frac{\alpha(p)}{\|p\|^s} \right)^n \frac{\left(e^{it/\sqrt{\mathbb{P}_{\mathcal{M}}(s)}} - 1 \right)^n}{n}$$

converges uniformly as $s \to 1^+$, and the limit as $s \to 1^+$ of $e^{it/\sqrt{\mathbb{P}_{\mathcal{M}}(s)}} - 1 = 0$, the second term in (3) tends to 1 as $s \to 1^+$. This proves that the characteristic function of

$$\frac{\omega_{\mathcal{M}}(X_s) - \mathbb{P}_{\mathcal{M}}(s)}{\sqrt{\mathbb{P}_{\mathcal{M}}(s)}}$$

converges to $e^{-t^2/2}$ as $s \to 1^+$, implying the theorem. □

The central limit theorem for $\omega_{\mathcal{M}}(X_s)$ can also be formulated as below.

Corollary 4.1. *As* $s \to 1^+$

$$\frac{\omega_{\mathcal{M}}(X_s) - \log \zeta_{\alpha,\mathcal{M}}(s)}{\sqrt{\log \zeta_{\alpha,\mathcal{M}}(s)}} \xrightarrow{d} \mathcal{N}(0,1)$$

Proof. This is immediate from Theorem 4.3 and Lemma 4.1. □

The following observation simplifies computations in the next corollary.

Proposition 4.4. *Let* X_s *be a random variable with the* $\zeta_{\alpha,\mathcal{M}}(s)$*-distribution. Then, as* $s \to 1^+$,

$$\frac{1}{\zeta_{\alpha,\mathcal{M}}(s)} \log \|X_s\| \xrightarrow{d} \mathcal{E},$$

where $\mathcal{E}$ *is a random variable with exponential distribution of parameter* $1/\kappa$. *Moreover, for* $D \gg C > 1$,

$$\lim_{s \to 1^+} Pr\left(\|X_s\| > D^{\zeta_{\alpha,\mathcal{M}}(s)} \right) = \frac{1}{D^{1/\kappa}} \tag{4}$$

and

$$\lim_{s \to 1^+} Pr\left(\|X_s\| < C^{\zeta_{\alpha,\mathcal{M}}(s)} \right) = 1 - \frac{1}{C^{1/\kappa}}. \tag{5}$$

Proof. Once again, we compare the characteristic function of the two random variables. First, note that

$$\mathbb{E}\left[e^{it \log \|X_s\|} \right] = \frac{1}{\zeta_{\alpha,\mathcal{M}}(s)} \sum_{m \in \mathcal{M}} \frac{e^{it \log \|m\|}}{\|m\|^s} = \frac{\zeta_{\alpha,\mathcal{M}}(s - it)}{\zeta_{\alpha,\mathcal{M}}(s)}.$$

Therefore, we have that

$$\mathbb{E}\left[\exp\left(\frac{it}{\zeta_{\alpha,\mathcal{M}}(s)} \log X_s \right) \right] = \frac{\zeta_{\alpha,\mathcal{M}}\left(s - \frac{it}{\zeta_{\alpha,\mathcal{M}}(s)} \right)}{\zeta_{\alpha,\mathcal{M}}(s)} \sim \frac{(s-1)}{s - 1 - \frac{it}{\zeta_{\alpha,\mathcal{M}}(s)}} \sim \frac{1}{1 - it\kappa},$$

which is the characteristic function of the exponential distribution of parameter $1/\kappa$. Equations (4) and (5) are now immediate. □

We establish a central limit theorem below from which deducing Theorem 1.1 will simply be an application of a Tauberian theorem.

Corollary 4.2. *As* $s \to 1^+$

$$\frac{\omega_{\mathcal{M}}(X_s) - \log\log \|X_s\|}{\sqrt{\log\log \|X_s\|}} \xrightarrow{d} \mathcal{N}(0,1)$$

Proof. In view of brevity of notation, we denote $\zeta^\star(s) = \zeta_{\alpha,\mathcal{M}}(s)$. The arguments below closely follow the proof in [5] with minor corrections. We include the details for the sake of completeness.

By Corollary 4.1, we know that

$$\psi_s(t) := \frac{1}{\zeta^\star(s)} \sum_{m\in\mathcal{M}} \frac{\alpha(m)\exp\left(it\,\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right)}{\|m\|^s} \to e^{-t^2/2}$$

as $s\to 1^+$. Choose $D \gg 1$ and set $C = \left(1-\frac{1}{D^{1/\kappa}}\right)^{-\kappa}$. We first divide the sum over $\mathcal{M}$ into three parts,

$$\psi_s(t) = \frac{1}{\zeta^\star(s)}\left\{\sum_{1\le\|m\|\le C^{\zeta^\star(s)}} + \sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} + \sum_{D^{\zeta^\star(s)}<\|m\|}\right\}\left(\frac{\alpha(m)\exp\left(it\,\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right)}{\|m\|^s}\right),$$

named as I_s, II_s and III_s respectively.

We estimate the first and the last sum as follows.

$$\begin{aligned}
|I_s| &\le \frac{1}{\zeta^\star(s)}\sum_{1\le\|m\|\le C^{\zeta^\star(s)}} \alpha(m)\,\|m\|^{-s}\left|\exp\left(it\,\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right)\right| \\
&= \frac{1}{\zeta^\star(s)}\sum_{1\le\|m\|\le C^{\zeta^\star(s)}} \frac{\alpha(m)}{\|m\|^s} = \Pr\left(X_s \le C^{\zeta^\star(s)}\right) \\
&\to 1-\frac{1}{C^{1/\kappa}} = \frac{1}{D^{1/\kappa}}, \qquad (6)
\end{aligned}$$

as $s\to 1^+$ by (5). Similarly,

$$\begin{aligned}
|III_s| &\le \frac{1}{\zeta^\star(s)}\sum_{D^{\zeta^\star(s)}<\|m\|} \alpha(m)\,\|m\|^{-s}\left|\exp\left(it\,\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right)\right| \\
&= \frac{1}{\zeta^\star(s)}\sum_{D^{\zeta^\star(s)}<\|m\|} \frac{\alpha(m)}{\|m\|^s} = \Pr\left(X_s > D^{\zeta^\star(s)}\right) \to \frac{1}{D^{1/\kappa}} \qquad (7)
\end{aligned}$$

as $s\to 1^+$ by (4).

For the sum II_s, observe that if $C^{\zeta^\star(s)} < \|m\| \le D^{\zeta^\star(s)}$, then

$$\log\zeta^\star(s) + \log\log C < \log\log\|m\| \le \log\log D + \log\zeta^\star(s).$$

Thus, if we set

$$\delta(m,s) := \log\zeta^\star(s) - \log\log\|m\|,$$

then

$$|\delta(m,s)| \le B_D, \qquad (8)$$

for $\|m\|$ in the above range, where B_D is a constant depending on D. Hence, we can write

$$II_s = \frac{1}{\zeta^\star(s)}\sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} \alpha(m)\,\|m\|^{-s}\exp\left(it\,\frac{\omega_{\mathcal{M}}(m)-\left(\log\log\|m\|+\delta(m,s)\right))}{\sqrt{\log\log\|m\|+\delta(m,s)}}\right).$$

Now, define

$$f_m(x) = \exp\left(it \, \frac{\omega_{\mathcal{M}}(m) - \big(\log\log\|m\| + x\big)}{\sqrt{\log\log\|m\| + x}} \right). \tag{9}$$

Then

$$f_m'(x) = -it\, f_m(x) \left[\frac{1}{\sqrt{\log\log\|m\| + x}} + \frac{1}{2}\left(\frac{\omega_{\mathcal{M}}(m) - \big(\log\log\|m\| + x\big)}{\big(\sqrt{\log\log\|m\| + x}\big)^3} \right) \right].$$

By a first order Taylor expansion, there is a $\xi(m,s) \in (0, \delta(m,s))$, in particular, $\xi(m,s) \le B_D$, such that

$$f_m\left(\delta(m,s)\right) = f_m(0) + f_m'\left(\xi(m,s)\right)\, \delta(m,s).$$

Using this in the previous expression of II_s gives

$$\begin{aligned} II_s = {} & \frac{1}{\zeta^\star(s)} \sum_{C^{\zeta^\star(s)} < \|m\| \le D^{\zeta^\star(s)}} \alpha(m)\, \|m\|^{-s} \exp\left(it\, \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \right) \\ & - \frac{it}{\zeta^\star(s)} \sum_{C^{\zeta^\star(s)} < \|m\| \le D^{\zeta^\star(s)}} \alpha(m) \cdot \frac{f_m(\xi(m,s))}{\|m\|^s} \cdot \frac{\delta(m,s)}{\sqrt{\log\log\|m\| + \xi(m,s)}} \\ & - \frac{it}{2\zeta^\star(s)} \sum_{C^{\zeta^\star(s)} < \|m\| \le D^{\zeta^\star(s)}} \alpha(m) \cdot \frac{f_m(\xi(m,s))}{\|m\|^s} \cdot \delta(m,s) \cdot \left(\frac{\omega_{\mathcal{M}}(m) - \big(\log\log\|m\| + \xi(m,s)\big)}{\big(\sqrt{\log\log\|m\| + \xi(m,s)}\big)^3} \right). \end{aligned}$$

We call the above sums as IV_s, V_s and VI_s respectively. It is easy to see that

$$IV_s = \frac{1}{\zeta^\star(s)} \sum_{m \in \mathcal{M}} \alpha(m)\, \|m\|^{-s} \exp\left(it\, \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \right) + O\left(\frac{1}{D^{1/\kappa}} \right), \tag{10}$$

as the terms with $\|m\|$ close to 1 or very large will contribute only $O(D^{-1/\kappa})$.

The denominator in V_s and VI_s can be bounded below since

$$\begin{aligned} |\log\log\|m\| + \xi(m,s)| = |\log\zeta^\star(s) - \delta(m,s) + \xi(m,s)| &\ge |\log\zeta^\star(s)| - |\delta(m,s) - \xi(m,s)| \\ &\ge \log\zeta^\star(s) - 2B_D \end{aligned} \tag{11}$$

by (8).

For the sum V_s, using (11) and (9), we get

$$\begin{aligned} |V_s| &\le \frac{t\, B_D}{\zeta^\star(s)} \sum_{C^{\zeta^\star(s)} < \|m\| \le D^{\zeta^\star(s)}} \frac{\|m\|^{-(s-\theta)}}{\sqrt{\log\log\|m\| + \xi(m,s)}} \\ &\le \frac{tB_D}{\zeta^\star(s)\sqrt{\log\zeta^\star(s) - 2B_D}} \sum_{C^{\zeta^\star(s)} < \|m\| \le D^{\zeta^\star(s)}} \frac{1}{\|m\|^{s-\theta}} \le \frac{tB_D}{\zeta^\star(s)\sqrt{\log\zeta^\star(s) - 2B_D}} \to 0 \end{aligned}$$

as $s \to 1^+$. Along the same lines, for the sum VI_s, we use (8) and (11) to get

$$
\begin{aligned}
&|VI_s| \\
&\le \frac{t}{2\zeta^\star(s)} \sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} \alpha(m)\cdot \frac{|f_m(\xi(m,s))|}{\|m\|^s}\cdot \delta(m,s)\cdot \left|\frac{\omega_{\mathcal{M}}(m)-\big(\log\log\|m\|+\xi(m,s)\big)}{\left(\sqrt{\log\log\|m\|+\xi(m,s)}\right)^3}\right| \\
&= \frac{tB_D}{2\zeta^\star(s)} \sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} \frac{\alpha(m)}{\|m\|^s}\cdot \left|\frac{\omega_{\mathcal{M}}(m)-\big(\log\zeta^\star(s)-\delta(m,s)+\xi(m,s)\big)}{\left(\sqrt{\log\log\|m\|+\xi(m,s)}\right)^3}\right| \\
&\le \frac{tB_D}{2}\cdot \frac{\sqrt{\log\zeta^\star(s)}}{(\log\zeta^\star(s)-2B_D)^{3/2}} \left(\frac{1}{\zeta^\star(s)} \sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} \frac{\alpha(m)}{\|m\|^s}\left|\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right|\right) \\
&\qquad\qquad + tB_D^2\cdot \frac{1}{(\log\zeta^\star(s)-2B_D)^{3/2}}\left(\frac{1}{\zeta^\star(s)} \sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} \frac{\alpha(m)}{\|m\|^s}\right).
\end{aligned}
$$

The second term is above is

$$
\le tB_D^2\cdot \frac{1}{(\log\zeta^\star(s)-2B_D)^{3/2}} \to 0 \qquad \text{as} \qquad s\to 1^+.
$$

On the other hand, in the first term,

$$
\frac{1}{\zeta^\star(s)} \sum_{C^{\zeta^\star(s)}<\|m\|\le D^{\zeta^\star(s)}} \frac{\alpha(m)}{\|m\|^s}\left|\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right| \le \mathbb{E}\left[\left|\frac{\omega_{\mathcal{M}}(m)-\log\zeta^\star(s)}{\sqrt{\log\zeta^\star(s)}}\right|\right] = O(1),
$$

as $s \to 1^+$ by the central limit theorem for $\omega_{\mathcal{M}}(X_s)$ with the zeta-distribution (Corollary 4.1). Here $\mathbb{E}$ denotes the expectation with respect to the zeta-distribution. Thus, it can be seen that the first term also tends to zero as $s \to 1^+$.

Putting everything together, except for sums I_s, II_s and IV_s, all other terms $\to 0$ as $s \to 1^+$. Moreover, from (6), (7) and (10), we deduce that

$$
\psi_s(t) = \frac{1}{\zeta^\star(s)} \sum_{m\in\mathcal{M}} \alpha(m)\,\|m\|^{-s} \exp\left(it\,\frac{\omega_{\mathcal{M}}(m)-\log\log\|m\|}{\sqrt{\log\log\|m\|}}\right) + O\left(\frac{1}{D^{1/\kappa}}\right)
$$

as $s \to 1^+$. Choosing D arbitrarily large and recalling that $\psi_s(t) \to e^{-t^2/2}$ as $s \to 1^+$, along with Lévy's continuity theorem proves the claim. □

4.2. **Deducing Theorem 1.1.** We now apply a Tauberian theorem to establish the Erdős-Kac theorem in our setup.

Corollary 4.2 is equivalent to

$$
\lim_{s\to 1^+}(s-1)\sum_{m\in\mathcal{M}} \frac{\alpha(m)\,\exp\left(\frac{it(\omega_{\mathcal{M}}(m)-\log\log\|m\|)}{\sqrt{\log\log\|m\|}}\right)}{\|m\|^s} = \kappa\, e^{-t^2/2},
$$

Applying the Hardy-Littlewood-Karamata theorem (Corollary 3.1), we deduce that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{m\in\mathcal{F}_N} \frac{\alpha(m)\, \exp\left(\frac{it(\omega_{\mathcal{M}}(m)-\log\log\|m\|)}{\sqrt{\log\log\|m\|}}\right)}{\|m\|} = \kappa\, e^{-t^2/2},$$

which by (1) is same as

$$\lim_{N\to\infty} \frac{1}{H_{\alpha,N}} \sum_{m\in\mathcal{F}_N} \frac{\alpha(m)\, \exp\left(\frac{it(\omega_{\mathcal{M}}(m)-\log\log\|m\|)}{\sqrt{\log\log\|m\|}}\right)}{\|m\|} = e^{-t^2/2}.$$

This proves Theorem 1.1.

5. **Support on Square-free elements**

This section is dedicated to proving Theorem 1.2. As in the previous section, we will first establish a central limit theorem for the corresponding zeta-distribution. Since the arguments in this section are modeled along the lines of the proof of Theorem 1.1, we omit the routine details.

Recall that $\eta : \mathcal{M} \to \mathbb{R}_{\geq 0}$ is a multiplicative function and $\|\cdot\| : \mathcal{M} \to \mathbb{R}_{\geq 0}$ is a 'norm' map. Associated to this data, we have the zeta-function

$$\zeta_{\eta,\mathcal{M}}(s) := \sum_{m\in\mathcal{M}} \frac{\eta(m)}{\|m\|^s} = \prod_{p\in\mathcal{P}} \left(1 + \frac{\eta(p)}{\|p\|^s}\right),$$

which converges in $\mathrm{Re}(s) > 1$. By hypothesis (i) of Theorem 1.2, we have that

$$\lim_{s\to 1^+} (s-1)\, \zeta_{\eta,\mathcal{M}}(s) = \kappa.$$

Thus, the Hardy-Littlewood-Karamata theorem (Corollary 3.1) gives

$$H_{\eta,N} = \sum_{m\in\mathcal{F}_N} \frac{\eta(m)}{\|m\|} = (\kappa + o(1))\, \log N. \tag{12}$$

As before, we define a random variable Y_s with the $\zeta_{\eta,\mathcal{M}}(s)$-distribution for $s > 1$ by

$$\Pr(Y_s = m) = \left(\frac{\eta(m)}{\|m\|^s}\right) \times \frac{1}{\zeta_{\eta,\mathcal{M}}(s)},$$

which is expressible as a product of random variables

$$Y_s = \prod_{p\in\mathcal{P}} p^{d_p(s)},$$

where $d_p(s) = 0$ or 1 since Y_s only takes squarefree values. Note that

$$\begin{aligned}\Pr(d_p(s) = 1) &= \frac{1}{\zeta_{\eta,\mathcal{M}}(s)} \sum_{\substack{m\in\mathcal{M}\\ \nu_p(m)=0}} \frac{\eta(p)\,\eta(m)}{\|pm\|^s} = \left(\frac{\eta(p)}{\|p\|^s}\right) \frac{\prod_{\ell\in\mathcal{P},\ell\neq p}\left(1 + \frac{\eta(\ell)}{\|\ell\|^s}\right)}{\prod_{\ell\in\mathcal{P}}\left(1 + \frac{\eta(\ell)}{\|\ell\|^s}\right)} \\ &= \frac{\frac{\eta(p)}{\|p\|^s}}{1 + \frac{\eta(p)}{\|p\|^s}} =: \psi(p,s). \end{aligned} \tag{13}$$

Moreover, for distinct primes p_1 and $p_2 \in \mathcal{P}$,

$$\Pr(d_{p_1}(s)=1,\, d_{p_2}(s)=1) = \frac{1}{\zeta_{\eta,\mathcal{M}}(s)} \sum_{\substack{m\in\mathcal{M},\\ \nu_{p_1}(m)=0\\ \nu_{p_2}(m)=0}} \frac{\eta(p_1)\,\eta(p_2)\,\eta(m)}{||p_1\,p_2\,m||^s} = \Pr(d_{p_1}(s)=1)\Pr(d_{p_2}(s)=1)$$

by an argument similar to above. Therefore, the random variables $d_{p_j}(s)$'s are independent for $p_j \in \mathcal{P}$. Also, we have

$$\omega_{\mathcal{M}}(Y_s) = \sum_{p\in\mathcal{P}} \mathbb{1}_{d_p(s)>1}.$$

For any $s>1$, set

$$\mathbb{P}_{\mathcal{M}^*}(s) := \sum_{p\in\mathcal{P}} \frac{\eta(p)}{||p||^s} \qquad \text{and} \qquad \mathbb{T}_{\mathcal{M}^*}(s) := \sum_{p\in\mathcal{P}} \frac{\frac{\eta(p)}{||p||^s}}{1+\frac{\eta(p)}{||p||^s}}.$$

The statements below can be proved in a similar manner as earlier.

Lemma 5.1. *The random variable Y_s satisfies the properties below.*

(a) As $s\to 1^+$, $\log\zeta_{\eta,\mathcal{M}}(s) = \mathbb{P}_{\mathcal{M}^*}(s) + O(1)$.

(b) As $s\to 1^+$,

$$\mathbb{T}_{\mathcal{M}^*}(s) = \mathbb{P}_{\mathcal{M}^*}(s) + O(1).$$

(b) The mean and variance of $\omega_{\mathcal{M}}(Y_s)$ is

$$\mathbb{E}\left[\omega_{\mathcal{M}}(Y_s)\right] = \mathbb{T}_{\mathcal{M}^*}(s) \qquad \textit{and} \qquad \mathit{var}\left(\omega_{\mathcal{M}}(Y_s)\right) = \mathbb{T}_{\mathcal{M}^*}(s) + O(1)$$

as $s\to 1^+$.

(c) The characteristic function of $\omega_{\mathcal{M}}(Y_s)$ is

$$\prod_{p\in\mathcal{P}} \left(1+\left[\left(e^{it}-1\right)\psi(p,s)\right]\right).$$

Proof. Taking logarithm of the Euler product of $\zeta_{\eta,\mathcal{M}}(s)$ gives

$$\log\zeta_{\eta,\mathcal{M}}(s) = \sum_{p\in\mathcal{P}} \log\left(1+\frac{\eta(p)}{||p||}\right) = \sum_{p\in\mathcal{P}}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\eta(p)^n}{n\,||p||^{ns}} = \mathbb{P}_{\mathcal{M}^*}(s) + O\left(\sum_{p\in\mathcal{P}}\sum_{n=2}^{\infty}\frac{|\eta(p)|}{n||p||^{ns}}\right),$$

where the series in the big O-term converges as $s\to 1^+$ because $|\eta(p)| < ||p||^{1/2}$ by hypothesis. This proves part (a). To deduce part (b), observe that

$$\begin{aligned}\mathbb{T}_{\mathcal{M}^*}(s) = \sum_{p\in\mathcal{P}}\psi(p,s) &= \sum_{p\in\mathcal{P}}\frac{\eta(p)}{||p||^s}\left(1-\frac{\eta(p)}{||p||^s}+\frac{\eta(p)^2}{||p||^{2s}}-\frac{\eta(p)^3}{||p||^3}+\cdots\right)\\ &= \sum_{p\in\mathcal{P}}\frac{\eta(p)}{||p||^s} + O\left(\sum_{p\in\mathcal{P}}\sum_{k=2}^{\infty}\frac{|\eta(p)|^k}{||p||^{ks}}\right),\end{aligned}$$

where the series in the big O-term once again converges as $s\to 1^+$ because $|\eta(p)| < ||p||^{1/2}$. To find the mean, use (13) to get

$$\mathbb{E}\left[\omega_{\mathcal{M}}(Y_s)\right] = \sum_{p\in\mathcal{P}}\mathbb{E}[\mathbb{1}_{d_p(s)=1}] = \sum_{p\in\mathcal{P}}\psi(p,s) = \mathbb{T}_{\mathcal{M}^*}(s).$$

For the variance, we have

$$\begin{aligned} var\left(\omega_{\mathcal{M}}(Y_s)\right) &= \mathbb{E}\left[\omega_{\mathcal{M}}(Y_s)^2\right] - \mathbb{E}\left[\omega_{\mathcal{M}}(Y_s)\right]^2 \\ &= \sum_{\substack{p,q\in\mathcal{P}\\ p\neq q}} \mathbb{E}\left[\mathbb{1}_{d_p(s)=1}, \mathbb{1}_{d_q(s)=1}\right] + \sum_{p\in\mathcal{P}} \mathbb{E}\left[\mathbb{1}_{d_p(s)=1}\right] - \mathbb{T}^2_{\mathcal{M}^*}(s) \\ &= \sum_{\substack{p,q\in\mathcal{P}\\ p\neq q}} \mathbb{E}\left(\left[\mathbb{1}_{d_p(s)=1}\right] \mathbb{E}\left[\mathbb{1}_{d_q(s)=1}\right]\right) - \mathbb{T}^2_{\mathcal{M}^*}(s) + \mathbb{T}_{\mathcal{M}^*}(s) \\ &= \left(\sum_{p\in\mathcal{P}} \mathbb{E}\left[\mathbb{1}_{d_p(s)=1}\right]\right)^2 - \mathbb{T}^2_{\mathcal{M}^*}(s) - \sum_{p\in\mathcal{P}} \mathbb{E}\left[\mathbb{1}_{d_p(s)=1}\right]^2 + \mathbb{T}_{\mathcal{M}^*}(s) \\ &= \mathbb{T}_{\mathcal{M}^*}(s) - \sum_{p\in\mathcal{P}} \left(\frac{\frac{\eta(p)}{\|p\|^s}}{1+\frac{\eta(p)}{\|p\|^s}}\right)^2. \end{aligned}$$

In the series above, one can bound the summand by $(\eta(p)/\|p\|^s)^2$ and use that the sum

$$\sum_{p\in\mathcal{P}} \left(\frac{\eta(p)}{\|p\|^s}\right)^2 \leq \sum_{p\in\mathcal{P}} \frac{1}{\|p\|^{2s-2\theta}},$$

which converges as $s \to 1^+$.

Now, to compute the characteristic function of $\omega_{\mathcal{M}}(Y_s)$, we use the independence of the $d_p(s)$'s to obtain

$$\begin{aligned} \Phi^{\star}_s(t) = \mathbb{E}\left(e^{it\sum_{p\in\mathcal{P}} \mathbb{1}_{d_p(s)=1}}\right) &= \prod_{p\in\mathcal{P}} \mathbb{E}\left(e^{it\,\mathbb{1}_{d_p(s)=1}}\right) = \prod_{p\in\mathcal{P}} \left(e^{it}\left(\frac{\frac{\eta(p)}{\|p\|^s}}{1+\frac{\eta(p)}{\|p\|^s}}\right) + \frac{1}{1+\frac{\eta(p)}{\|p\|^s}}\right) \\ &= \prod_{p\in\mathcal{P}} \left(\frac{1+\left(e^{it}\,\frac{\eta(p)}{\|p\|^s}\right)}{1+\frac{\eta(p)}{\|p\|^s}}\right) = \prod_{p\in\mathcal{P}} \left(1+\left[\left(e^{it}-1\right)\,\psi(p,s)\right]\right). \end{aligned}$$

□

We now establish an Erdős-Kac theorem for $\omega_{\mathcal{M}}(Y_s)$,

Theorem 5.1. *As $s \to 1^+$,*

$$\frac{\omega_{\mathcal{M}}(Y_s) - \mathbb{T}_{\mathcal{M}}(s)}{\sqrt{\mathbb{T}_{\mathcal{M}}(s)}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution with mean 0 and variance 1.

Proof. By part (d) of Lemma 5.1, we get

$$\mathbb{E}\left[\exp\left(\frac{it(\omega_{\mathcal{M}}(Y_s) - \mathbb{T}_{\mathcal{M}}(s))}{\sqrt{\mathbb{T}_{\mathcal{M}}}(s)}\right)\right] = e^{-it\sqrt{\mathbb{T}_{\mathcal{M}}(s)}}\,\mathbb{E}\left[\exp\left(\frac{it\,\omega_{\mathcal{M}}(Y_s)}{\sqrt{\mathbb{T}_{\mathcal{M}}(s)}}\right)\right] = e^{-it\sqrt{\mathbb{T}_{\mathcal{M}}(s)}}\,\Phi^{\star}_s\left(\frac{t}{\sqrt{\mathbb{T}_{\mathcal{M}}(s)}}\right).$$

As $|\alpha(p)| < 2\|p\|$, we can write

$$\begin{aligned}\Phi_s^\star(t) &= \prod_{p\in P} \exp\left(\log\left(1 + \left[\left(e^{it} - 1\right)\, \psi(p,s)\right]\right)\right)\\ &= \exp\left(-\sum_{p\in\mathcal{P}} \sum_{n=1}^{\infty} (-\psi(p,s))^n \, \frac{(e^{it}-1)^n}{n}\right)\\ &= \exp\left(\mathbb{T}_{\mathcal{M}}(s)(e^{it}-1)\right) \times \exp\left(-\sum_{p\in\mathcal{P}} \sum_{n=2}^{\infty} (-\psi(p,s))^n \frac{(e^{it}-1)^n}{n}\right).\end{aligned}$$

Therefore, using $\Phi_s^\star\left(t/\sqrt{\mathbb{T}_{\mathcal{M}^*}(s)}\right)$, we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{it(\omega_{\mathcal{M}}(X_s) - \mathbb{T}_{\mathcal{M}}(s))}{\sqrt{\mathbb{T}_{\mathcal{M}}}(s)}\right)\right]\\ &= \exp\left(\mathbb{T}_{\mathcal{M}}(s)\left(e^{it/\sqrt{\mathbb{T}_{\mathcal{M}}(s)}} - 1\right) - it\sqrt{\mathbb{T}_{\mathcal{M}}(s)}\right)\\ &\qquad\qquad \times \exp\left(-\sum_{p\in\mathcal{P}} \sum_{n=2}^{\infty} (-\psi(p,s))^n \frac{\left(e^{it/\sqrt{\mathbb{T}_{\mathcal{M}}(s)}} - 1\right)^n}{n}\right). \end{aligned} \tag{14}$$

Taylor series expansion implies

$$\mathbb{T}_{\mathcal{M}}(s)\left(e^{it/\sqrt{\mathbb{T}_{\mathcal{M}}(s)}} - 1\right) - it\sqrt{\mathbb{T}_{\mathcal{M}}(s)} = -\frac{t^2}{2} + O\left(\frac{1}{\sqrt{\mathbb{T}_{\mathcal{M}}(s)}}\right),$$

as $s \to 1^+$. Furthermore, since the series

$$-\sum_{p\in\mathcal{P}} \sum_{n=2}^{\infty} (-\psi(p,s))^n \frac{\left(e^{it/\sqrt{\mathbb{T}_{\mathcal{M}}(s)}} - 1\right)^n}{n}$$

converges uniformly as $s \to 1^+$, and $e^{it/\sqrt{\mathbb{T}_{\mathcal{M}}(s)}} - 1 \to 0$, the second term in (3) tends to 1 as $s \to 1^+$. This proves that the characteristic function of

$$\frac{\omega_{\mathcal{M}}(X_s) - \mathbb{T}_{\mathcal{M}}(s)}{\sqrt{\mathbb{T}_{\mathcal{M}}(s)}}$$

converges to $e^{-t^2/2}$ as $s \to 1^+$, implying the theorem. □

Corollary 5.1. *As* $s \to 1^+$

$$\frac{\omega_{\mathcal{M}}(Y_s) - \log\zeta_{\eta,\mathcal{M}}(s)}{\sqrt{\log\zeta_{\eta,\mathcal{M}}(s)}} \xrightarrow{d} \mathcal{N}(0,1)$$

Proof. This is immediate from Theorem 5.1 and parts (a) and (b) of Lemma 5.1. □

The derivation of the central limit theorem for

$$\frac{\omega_{\mathcal{M}}(Y_s) - \log\log\|Y_s\|}{\sqrt{\log\log\|Y_s\|}}$$

and Theorem 1.2 from Theorem 5.1 via the application of the Tauberian theorem (Corollary 3.1) follows verbatim as in the previous section. The details are left to the reader.

6. **Erdős-Kac with uniform measure**

From the Erdős-Kac for the zeta-distribution, one can obtain the Erdős-Kac for the monoid with the uniform measure, conditional upon the analytic continuation of a certain series. This is a consequence of the application of the Wiener-Ikehara Tauberian theorem (Theorem 3.3) instead of Corollary 3.1. We make a note of this observation in the theorems below.

Theorem 6.1. *Let $\alpha : \mathcal{M} \to \mathbb{R}_{\geq 0}$ be a completely multiplicative function such that*

(i) $\lim_{s\to 1^+}(s-1)\,\zeta_{\alpha,\mathcal{M}}(s) = \kappa$ *for some* $\kappa \in \mathbb{C}$

(ii) $|\alpha(p)| \leq \|p\|^\theta$ *for some* $0 < \theta < 1/2$ *and all* $p \in \mathcal{P}$

(iii) the Dirichlet series

$$\mathbb{F}(s) = \sum_{m\in\mathcal{M}} \frac{\alpha(m)}{\|m\|^s}\, e^{it\left(\frac{\omega_{\mathcal{M}}(m)-\log\log\|m\|}{\sqrt{\log\log\|m\|}}\right)}$$

has analytic continuation to an open neighborhood of $Re(s) \geq 1$, *with a simple pole at* $s = 1$.

Then

$$\lim_{N\to\infty} \frac{1}{\#\mathcal{F}_N}\, \#\left\{m \in \mathcal{F}_N \,:\, \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

for every real number x.

Theorem 6.2. *Denote by*

$$\mathcal{F}_{N,sf} = \{m \in \mathcal{M} \,:\, \nu_p(m) \leq 1 \text{ for all } p \in \mathcal{P}\}\,.$$

Let $\eta : \mathcal{M} \to \mathbb{R}_{\geq 0}$ be a multiplicative function supported on the squarefree elements of $\mathcal{M}$ such that

(i) $\lim_{s\to 1^+}(s-1)\,\zeta_{\eta,\mathcal{M}}(s) = \kappa$ *for some* $\kappa \in \mathbb{R}$.

(ii) $|\eta(p)| \leq \|p\|^\theta$ *for some* $0 < \theta < 1/2$ *and all* $p \in \mathcal{P}$

(iii) the Dirichlet series

$$\mathbb{G}(s) = \sum_{m\in\mathcal{M}} \frac{\eta(m)}{\|m\|^s}\, e^{it\left(\frac{\omega_{\mathcal{M}}(m)-\log\log\|m\|}{\sqrt{\log\log\|m\|}}\right)}$$

has analytic continuation to an open neighborhood of $Re(s) \geq 1$, *with a simple pole at* $s = 1$.

Then

$$\lim_{N\to\infty} \frac{1}{\#\mathcal{F}_{N,sf}}\, \#\left\{m \in \mathcal{F}_{N,sf} \,:\, \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

for every real number x.

7. **Concluding remarks**

It is crucial to note that Erdős-Kac theorem with respect to the harmonic measure is implied by the Erdős-Kac theorem with the uniform measure. Indeed, by the classical Erdős-Kac theorem, we have

$$\mathcal{S}(N) := \sum_{n=1}^{N} e^{it\left(\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\right)} = \left(e^{-t^2/2} + o(1)\right) N.$$

By partial summation, we get,

$$\sum_{n=1}^{N} \frac{\exp\left(it\left(\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\right)\right)}{n} = \frac{\mathcal{S}(N)}{N} + \int_1^N \frac{\mathcal{S}(x)}{x^2}\,dx$$
$$= \frac{\mathcal{S}(N)}{N} + e^{-t^2/2}\int_1^N \frac{1}{x}\,dx + o\left(\int_1^N \frac{1}{x}\,dx\right)$$
$$= \frac{\mathcal{S}(N)}{N} + e^{-t^2/2}\log N + o\left(\log N\right),$$

which implies that

$$\lim_{N\to\infty} \frac{1}{H_N}\sum_{n=1}^{N} \frac{e^{it\left(\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\right)}}{n} = e^{-t^2/2},$$

giving the Erdős-Kac theorem for harmonic distribution. Recall that $H_N = \sum_{n=1}^N n^{-1}$. In fact, the same partial summation argument leads to the following intriguing proposition.

Proposition 7.1. *Let $\mathcal{P}$ be a set of countably infinite elements, $\mathcal{M}$ be the monoid generated by them and $\|\cdot\| : \mathcal{M} \to \mathbb{R}_{\geq 0}$ be a monoid homomorphism as in Theorem 1.1. Let $\mathcal{F}_N := \{m \in \mathcal{M} : \|m\| \leq N\}$ and ν_N denote the uniform measure on $\mathcal{M}$. Fix a $0 < \theta \leq 1$. Set*

$$\mathcal{H}_{\theta,N} := \sum_{m\in\mathcal{F}_N} \frac{1}{\|m\|^\theta}$$

and define a θ-measure $\mu_{\theta,N}$ on $\mathcal{F}_N$ by

$$\mu_{\theta,N}\left(\{m\}\right) = \frac{1}{\mathcal{H}_{\theta,N}} \times \frac{1}{\|m\|^\theta} \qquad \text{for each} \qquad m \in \mathcal{F}_N.$$

Suppose that

$$\lim_{N\to\infty} \nu_N\left(\left\{m \in \mathcal{F} : \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \leq x\right\}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Then,

$$\lim_{N\to\infty} \mu_{\theta,N}\left(\left\{m \in \mathcal{F} : \frac{\omega_{\mathcal{M}}(m) - \log\log\|m\|}{\sqrt{\log\log\|m\|}} \leq x\right\}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

However, considerably weaker hypothesis are needed to establish Erdős-Kac with the harmonic measure. For instance, one can compare Theorem 1.1 with the axiomatization of the Erdős-Kac theorem for uniform distribution by Yu-Ru Liu [24]. The condition required for [24, Theorem 1], namely that

$$\#\{M : m \in \mathcal{M}, m \leq N\} = \kappa N + O(N^\theta) \qquad \text{for some} \qquad 0 \leq \theta < 1$$

is stronger than the hypothesis (i) of Theorem 1.1 in this paper.

Similarly, in the context of Beurling primes, the prime number theorem [1] is known to hold provided

$$N_{\mathcal{B}}(x) = Ax + O\left(\frac{x}{(\log x)^\lambda}\right) \qquad \text{for} \qquad \lambda > 3/2,$$

and the Erdős-Kac is known [32] under the assumption of

$$N_{\mathcal{B}}(x) = Ax + O\left(x^\theta\right) \qquad \text{for} \qquad 0 < \theta < 1.$$

In contrast to these, the hypothesis (i) in Theorem 1.1, which is equivalent to

$$N_{\mathcal{B}}(x) = Ax + o(x),$$

is significantly weaker. Thus, there is a possibility of the existence of Beurling systems for which the prime number theorem as well as the Erdős-Kac theorem with uniform measure fail, but the Erdős-Kac with harmonic measure holds! Finding explicit examples of such systems is an interesting problem, which we relegate to future research.

It is not difficult to adapt the technique of M. Cranston and T. Mountford outlined in this paper to derive central limit theorems for $\Omega_{\mathcal{M}}(m)$ and more generally for strongly additive functions on monoids. One can also deduce 'localized' versions of Theorem 1.1 and Theorem 1.2 following the arguments detailed in [5, Section 5]. We leave these as exercises for the interested reader.

The methods utilized in this paper are restrictive because of the requirement of complete multiplicativity of the associated weights. This is essential in deducing the independence of the $c_p(s)$ random variables and the expression for the characteristic function of $\omega(X_s)$ as an Euler product. In order to make the method work for multiplicative weights, one perhaps requires more knowledge of error terms. This avenue warrants further investigation. However, there is a class of Erdős-Kac theorems for which this technique may not be amenable, namely the results of Halberstam [14] for $\omega(p+a)$ for a fixed natural number a, Murty-Saidak [29] for $\omega(f_a(p))$ where a is a fixed natural number and $f_a(p)$ is the order of $a \bmod p$, Murty-Murty [26] [27] for $\omega(\tau(p))$ where $\tau(n)$ denotes the Ramanujan tau-function and Murty-Dixit [8] for $\omega_y(p+a)$. For a combined approach to these, one must resort to the crucial tool of sieve theory which has been established in [30].

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